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AN EXAMPLE CONCERNING EQUIVARIANT DEFORMATIONS

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Dedicated to the memory of Juliusz P. Schauder

Abstract. We give an example of Z_2-space X with a property that the identity map $\mathrm{id}_X : X \to X$ as well as its restriction to the fixed point set of the group action $\mathrm{id}^{Z_2} : X^{Z_2} \to X^{Z_2}$ are deformable to fixed point free maps whereas there is no fixed point free map in the equivariant homotopy class of the identity $[\mathrm{id}_X]_{Z_2}$.

1. Introduction

Let X and Y be spaces on which an action of a finite group G is defined. There are some equivariant properties of a G-map $f : X \to Y$ which are satisfied provided the same properties hold without any group action for all restrictions f^H of f to the fixed point sets

$$X^H = \{x \in X \mid gx = x \text{ if } g \in H,\ H \text{ is a subgroup of } G\}.$$

This is, for example, the case of f to be a G-homotopy equivalence or f to be a G-fibration if the spaces involved are sufficiently nice, say compact G-ENRs. In this note we wish to investigate the property of a G-selfmap $f : X \to X$ to be equivariantly deformable to a fixed point free map. Roughly speaking, it will be seen by our example that also for G-ENR's the equivariant information is not completely contained in the fixed point sets X^H. Classical deformability to fixed

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point free maps on the fixed point sets does not imply equivariant deformability on X to a fixed point free map.

2. The Example

Let X be a double torus T_2 with a two sphere S^2 inside glued to it along the equator as showed in the picture below.

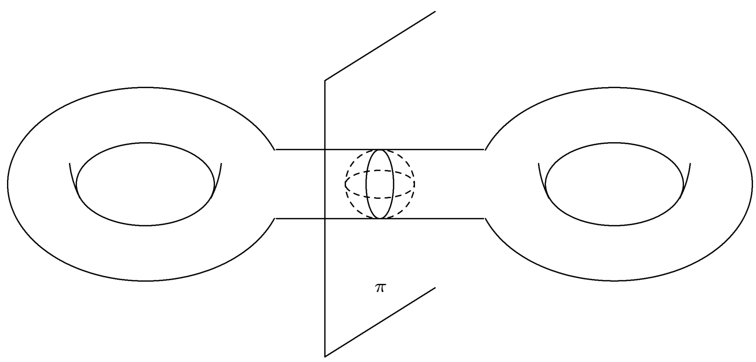

Figure 1

This space can be viewed as a Z_2-space if we define the action of Z_2 as the reflection with respect to the plane in the figure. Observe, that the fixed point set X^{Z_2} is a circle. We have that the identity map id_X is deformable to a fixed point free map on each fixed point sets X^{Z_2} and X. The last, because X has no local separating points and the Euler characteristic $\chi(X) = 0$ (cf. [2]). Now, we show that the following proposition holds true

Proposition 2.1. *The identity map* id_X *is not* Z_2*-homotopic to a fixed point free map.*

Proof. Contrary to our claim, suppose that $f : X \to X$ is a fixed point free map Z_2-homotopic to id_X. Take an $\varepsilon > 0$ such that the distance $d(f(x), x) > 2\varepsilon$ for all $x \in X$. Since X^{Z_2} is an ENR we have an equivariant ε-homotopy $\{h_t\}$ such that $f \simeq h : X \to X$ with h fixed point free and taut over X^{Z_2} in some invariant neighbourhood U of X^{Z_2}. Now, the quotient of the free part $(X \setminus X^{Z_2})/Z_2$ has two components C_1 and C_2. Following Wilczyński in [4] we find that the trace of the composition

$$q_k \circ h^* \circ i_k^* : H^*(X, X \setminus p^{-1}(C_k)) \to H^*(X, X \setminus p^{-1}(C_k)), \quad k = 1, 2$$

is invariant under equivariant homotopies, where h^* and i_k^* are homomorphisms induced on cohomology by h and the inclusion map

$$i_k : (X, X^{Z_2}) \to (X, X \setminus p^{-1}(C_k))$$

respectively, p is the natural projection onto the orbit space and q_k is the homomorphism defined by projection on the corresponding factor in the direct sum

$$H^*(X, X^{Z_2}) \cong H^*(X, X \setminus p^{-1}(C_1)) \oplus H^*(X, X \setminus p^{-1}(C_2)).$$

Moreover, he has proved that for taut maps that trace is equal to a fixed point index $\operatorname{ind}(h_{|V_k} : V_k \to X)$, where V_k is any open subset such that

$$\overline{V_k} \subset Y_k \setminus X^{Z_2} \subset U \cup V_k, \quad Y_k = S^2 \text{ or } T_2, \quad k = 1, 2,$$

and these two indices are zero, because h is fixed point free. On the other hand, by the equivariant homotopy invariance we derive

$$\operatorname{tr}(q_k \circ h^* \circ i_k^*) = \operatorname{tr}(q_k \circ i_k^*) = \chi(X, Y_k) = \pm 2$$

which contradicts the fact the indices above are zero. □

3. Remarks and Questions

Our example gives a negative answer for the question posed by the authors in [1] (see Problem 3.6). Compare also our example with the work of P. Wong [5], where for a finite G-Wecken complex the fixed point free equivariant class of the identity is characterized.

If we are dealing with closed smooth manifolds with a smooth action of a finite group G, B. Jiang has proved in [3] that in this category such an example for the identity map is impossible. For arbitrary f and a free action on a closed manifold equivariant deformability of f to a fixed point free map is equivalent to a classical deformability of f to a fixed point free map. This can be easily proved via the Nielsen fixed point theory when $\dim(X) \geq 3$. For free actions on surfaces we expect that this is also true. One only needs to consider each type of a closed surface separately. We do not know the answer for general f and an arbitrary action case.

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