Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 19, 2002, 123–152

ZEROS OF CLOSED 1-FORMS, HOMOCLINIC ORBITS AND LUSTERNIK–SCHNIRELMAN THEORY

Michael Farber

Abstract. In this paper we study topological lower bounds on the number of zeros of closed 1-forms without Morse type assumptions. We prove that one may always find a representing closed 1-form having at most one zero. We introduce and study a generalization $\mathrm{cat}\,(X,\xi)$ of the notion of the Lusternik–Schnirelman category, depending on a topological space X and a 1-dimensional real cohomology class $\xi \in H^1(X;\mathbb{R})$. We prove that any closed 1-form ω in class ξ has at least $\mathrm{cat}\,(X,\xi)$ zeros assuming that ω admits a gradient-like vector field with no homoclinic cycles. We show that the number $\mathrm{cat}\,(X,\xi)$ can be estimated from below in terms of the cup-products and higher Massey products.

This paper corrects some my statements made in [6], [7].

1. Introduction

The Novikov inequalities [12], [13] estimate the numbers of zeros of different indices of closed 1-forms ω on manifolds lying in a given cohomology class, assuming that all the zeros are non-degenerate in the sense of Morse. Applications of the Novikov inequalities in mechanics, in geometry, and in symplectic topology are well-known, see [3], [9], [10], [14], [15].

2000 *Mathematics Subject Classification.* 37C29, 58E05.

Key words and phrases. Morse theory, Lusternik–Schnirelman theory, closed 1-form, Massey products, homoclinic orbits.

Partially supported by the US–Israel Binational Science Foundation; part of this work was done while the author visited Max-Planck Institute for Mathematics in Bonn.

In this paper we show that one may always realize a nonzero cohomology class by a closed 1-form with at most one (degenerate) zero. In the proof we use the technique of rearrangements of critical points and the result of F. Takens [18], which describes the conditions when several critical points of a function can be collided into one.

Central role in this paper plays a suitable generalization of the notion of the Lusternik–Schnirelman category. For any pair (X, ξ), consisting of a topological space X and a real cohomology class $\xi \in H^1(X; \mathbb{R})$, we define a non-negative integer $\operatorname{cat}(X, \xi)$, the category of X with respect to the cohomology class ξ. The definition of $\operatorname{cat}(X, \xi)$ is similar in spirit to the definition of $\operatorname{cat}(X)$; it deals with open covers of X with certain homotopy properties. We show that $\operatorname{cat}(X, \xi)$ depends only on the homotopy type of (X, ξ) and coincides with $\operatorname{cat}(X)$ in the case $\xi = 0$. If $\xi \neq 0$ then $\operatorname{cat}(X, \xi) < \operatorname{cat}(X)$; we show by examples that the difference, $\operatorname{cat}(X) - \operatorname{cat}(X, \xi)$, may be an arbitrary positive integer.

The main theorem of the paper (Theorem 4.1) states that any smooth closed 1-form ω on a smooth closed manifold X must have at least $\operatorname{cat}(X, \xi)$ geometrically distinct zeros, where $\xi = [\omega] \in H^1(X; \mathbb{R})$ denotes the cohomology class of ω, assuming that ω admits a gradient-like vector field with no homoclinic cycles. Recall that *a homoclinic orbit* is defined as a trajectory $\gamma(t)$, $t \in \mathbb{R}$, such that both limits $\lim_{t\to\infty} \gamma(t)$ and $\lim_{t\to-\infty} \gamma(t)$ exist and are equal. More generally, *a homoclinic cycle of length* n is a sequence of orbits $\gamma_1, \dots, \gamma_n$ such that

$$\lim_{t\to\infty} \gamma_i(t) = \lim_{t\to-\infty} \gamma_{i+1}(t)$$

for $i = 1, \dots, n-1$ and

$$\lim_{t\to\infty} \gamma_n(t) = \lim_{t\to-\infty} \gamma_1(t).$$

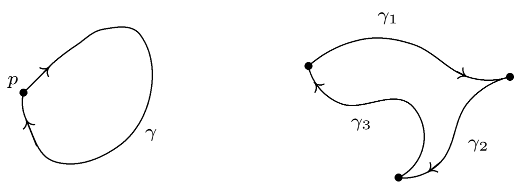

FIGURE 1. Homoclinic orbit (left) and homoclinic cycle (right)

Viewed differently, the main theorem of the paper claims that any gradient-like vector field of a closed 1-form ω has a homoclinic cycle if the number of zeros of ω is less than $\operatorname{cat}(M, \xi)$.

The homoclinic orbits were discovered by H. Poincaré and were studied by S. Smale. In the mathematical literature there are many results about existence of homoclinic orbits in Hamiltonian systems. Homoclinic orbits may not exist in the gradient systems for functions, i.e. in the case $\xi = 0$, corresponding to the classical Lusternik–Schnirelman theory.

In recent papers [5]–[7] we described cohomological cup-length type estimates on the number of zeros of closed 1-forms. Unfortunately, they are incorrect as stated. Purely algebraic Proposition 3 of [6] and a similar in character Lemma 6.6 of [7] are incorrect. These algebraic statements hold under slightly stronger assumptions; in order to meet these assumptions in the main theorems, however, one has to make extra assumptions on the closed 1-form. This paper gives a different (more geometric) approach to show that the main results of [7] hold for closed 1-forms having a gradient-like vector field with no homoclinic cycles. The cohomological lower bounds of [6] also require some additional assumption, which will be discussed elsewhere.

I would like to thank Octav Cornea, Pierre Milman, Kaoru Ono and Shmuel Weinberger for useful discussions and help.

2. Colliding the critical points

In this section we will prove the following realization result.

Theorem 2.1. *Let M be a closed connected n-dimensional smooth manifold, and let $\xi \in H^1(M;\mathbb{Z})$ be a nonzero cohomology class. Then there exists a smooth closed 1-form ω in class ξ having at most one zero.*

Proof. We will assume that the class $\xi \in H^1(M;\mathbb{Z})$ is indivisible, i.e. is not a multiple of another integral class. Our statement clearly follows from this special case.

Our purpose is to show that we may find a smooth map $\phi\colon M \to S^1$ with Morse critical points, having the following properties:

- (A) *The cohomology class of the closed 1-form $\widetilde{\omega} = \phi^*(d\theta)$ coincides with ξ, where $d\theta$ denotes the standard angular form on the circle S^1.*
- (B) *All fibers $\phi^{-1}(b)$ are connected, where $b \in S^1$.*
- (C) *The map ϕ has at most one critical value $b_0 \in S^1$. In other words, all critical points of ϕ lie in the same fiber $f^{-1}(b_0)$.*

Having achieved this, we may apply the technique of F. Takens [18, pp. 203–206] which allows to collide the critical points of ϕ (equivalently, the zeros of closed 1-form $\widetilde{\omega}$) into a single degenerate critical point of a closed 1-form ω lying in the same cohomology class. Namely, first, we may find a piecewise smooth tree $\Gamma \subset \phi^{-1}(b_0)$ containing all the critical points of ϕ. Secondly, we may find a continuous map $\Psi\colon M \to M$ with the following properties:

- $\Psi(\Gamma)$ *is a single point* $p \in \Gamma$,
- $\Psi|_{M-\Gamma}$ *is a diffeomorphism onto* $M - p$,

- Ψ *is the identity map on the complement of a small neighbourhood of* Γ*; in particular,* Ψ *is homotopic to the identity map* $M \to M$*.*

The circle-valued map $\phi \circ \Psi^{-1}$ is well-defined and is continuous. Moreover, $\phi \circ \Psi^{-1}$ is smooth on $M - p$. Applying Theorem 2.7 from [18] we see that we can replace the map $\phi \circ \Psi^{-1}$ in a small neighbourhood of p by a smooth map $\psi: M \to S^1$ having a single critical point. ψ is homotopic to ϕ and thus closed 1-form $\omega = \psi^*(d\theta)$ lies in the cohomology class ξ and has possibly a single zero.

In the rest of the proof we will show that we can find a smooth Morse map $\psi: M \to S^1$ with properties (A)–(C) above.

It is well known that any indivisible class $\xi \neq 0$ may be realized by a connected codimension one submanifold $V \subset M$ with an oriented normal bundle. Cutting M along V produces a compact cobordism N with $\partial N = \partial_+ N \cup \partial_- N$, a disjoint union of two copies of V. Consider a Morse function $f: N \to [0,1]$ having 0 and 1 as regular values and $f^{-1}(0) = \partial_+ N$, $f^{-1}(1) = \partial_- N$. We may assume that f has no critical points of indices 0 and $n = \dim M$. Moreover, we may construct f, such that all level sets $f^{-1}(c)$, where $c \in [0,1]$, are connected and having the self-indexing property: all critical points of f having Morse index i lie in $f^{-1}(i/n)$, where $i = 1, \dots, n-1$. The map $N \to S^1$, where $x \mapsto \exp(2\pi i f(x))$, defines a smooth map $\phi_1: M \to S^1$ in the cohomology class ξ with connected fibers having the following property: for any critical point $m \in M$, $d\phi_1(m) = 0$, with Morse index i, the image $\phi_1(x)$ equals $\exp(2\pi i/n)$. In other words, the critical points of ϕ_1 with the same Morse index lie in the same fiber, and these critical fibers $\phi_1^{-1}(b)$ appear in the order of their Morse indices, while the point b moves in a positive direction along the circle S^1.

For points $b_1, \dots, b_r \in S^1$ on the circle S^1 we will write

$$b_1 < b_2 < \dots < b_r < b_1$$

to denote that moving from point b_1 in a positive direction along the circle S^1, we first meet b_2, then $b_3, \dots$, until we again meet b_1.

Let us now formulate the following approximation to property (C):

(C_j) *All critical points of a smooth Morse map* $\phi: M \to S^1$ *with Morse index* i *lie in the same fiber* $\phi^{-1}(b_i)$*, where* $b_i \in S^1$*,* $i = 1, \dots, n-1$*, and*

$$b_1 < b_1 < \dots < b_j = b_{j+1} = \dots = b_{n-1} < b_1.$$

In particular, the critical values $b_j = b_{j+1} = \dots = b_{n-1}$ *coincide.*

Note that (C_1) is equivalent to (C), which is our purpose.

We have found above a smooth map $\phi_1: M \to S^1$, satisfying (A), (B), and (C_{n-1}). In the next step we will show that we may replace ϕ_1 by a smooth Morse map $\phi_2: M \to S^1$ with properties (A), (B) and (C_{n-2}). Let $b_1 < \dots < b_{n-1} < b_1$ be the critical values of ϕ_1. Consider a point $c \in S^1$, lying between b_{n-2} and

b_{n-1}. Cut M along the submanifold $\phi_1^{-1}(c)$ and consider the obtained cobordism N and a Morse function g from N to an interval, obtained by cutting the circle S^1 at point c. All level sets of g are connected. Moving from the bottom of this cobordism to the top, we meet $n-1$ critical levels; first we meet the level containing the critical points of Morse index $n-1$, then the levels containing the critical points with Morse indices $1, \dots, n-2$. We will use the theory of S. Smale of rearrangement of critical points. Choose a generic gradient-like vector field v for g. Then all integral trajectories of $\pm v$, which go out of the critical points of index $n-1$, reach ∂N without interaction with the other critical points. Therefore we may slide the critical points of index $n-1$ up some distance, putting them on the same level as the critical point of index $n-2$, see [11, Theorem 4.1]. In other words, we may replace g by a Morse function g', which coincides with g near ∂N, and has the same critical points, but the value at the critical points of index $n-1$ equals the value at the critical points of index $n-2$. Note that the level sets of g' are all connected:

(a) the bottom level ${g'}^{-1}(0)$ is unchanged and so it is connected,
(b) passing the critical levels with Morse indices $1, \dots, n-3$ may not create nonconnected level sets,
(c) in principle, nonconnected level sets may appear after passing the top critical value containing the critical points of indices $n-2$ and $n-1$; however in our situation all higher upper level sets are the same as for the previous function g, and so they are all connected.

Folding this cobordism back, gives a smooth map $\phi_2 \colon M \to S^1$ having properties (A), (B), and (C_{n-2}).

We may proceed similarly to find a smooth map $\phi_3 \colon M \to S^1$ with properties (A), (B), and (C_{n-3}). The critical values of ϕ_2 are $b_1 < \ldots < b_{n-2} = b_{n-1} < b_1$. We find a point $c \in S^1$ between b_{n-3} and $b_{n-2} = b_{n-1}$ and cut M along $\phi_2^{-1}(c)$. The Morse function on the obtained cobordism will have $n-2$ critical levels. The lowest will be the level containing all critical points of indices $n-2$ and $n-1$ and then the critical levels of points with the Morse indices $1, \dots, n-3$. Repeating the above procedure, we may slide the critical points of indices $n-2$ and $n-3$ up the same distance putting them on the same level as the critical points of index $n-3$.

Proceeding in this way inductively we arrive at a smooth map $\phi_{n-1} \colon M \to S^1$ having properties (A), (B), and (C_1) = (C). This completes the proof. $\square$

According to a remark in [3], a statement in the spirit of Theorem 2.1 was made by Yu. Chekanov at a seminar talk in 1996. As far as I know, no written account of his work is available.

3. Category of a space with respect to a cohomology class

3.1. Definition of $\mathrm{cat}(X,\xi)$**.** Let X be a finite CW-complex and let $\xi \in H^1(X;\mathbb{R})$ be a real cohomology class. We will define below a numerical invariant $\mathrm{cat}(X,\xi)$ (*the category of X with respect to class ξ*), depending only on the homotopy type of the pair (X,ξ). It will turn into the classical Lusternik–Schnirelman category $\mathrm{cat}(X)$ in the case $\xi = 0$. The main property of $\mathrm{cat}\,(X,\xi)$ is that it gives a relation between the number of geometrically distinct zeros which have closed 1-forms realizing class ξ, and the homoclinic orbits of their gradient-like vector fields, see Theorem 4.1.

Fix a continuous closed 1-form ω on X representing the cohomology class ξ. See Appendix A for definitions.

Definition 3.1. We will define $\mathrm{cat}(X,\xi)$ to be the least integer k such that for any integer $N > 0$ there exists an open cover

$$X = F \cup F_1 \cup \ldots \cup F_k, \tag{3.1}$$

such that:

(a) Each inclusion $F_j \to X$ is null-homotopic, where $j = 1,\ldots,k$.
(b) There exists a homotopy $h_t\colon F \to X$, where $t \in [0,1]$, such that h_0 is the inclusion $F \to X$ and for any point $x \in F$,

$$\int_{\gamma_x} \omega \le -N, \tag{3.2}$$

where the curve $\gamma_x\colon [0,1] \to X$ is given by $\gamma_x(t) = h_t(x)$.

The meaning of the line integral $\int_\gamma \omega$, where $\gamma\colon [0,1] \to X$ is a continuous curve, is explained in Appendix A.

Intuitively, condition (b) means that in the process of the homotopy h_t every point of F makes at least N full twists (in the negative direction) with respect to ω. We want to emphasize that when N tends to infinity we will obtain a sequence of different coverings (3.1) with the same number k and with the set F becoming possibly more and more complicated, so that its limit could be wild, looking like a fractal. This explains the approximative nature of our Definition 3.1, which allows these difficulties to be avoided.

Observe that $\mathrm{cat}\,(X,\xi)$ does not depend on the choice of the continuous closed 1-form ω (which appears in Definition 3.1) and depends only on the cohomology class $\xi = [\omega]$. Indeed, if ω' is another continuous closed 1-form representing ξ then $\omega - \omega' = df$, where $f\colon X \to \mathbb{R}$ is a continuous function, and for any continuous curve $\gamma\colon [0,1] \to X$,

$$\left| \int_\gamma \omega - \int_\gamma \omega' \right| = |f(\gamma(1)) - f(\gamma(0))| \le C,$$

where the constant C is independent of γ. Here we have used the compactness of X. This shows that if we can construct open covers (3.1) of X such that (b) is satisfied with an arbitrary large $N > 0$ then the same is true with ω' replacing ω.

In general the following inequality

$$\operatorname{cat}(X, \xi) \leq \operatorname{cat}(X) \tag{3.3}$$

holds since we may always consider covers (3.1) with F empty. Here $\operatorname{cat}(X)$ denotes the classical Lusternik–Schnirelman category of X, i.e. the least integer k such that there is an open cover $X = F \cup F_1 \cup \ldots \cup F_k$, such that each inclusion $F_j \to X$ is null-homotopic, where $j = 1, \ldots, k$.

Inequality (3.3) can be improved to

$$\operatorname{cat}(X, \xi) \leq \operatorname{cat}(X) - 1 \tag{3.4}$$

assuming that X is connected and $\xi \neq 0$. This follows because under the above assumptions any open subset $F \subset X$, which is contractible to a point in X, satisfies (b) of Definition 3.1 for any N (since we may first contract F to a point and then rotate the point many times so that inequality (3.2) holds). Hence, given a categorical open cover $X = G_1 \cup \ldots \cup G_r$ we may set $F = G_1$ and $F_j = G_{j+1}$ for $j = 1, \ldots, r-1$, which gives a cover of X satisfying Definition 3.1. Observe also that

$$\operatorname{cat}(X, \xi) = \operatorname{cat}(X, \lambda\xi), \tag{3.5}$$

for $\lambda \in \mathbb{R}$, $\lambda > 0$, as follows clearly from the above definition.

3.2. A reformulation. Sometimes it will be convenient to use a different version of condition (b) of Definition 3.1, which we will now describe.

Let $p: \widetilde{X} \to X$ be the normal covering corresponding to the kernel of the homomorphism of periods (cf. Appendix A)

$$\pi_1(X, x_0) \to \mathbb{R}, \quad [\gamma] \mapsto \int_\gamma \omega. \tag{3.6}$$

The group of covering transformations Γ of this covering equals the image of the homomorphism (3.6). The induced closed 1-form $p^*\omega$ equals df, where $f: \widetilde{X} \to \mathbb{R}$ is a continuous function. Denote $p^{-1}(F)$ by $\widetilde{F}$; it is an open subset invariant under Γ. Then condition (b) is equivalent to:

(b') *There exists a homotopy $\widetilde{h}_t: \widetilde{F} \to \widetilde{X}$, where $t \in [0, 1]$, such that $\widetilde{h}_0$ is the inclusion $\widetilde{F} \to \widetilde{X}$, each $\widetilde{h}_t$ is Γ-equivariant, and for any point $x \in \widetilde{F}$,*

$$f(\widetilde{h}_1(x)) - f(x) \leq -N.$$

Condition (b') requires that under the homotopy $\widetilde{h}_t$ any point of $\widetilde{F}$ descends at least N units down, measured by function f. The homotopy $\widetilde{h}_t$ is a lift of the homotopy h_t, which exists because of the HLP (homotopy lifting property) of coverings. If $\widetilde{\gamma}_y$, where $y \in \widetilde{F}$, $p(y) = x$, denotes the path $\widetilde{\gamma}_y(t) = \widetilde{h}_t(y)$ in $\widetilde{X}$ then $\gamma_x = p_*\widetilde{\gamma}_y$ and

$$\int_{\gamma_x} \omega = \int_{p_*\widetilde{\gamma}_y} \omega = \int_{\widetilde{\gamma}_y} p^*\omega = \int_{\widetilde{\gamma}_y} df = f(\widetilde{h}_1(y)) - f(y).$$

This explains the quivalence between (b) and (b').

3.3. Examples.

Example 3.2. Consider first the case when $\xi = 0$. Let us show that then (3.3) is an equality. $\xi = 0$ implies $\omega = df$, where $f: X \to \mathbb{R}$ is continuous. Then for any continuous curve $\gamma: [0,1] \to X$ the integral $\int_\gamma \omega$ equals $f(\gamma(1)) - f(\gamma(0))$ and it cannot become smaller than the variation of f on X. Therefore, for $\xi = 0$, inequality (3.2) may be satisfied for large N only if $F = \emptyset$. This proves that $\operatorname{cat}(X, 0) = \operatorname{cat}(X)$.

Example 3.3. Let X be a mapping torus, i.e. X is obtained from a cylinder $Y \times [0,1]$, where Y is a compact polyhedron, by identifying points $(y, 0)$ and $(\phi(y), 1)$ for all $y \in Y$, where $\phi: Y \to Y$ is a continuous map. Note that we do not assume that ϕ is a homeomorphism or a homotopy equivalence. We will denote points of X by pairs $\langle y, s\rangle$, where $y \in Y$, $s \in [0,1]$, understanding that $\langle y, 0\rangle = \langle \phi(y), 1\rangle$. X admits a natural projection $q: X \to S^1$, where $q\langle y, s\rangle = \exp(2\pi i s)$, and we will denote by $\omega = q^*(d\theta)$ the pullback of the standard angular form $d\theta$ of S^1; ω is a closed 1-from on X.

Let us show that $\operatorname{cat}(X, \xi) = 0$, where $\xi = [\omega] \in H^1(X; \mathbb{R})$. Given a number $N > 0$, define a homotopy $h_t: X \to X$, where for $t \in [0,1]$

$$h_t\langle y, s\rangle = \langle \phi^{n(t,s)}(y), s - Nt + n(t,s)\rangle.$$

Here $n(t, s)$ denotes the number of integers contained in the semi-open interval $(s - Nt, s]$. It is easy to see that this formula defines a continuous homotopy, $h_0 = \mathrm{id}$ and $\int_{\gamma_x} \omega = -N$ for any point $x \in X$, where $\gamma_x(t) = h_t(x)$. This shows that $\operatorname{cat}(X, \xi) = 0$.

Note that the Lusternik–Schnirelman category $\operatorname{cat}(X)$ of a mapping torus may be arbitrarily large (for example, $\operatorname{cat}(T^n) = n + 1$). Hence the above example shows that the difference $\operatorname{cat}(X) - \operatorname{cat}(X, \xi)$ may be arbitrarily large.

Example 3.4. In Definition 3.1 the notions "up" and "down" appear non-symmetrically. Hence it may happen that $\operatorname{cat}(X, \xi) \neq \operatorname{cat}(X, -\xi)$. We will see such an example now.

Consider the mapping torus X, as described in the previous example, with Y being the sphere S^2 and with $\phi: Y \to Y$ a map of degree 2. We have seen that $\mathrm{cat}\,(X, \xi) = 0$, where $\xi \in H^1(X; \mathbb{R})$ denotes the cohomology class described above. Let us show that $\mathrm{cat}\,(X, -\xi) \geq 1$.

The universal covering $p: \widetilde{X} \to X$ is Milnor's telescope; it can be described as follows. $\widetilde{X}$ is obtained from the disjoint union $\amalg_{n \in \mathbb{Z}} X_n$, where X_n denotes $Y \times [n \times n+1]$, by identifying any point $(y, n) \in X_n$ with $(\phi(y), n) \in X_{n-1}$. The projection $p: \widetilde{X} \to X$ maps any point $(y, t) \in X_n$ to $\langle \phi^{[t-n]}(y), \{t\} \rangle$. If ω is the closed 1-form on X described in the previous example then $p^*\omega = df$ where $f: \widetilde{X} \to \mathbb{R}$ is the continuous function given by $f(y, t) = t$.

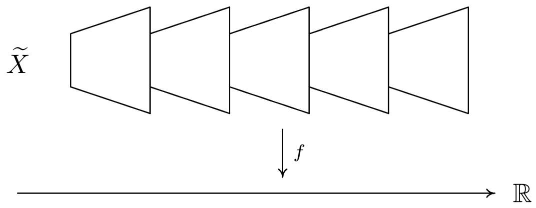

FIGURE 2. Covering of mapping torus

Let $X^k \subset \widetilde{X}$ denote $\bigcup_{n \geq k} X_n$. It is easy to see that $H_2(X^k; \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ with the fundamental class of the sphere $Y \times (k+1) \subset X_k$ as a generator. The inclusion $X^k \to X^{k-1}$ induces on the homology the homomorphism $H_2(X^k; \mathbb{Z}) \to H_2(X^{k-1}; \mathbb{Z})$ of multiplication by 2. Hence we see that $H_2(\widetilde{X}; \mathbb{Z})$ is isomorphic to the abelian group $\mathbb{Z}_{(2)}$ of rational numbers with denominators powers of 2 and the image of $H_2(X^k; \mathbb{Z})$ can be identified with $2^k\mathbb{Z} \subset \mathbb{Z}_{(2)}$. It follows that the sphere $Y \times k \subset X_k$ cannot be homotoped into X^k by a homotopy in $\widetilde{X}$. Using the condition (b') of Subsection 3.2 we see that there is no deformation taking the sphere $Y \times k$ up into X^k and this proves that $\mathrm{cat}\,(X, -\xi) > 0$.

On the contrary, there is a deformation of $\widetilde{X}$ taking all points arbitrarily far down (with respect to f), as we have seen in the previous example.

EXAMPLE 3.5. Consider a bouquet $X = Y \vee S^1$, where Y is a finite polyhedron, and assume that the class $\xi \in H^1(X, \mathbb{R})$ satisfies $\xi|_Y = 0$ and $\xi|_{S^1} \neq 0$. Let us show that then

$$\mathrm{cat}(X, \xi) = \mathrm{cat}(Y) - 1.$$

Consider an open cover $X = F \cup F_1 \cup \ldots \cup F_k$ satisfying conditions (a) and (b) of Definition 3.1. Let F' denote $F \cap Y$ and $F'_j = F_j \cap Y$. It is easy to show that F' is contractible in Y. This implies that $Y = F' \cup F'_1 \cup \ldots \cup F'_k$ is a categorical cover of Y, and hence $\mathrm{cat}\,(X, \xi) + 1 \geq \mathrm{cat}\,(Y)$.

The opposite inequality $\operatorname{cat}(X,\xi)+1 \le \operatorname{cat}(Y)$ is clear since for any categorical open cover $Y = G_0 \cup \ldots \cup G_r$ we may set $F = S^1 \cup G_0$ and $F_j = G_j$, where $j = 1, \ldots, r$. The set F satisfies (a) and (b) of Definition 3.1 for any integer $N > 0$.

This example shows that the integer $\operatorname{cat}(X,\xi)$ may assume arbitrary non-negative values.

3.4. Homotopy invariance.

LEMMA 3.6. *Let $\phi: X_1 \to X_2$ be a homotopy equivalence, $\xi_2 \in H^1(X_2;\mathbb{R})$, and $\xi_1 = \phi^*(\xi_2) \in H^1(X_1;\mathbb{R})$. Then*

$$\operatorname{cat}(X_1,\xi_1) = \operatorname{cat}(X_2,\xi_2). \tag{3.7}$$

PROOF. Let $\psi: X_2 \to X_1$ be a homotopy inverse of ϕ. Choose a closed 1-form ω_1 on X_1 in the cohomology class ξ_1. Then $\omega_2 = \psi^*\omega_1$ is a closed 1-form on X_2 lying in the cohomology class ξ_2.

Fix a homotopy $r_t: X_1 \to X_1$, where $t \in [0,1]$, such that $r_0 = \mathrm{id}_{X_1}$ and $r_1 = \psi \circ \phi$. Compactness of X_1 implies that there is a constant $C > 0$ such that $|\int_{\alpha_x} \omega_1| < C$ for any point $x \in X_1$, where α_x is the track of the point x under homotopy r_t, i.e. $\alpha_x(t) = r_t(x)$, where $t \in [0,1]$.

Suppose that $\operatorname{cat}(X_2,\xi_2) \le k$. Given any $N > 0$, there is an open covering $X_2 = F \cup F_1 \cup \ldots \cup F_k$, such that $F_1, \ldots, F_k$ are contractible in X_2 and there exists a homotopy $h_t: F \to X_2$, where $t \in [0,1]$, such that $\int_{\gamma_x} \omega_2 \le -N - C$ for any $x \in F$, where $\gamma_x(t) = h_t(x)$. Define

$$G = \phi^{-1}(F), \quad G_j = \phi^{-1}(F_j), \quad j = 1, \ldots, k.$$

These sets form an open cover of $X_1 = G \cup G_1 \cup \ldots \cup G_k$. Let us show that the set $G \subset X_1$ satisfies condition (b) of Definition 3.1. Define a homotopy $h'_t: G \to X_1$, where $t \in [0,1]$, by

$$h'_t(x) = \begin{cases} r_{2t}(x) & \text{for } 0 \le t \le 1/2, \\ \psi(h_{2t-1}(\phi(x))) & \text{for } 1/2 \le t \le 1. \end{cases} \tag{3.8}$$

Then h'_0 is the inclusion $G \to X_1$ and for any point $x \in G$,

$$\int_{\gamma'_x} \omega_1 \le -N$$

holds, where $\gamma'_x(t) = h'_t(x)$ is the track of x under homotopy h'_t. The following diagram

$$\begin{array}{ccc} G_j & \xrightarrow{\subset} & X_1 \\ {\scriptstyle\phi}\downarrow & & \uparrow{\scriptstyle\psi} \\ F_j & \xrightarrow[\subset]{} & X_2 \end{array}$$

is homotopy commutative and the horizontal map below is null-homotopic. This shows that the inclusion $G_j \to X_1$ is null-homotopic, where $j = 1, \dots, k$. The above argument proves that $\operatorname{cat}(X_1, \xi_1) \le \operatorname{cat}(X_2, \xi_2)$. The inverse inequality follows similarly. $\square$

4. Estimate of the number of zeros

In this section we will use invariant $\operatorname{cat}(X, \xi)$ to obtain a lower bound on the number of zeros of vector fields having no homoclinic orbits.

Let ω be a smooth closed 1-form on a connected closed smooth manifold M. We will assume that ω has finitely many zeros. A smooth vector field v is a *gradient-like* vector field for ω on M if the following two conditions hold: (1) $\omega(v) > 0$ on the complement of the set of zeros of v, and (2) in a neighbourhood $U_p \subset M$ of any zero $p \in M$, $\omega_p = 0$, the field $v|_{U_p}$ coincides with the gradient of the 1-form ω with respect to a Riemannian metric on U_p.

An integral trajectory $\gamma(t)$ of vector field v is *a homoclinic orbit* if both limits $\lim_{t\to\pm\infty} \gamma(t) = \gamma(\pm\infty)$ exist and are equal. The point $p = \gamma(\pm\infty)$ is then a zero of v. Note that $\int_\gamma \omega > 0$ for any homoclinic orbit γ. Hence, homoclinic orbits do not exist in the case $\xi = 0$, i.e. in the classical gradient systems.

More generally, a *homoclinic cycle with n edges* is a sequence of trajectories $\gamma_1, \dots, \gamma_n$ of the field v, such that all the limits $\lim_{t\to\pm\infty} \gamma_i(t)$ exist, where $i = 1, \dots, n$, and

$$\lim_{t\to\infty} \gamma_i(t) = \lim_{t\to-\infty} \gamma_{i+1}(t), \quad i = 1, \dots, n.$$

For $i = n$ this means that $\lim_{t\to\infty} \gamma_n(t) = \lim_{t\to-\infty} \gamma_1(t)$, and so the union of the curves γ_i form a closed cycle.

THEOREM 4.1. *Let ω be a smooth closed 1-form on a closed manifold M and let $\xi = [\omega] \in H^1(M; \mathbb{R})$ denote the cohomology class of ω. If ω admits a gradient-like vector field v with no homoclinic cycles, then ω has at least* $\operatorname{cat}(M, \xi)$ *geometrically distinct zeros.*

Here is a different formulation of the above theorem:

THEOREM 4.2. *If the number of zeros of a smooth closed 1-form ω is less than* $\operatorname{cat}(M, \xi)$*, where $\xi = [\omega] \in H^1(M; \mathbb{R})$ denotes the cohomology class of ω, then any gradient-like vector field for ω has a homoclinic cycle.*

Combined with Theorem 2.1, this shows that there may exist homoclinic cycles which cannot be destroyed while perturbing the gradient-like vector field. This "*focusing effect*" starts when the number of zeros of a closed 1-form becomes less than the number $\operatorname{cat}(M, \xi)$. It is a new phenomenon, not occuring in the Novikov theory: assuming that the zeros of ω are all Morse it is always possible to find a gradient-like vector field v for ω such that any integral trajectory connects a zero with higher Morse index with a zero with lower Morse index (by the Kupka–Smale Theorem [16]).

4.1. Proof of Theorem 4.1. Let $p_1, \dots, p_k \in M$ denote all the zeros of ω. Assume that there exists a gradient-like vector field v for ω with no homoclinic orbits. Our purpose is to show that then $\operatorname{cat}(M, \xi) \le k$.

Fix a number $N > 0$. Consider the flow $M \times \mathbb{R} \to \mathbb{R}$, where $(m, t) \mapsto m \cdot t$, generated by the field $-v$.

Choose small closed disks U_j around each point p_j, where $j = 1, \dots, k$. We will assume that for $i \neq j$ the disks U_i and U_j are disjoint. Also, we will fix a Riemannian metric on M such that in the disks U_i, where $i = 1, \dots, k$, the field v is the gradient of ω with respect to this metric.

We claim that: *one may choose closed disks V_i, where $i = 1, \dots, k$, such that:*

(a) $p_i \in \operatorname{Int} V_i$ *and* $V_i \subset \operatorname{Int} U_i$.

(b) *The disk V_i is gradient-convex* (*see Appendix* A) *in the following sense. Consider the covering $\pi: \widetilde{M} \to M$ corresponding to the kernel of the homomorphism of periods* (3.6). *Then the form ω lifts to $\widetilde{M}$ as a smooth function $f: \widetilde{M} \to M$, i.e. $\pi^*\omega = df$, and the field v lifts to a gradient-like vector field $\tilde{v}$ of function $-f$. We require any lift to $\widetilde{M}$ of the disk V_i to be gradient-convex, see Appendix* B.

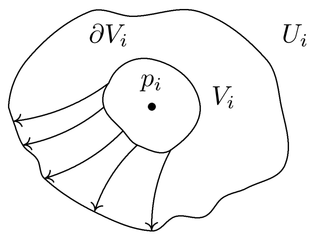

FIGURE 3. Disks U_i and V_i

(c) *Let $\partial_- V_i$ denote the set of points $p \in \partial V_i$, such that for all sufficiently small $\tau > 0$, $p \cdot \tau \notin V_i$ holds. Then we require that for any $p \in \partial_- V_i$ there exists no real number $t_p > 0$, such that: the point $p \cdot t_p$ belongs to* $\operatorname{Int} V_i$, *and $\int_p^{p \cdot t_p} \omega \ge -N$, where the integral is calculated along the integral trajectory $\sigma_p: [0, t_p] \to M$, $\sigma_p(t) = p \cdot t$.*

Assume first that V_i is any neighbourhood satisfying (a) and (b). Then any trajectory $p \cdot t$, where $p \in \partial_- V_i$, leaves U_i before it can re-enter V_i. This follows from the gradient convexity of V_i, since the disk U_i also lifts to the covering $\widetilde{M} \to M$. Hence:

(i) *There exists* $a > 0$, *such that for any* $p \in \partial_- V_i$ *and* $t > 0$ *with* $p \cdot t \in V_j$ *holds* $t \geq a$.

We may take $a = \min\{l_i v_i^{-1} : i = 1, \ldots, k\}$, where $l_i > 0$ denotes the distance between V_i and $M - \operatorname{Int} U_i$, and $v_i = \max |v(x)|$ for $x \in U_i - \operatorname{Int} V_i$.

Note that if we shrink the disks V_i the number $a > 0$ may only increase, assuming that V_i are sufficiently small.

(ii) *There exists* $b > 0$, *such that for any* $p \in \partial_- V_i$ *and* $t > 0$ *with* $p \cdot t \in V_j$ *holds*

$$\int_p^{p \cdot t} \omega < -b.$$

We may take $b = \min\{\varepsilon_i a : i = 1, \ldots, k\}$, where $\varepsilon_i = \min\{\omega(v)(x)\}$ for $x \in U_i - \operatorname{Int} V_i$. Suppose that we can never achieve (c) by shrinking the disk V_i, satisfying conditions (a) and (b). Then there exists an infinite sequence of points $p_{i,n} \in \partial_- V_i$, where $n = 1, 2, \ldots$, and two sequences of real numbers $t_{i,n} > 0$ and $s_{i,n} < 0$ such that:

(1) the set $p_{i,n} \cdot [s_{i,n}, 0]$ is contained in the disk V_i and the point $p_{i,n} \cdot s_{i,n}$ converges to p_i as n tends to ∞,

(2) $p_{i,n} \cdot t_{i,n}$ converges to p_i,

(3)
$$\int\limits_{p_{i,n}}^{p_{i,n} \cdot t_{i,n}} \omega \geq -N.$$

Let $\gamma_{i,n} \in H_1(M)$ denote the homology class of the loop obtained as follows. Start at the point $p_{i,n} \in \partial_- V_i$, follow the trajectory $p_{i,n} \cdot [0, t_{i,n}]$, and then connect the endpoint $p_{i,n} \cdot t_{i,n}$ with $p_{i,n}$ by a path inside V_i.

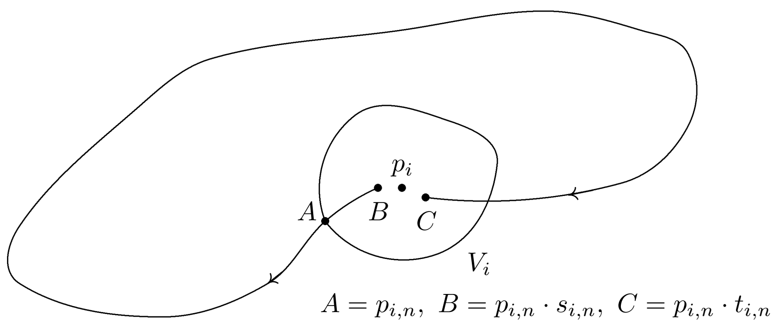

Figure 4. An orbit returning to the original disk V_i

We claim that *the set* $\{\gamma_{i,n}\}_{n\geq 1} \subset H_1(M)$ *of thus obtained homology classes is finite.* This would follow once we show that the total length $L_{i,n}$ of the parts of any trajectory of the form $p_{i,n} \cdot [0, t_{i,n}]$, spent outside the union of the neighbourhoods $U_1 \cup \ldots \cup U_k$, is bounded above by a constant independent of n. Writing

$$\int_\gamma \omega = \int_a^b \omega(\dot\gamma(t))\, dt = \int_a^b \frac{\omega(\dot\gamma(t))}{|\dot\gamma(t)|} \cdot |\dot\gamma(t)|\, dt$$

and using (3) above, it is easy to see that

$$L_{i,n} \leq N \cdot c^{-1},$$

where the constant $c > 0$ is given by

$$c = \min \left\{ \omega(v_x) \cdot |v_x|^{-1} : x \in M - \bigcup_{i=1}^{k} \operatorname{Int} U_i \right\}.$$

Passing to a subsequence, we may assume that $p_{i,n}$ converges to a point $q_i \in \partial_- V_i$ and the sequences $s_{i,n}$ and $t_{i,n}$ have finite or infinite limits, which we denote by s_i and t_i correspondingly. We may also assume that the homology class $\gamma_{i,n} \in H_1(M)$ is independent of n. Properties (1) and (2) imply that $t_i = \infty$ and $s_i = -\infty$.

First we want to show that $\lim_{t\to-\infty} q_i \cdot t = p_i$. We will identify V_i with one of its lifts to the covering $\widetilde{M}$. If there exists $s < 0$ such that $f(q_i \cdot s) > f(p_i) + \varepsilon$ for some $\varepsilon > 0$, then the set $\{(x,s) \in \widetilde{M} \times \mathbb{R}; f(x \cdot s) > f(p_i) + \varepsilon\}$ (which is open) contains $(p_{i,n}, s_{i,n})$ for all n sufficiently large, which contradicts (1). This shows that on the covering $\widetilde{M}$, $q_i \cdot s \to p_i$ holds for $s \to -\infty$. Hence the same relation holds on the initial manifold M as well.

Now we want to understand the limit $\lim_{t\to\infty} q_i \cdot t$. Consider the above lift of V_i to the covering $\widetilde{M}$, such that the flow becomes the gradient-like flow $\widetilde{v}$ of function $-f\colon \widetilde{M} \to \mathbb{R}$. The points $p_{i,n} \cdot t_{i.\,n}$ all belong to a translate gV_i of the disk V_i, where g is independent of n. The trajectory $q_i \cdot t$ in $\widetilde{M}$ for large t may either reach the neighbourhood gV_i, or it may be "caught" by some other critical point of f on the way.

Let us show that in the first case the point $q_i \cdot t$ tends to gp_i, as t tends to ∞ and hence the vector field v on M has a homoclinic orbit, starting and ending at p_i. If it is not true that $\lim_{t\to\infty} q_i \cdot t = gp_i$, then $f(q_i \cdot t) < f(gp_i) - \varepsilon$ for some t and $\varepsilon > 0$. Then the open set $\{(x,t) \in \widetilde{M} \times \mathbb{R}; f(x \cdot s) < f(gp_i) - \varepsilon\}$ would contain $(p_{i,n}, t_{i,n})$ for all n sufficiently large, which contradicts (2). This shows that on the covering $\widetilde{M}$, $q_i \cdot t \to gp_i$ holds for $t \to \infty$. Hence on M, the trajectory $q_i \cdot t$ tends to p_i for $t \to \infty$.

Consider now the second possibility, when the trajectory $q_i \cdot t$, after leaving V_i, tends, as t tends to ∞, to another zero $p_j \neq p_i$ of ω. We will show that in this case the vector field v has a homoclinic cycle with the number of edges greater

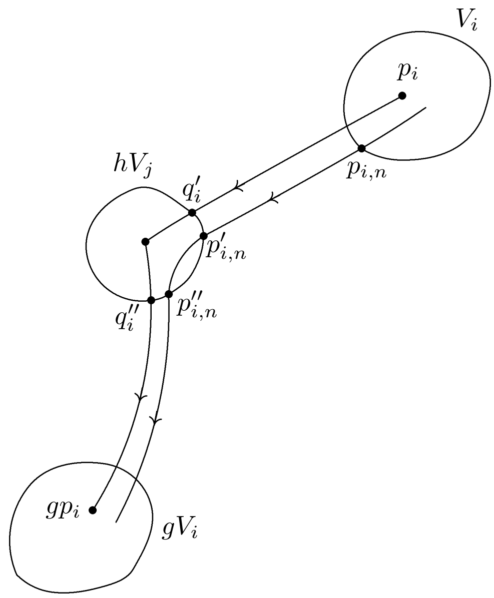

FIGURE 5. Creation of homoclinic cycle; picture in $\widetilde{M}$

than 1. Consider the flow determined by v on the covering $\widetilde{M}$, corresponding to the kernel of the homomorphism of periods (3.6). Under our assumptions, the limit of $q_i \cdot t$ equals hp_j for $t \to \infty$, where h is an element of the group of periods of ω. Statement (ii) (see above) gives a positive constant $b > 0$, such that

$$f(gp_i) + b < f(hp_j) \quad \text{and} \quad f(hp_j) + b < f(p_i).$$

Then, for large n, the trajectory, starting at $p_{i,n}$, enters the neighbouurhood hV_j at some point $p'_{i,n} \in \partial(hV_j)$ and leaves it at some point $p''_{i,n} \in \partial_-(hV_j)$. We have $p'_{i,n} \cdot \tau_{i,n} = p''_{i,n}$, where $\tau_{i,n} > 0$. Passing to a subsequence, we may assume that the sequences $p'_{i,n}$, $p''_{i,n}$, $\tau_{i,n}$ converge. Denote by $q'_i \in \partial(hV_j)$ and $q''_i \in \partial_-(hV_j)$ the limits of $p'_{i,n}$ and $p''_{i,n}$ correspondingly. Clearly, $\lim \tau_{i,n} = \infty$. Repeating the arguments above, we find that

$$\lim_{t\to-\infty} q'_i \cdot t = p_i, \quad \lim_{t\to\infty} q'_i \cdot t = hp_j = \lim_{t\to-\infty} q''_i \cdot t.$$

The limit $\lim_{t\to\infty} q''_i \cdot t$ either equals gp_i, or it equals $h_1 p_{j_1} \in \widetilde{M}$, where

$$f(gp_i) + b < f(h_1 p_{j_1}) \quad \text{and} \quad f(h_1 p_{j_1}) + b < f(hp_j).$$

Continuing by induction, we obtain downstairs (i.e. on the manifold M) a homoclinic cycle "starting and ending"at p_i. The number of steps in the above process is finite (at most $[N/b]$).

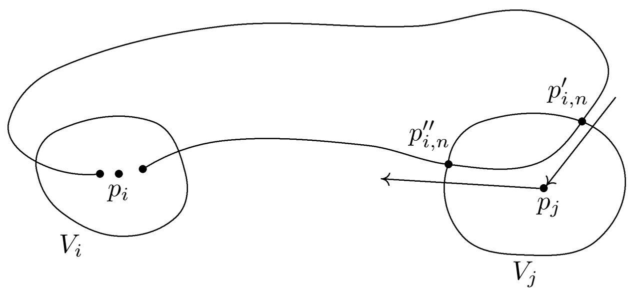

FIGURE 6. Creation of homoclinic cycle; picture in M

This proves the existence of the disks V_i with properties (a), (b), (c), assuming that the vector field v has no homoclinic cycles.

Next we will construct an open cover $F \cup F_1 \cup \ldots \cup F_k = M$. We will define F as the set of all points $p \in M$ such that there exists a positive number $t_p > 0$, so that the integral curve $\sigma_p: [0, t_p] \to M$, where $\sigma_p(t) = p \cdot t$, satisfies $\int_{\sigma_p} \omega = -N$. It is clear that F is open, and $p \mapsto t_p$ is a continuous real-valued function on F. We may define a homotopy

$$h_\tau: F \to F, \quad \text{by} \quad h_\tau(p) = p \cdot (\tau t_p), \quad \tau \in [0, 1].$$

This homotopy satisfies condition (b) of Definition 3.1.

Now we will define the sets F_j, where $j = 1, \ldots, k$. We say that $p \in F_j$ if for some $t_p > 0$ the point $p \cdot t_p$ belongs to the interior of V_j, and

$$\int_{\sigma_p} \omega > -N,$$

where $\sigma_p: [0, t_p] \to M$ is given by $\sigma_p(t) = p \cdot t$. It is clear that F_j is open. The sets $F, F_1, \ldots, F_k$ cover M. Indeed, for any point $p \in M$ either

$$\int_p^{p \cdot t} \omega < -N$$

for some $t > 0$, or the trajectory $\gamma(t) = p \cdot t$ "enters", a zero p_j, so that

$$\lim_{t \to \infty} \int_p^{p \cdot t} \omega \geq -N.$$

In the first case p belongs to F, and in the second case p belongs to F_j.

Now we will show that the set F_j, where $j = 1, \ldots, k$, is contractible in M. For any point $p \in M$ let $J_p \subset \mathbb{R}$ denote the set $J_p = \{t \geq 0; p \cdot t \in V_j\}$. Because of our assumption about the gradient-convexity of neighbourhood V_j, the set J_p is a union of disjoint closed intervals, and some of these intervals may degenerate to a point. Consider the first interval $[\alpha_p, \beta_p] \subset J_p$. If this interval degenerates to a point (i.e. the trajectory through p touches V_j), then p does not belong to the set F_j, according to our assumption (c). By the same reason, points of $\partial_- V_j$ do not belong to F_j.

Assume now that $p \in F_j$ and $p \notin \operatorname{Int} V_j$. Then the point $p \cdot t$ lies in the interior of V_j for $\alpha_p < t < \beta_p$. Also, we have

$$\int_p^{p\cdot\alpha_p} \omega > -N.$$

The function $\phi_j : F_j \to \mathbb{R}$, given by

$$\phi_j(p) = \begin{cases} 0 & \text{for } p \in \operatorname{Int} V_j, \\ \alpha_p & \text{for } p \in F_j - \operatorname{Int} V_j, \end{cases}$$

is continuous. To show this, suppose that a sequence of points $x_n \in F_j$, where $n = 1, 2, \dots,$ converges to $x_0 \in F_j$. First consider the case $x_0 \in V_j$. Since $x_0 \in F_j$, we conclude that $x_0 \notin \partial_- V_j$ (because of condition (c)), i.e. $x_0 \cdot \tau$ belongs to $\operatorname{Int} V_j$ for all $\tau \in (0, \varepsilon)$. It follows that for any $\tau \in (0, \varepsilon)$, $x_n \cdot \tau \in \operatorname{Int} V_j$ holds for all large n. Therefore, $\alpha_{x_n} < \tau$ for all large n. Hence, the sequence α_{x_n} converges to 0. Consider now the case when x_0 does not belong to V_j. Then the trajectory $x_0 \cdot t$ does not touch V_j for $t < \alpha_{x_0}$ (again, because of condition (c)) and $t = \alpha_{x_0}$ is the first moment when the trajectory penetrates V_j. We know also that the velocity vector v_x is transversal to the boundary ∂V_j, where $x = x_0 \cdot \alpha_{x_0}$. Then the sequence α_{x_n} converges to α_{x_0}, as follows from the continuity of solutions of ordinary differential equations with respect to the initial conditions.

We may define a homotopy

$$h_\tau : F_j \to M, \quad h_\tau(p) = p \cdot (\tau \phi_j(p)), \quad p \in F_j, \quad \tau \in [0, 1].$$

Here h_0 is the inclusion $F_j \to M$ and h_1 maps F_j into the disk V_j. This completes the proof. □

5. Moving homology classes

Here we study the effect of condition (b') of Subsection 3.2 on homology classes. For simplicity we assume that the group of periods $\Gamma \subset \mathbb{R}$ is infinite cyclic. $\mathbf{k}$ will denote a field. Let $\widetilde{X} \to X$ be an infinite cyclic covering, i.e. a regular covering of a finite CW-complex having an infinite cyclic group of covering transformations. In this section we will denote by $\tau : \widetilde{X} \to \widetilde{X}$ a fixed generator of this group.

Let $K \subset \widetilde{X}$ be a compact subset such that $\widetilde{X}$ is the union of the translates $\tau^i(K)$, where $i \in \mathbb{Z}$.

Definition 5.1. We will say that a homology class $z \in H_q(\widetilde{X};\mathbf{k})$ is *movable to* ∞ if for any integer $N \in \mathbb{Z}$ there exists a cycle in $\bigcup_{i>N} \tau^i(K)$ representing z. Similarly, a homology class $z \in H_q(\widetilde{X};\mathbf{k})$ is *movable to* $-\infty$ if for any integer $N \in \mathbb{Z}$ there exists a cycle in $\bigcup_{i<N} \tau^i(K)$ representing z.

It is clear that the above properties of the homology class $z \in H_q(\widetilde{X};\mathbf{k})$ do not depend on the choice of compact K.

The following lemma gives an approximative condition of movability. Roughly, it claims: if a cycle can be moved a sufficiently large distance away then it may be moved arbitrarily far away. The word "*cycle*" means "*singular cycle with coefficients in* $\mathbf{k}$".

Lemma 5.2. *Let $K \subset \widetilde{X}$ be a compact subset such that $\bigcup_{j\in\mathbb{Z}} \tau^j(K)$ coincides with $\widetilde{X}$. Then there exists an integer $N > 0$ (depending on K), such that the following properties hold:*

(a) *Let a cycle c in K be homologous in $\widetilde{X}$ to a cycle in $\bigcup_{j\geq N} \tau^j(K)$. Then the homology class $[c] \in H_q(\widetilde{X};\mathbf{k})$ is movable to ∞.*
(b) *Let a cycle c in K be homologous in $\widetilde{X}$ to a cycle in $\bigcup_{j\leq -N} \tau^j(K)$. Then the homology class $[c] \in H_q(\widetilde{X};\mathbf{k})$ is movable to $-\infty$.*

Proof. Denote

$$V_r = \mathrm{im}[H_q(K;\mathbf{k}) \to H_q(\widetilde{X};\mathbf{k})] \cap \mathrm{im}\left[H_q\left(\left(\bigcup_{j\geq r} \tau^j K\right);\mathbf{k}\right) \to H_q(\widetilde{X};\mathbf{k})\right],$$

where $r = 1, 2, \dots$. Then $V_1 \supset V_2 \supset \dots$ is a decreasing sequence of finite-dimensional vector spaces. Hence, there exists an integer N, such that V_N coincides with $V_\infty = \cap_{r>0} V_r$. Any homology class $z \in V_\infty$ is movable to ∞. Thus, this N satisfies (a). Similarly, we may increase N, if necessary, such that (b) is satisfied as well. $\square$

Lemma 5.3. *Given a homology class $z \in H_q(\widetilde{X};\mathbf{k})$, the following conditions are equivalent:*

(i) *z is movable to ∞,*
(ii) *z is movable to $-\infty$,*
(iii) *z is a torsion element of $\mathbf{k}[\tau,\tau^{-1}]$-module $H_q(\widetilde{X};\mathbf{k})$, i.e. there exists a nontrivial Laurent polynomial $p(\tau) \in \mathbf{k}[\tau,\tau^{-1}]$, such that $p(\tau)z = 0$.*

Proof. We will show that (i) implies (iii) and that (iii) implies both (i) and (ii). The implication (ii)⇒(iii) follows similarly.

Assume that a class $z \in H_q(\widetilde{X};\mathbf{k})$ is movable to ∞. Realize z by a cycle c in $\widetilde{X}$ and specify a compact subset $K \subset \widetilde{X}$ containing c and such that $\widetilde{X} = \bigcup_{i\in\mathbb{Z}} \tau^i(K)$. Assume that $N > 0$ is large enough, so that it satisfies Lemma 5.3 and the subset $\tau^N(K)$ is disjoint from K. Let us show that for any integer $r \geq N$

the homology class z may be realized by a cycle in $\tau^r(K)$. Since z is movable to ∞, there exists a cycle c' in $U = \bigcup_{j\geq 2r}\tau^j(K)$ representing z. Write $\widetilde{X} = B\cup C$, where B contains K, C contains U and $B\cap C = \tau^r K$. In the Mayer–Vietoris sequence

$$H_q(B\cap C;\mathbf{k}) \to H_q(B;\mathbf{k})\oplus H_q(C;\mathbf{k}) \to H_q(\widetilde{X};\mathbf{k}),$$

the difference $[c]-[c']$ goes to zero. Hence there is a cycle in $B\cap C = \tau^r K$, which is homologous to c in B and homologous to c' in C. This proves our claim.

Consider

$$V = \bigcap_{r\geq N} \mathrm{im}[H_q(\tau^r K;\mathbf{k}) \to H_q(\widetilde{X};\mathbf{k})] \subset H_q(\widetilde{X};\mathbf{k}).$$

It is a finite dimensional $\mathbf{k}$-linear subspace; we observe that V is invariant under τ^{-1}. Hence, by the Caley–Hamilton Theorem, there exists a polynomial $p(\tau)\in \mathbf{k}[\tau]$, such that $p(\tau^{-1})$ acts trivially on V. Since z belongs to V, we obtain $p(\tau^{-1})z = 0$. This shows that (i) implies (iii).

Let us show that (iii) implies (i) and (ii). Suppose that $z\in H_q(\widetilde{X};\mathbf{k})$ is such that $p(\tau)z = 0$ for a Laurent polynomial

$$p(\tau) = \sum_{i=r}^{r+\ell} a_i\tau^i, \quad a_i\in\mathbf{k}, \quad \text{where} \quad a_r\neq 0,\ a_{r+\ell}\neq 0.$$

Consider the ring $R = \mathbf{k}[\tau,\tau^{-1}]/J$, where J is the ideal generated by $p(\tau)$. The powers τ^i, where $i = r, r+1,\dots, r+\ell-1$, form an additive basis of R. Since multiplication by τ is an automorphism of R, we obtain that for any integer N, the powers $\tau^N,\dots,\tau^{N+\ell-1}$ form a linear basis of R as well. In particular, we may express $1\in R$ as a linear combination of $\tau^N,\dots,\tau^{N+\ell-1}$ in R. This means that for any N we may find numbers $b_j\in\mathbf{k}$, where $j = N,\dots,N+\ell-1$, such that

$$z = \sum_{j=N}^{N+\ell-1} b_j\tau^j z \quad \text{in } H_q(\widetilde{X};\mathbf{k}).$$

Assume that $K\subset\widetilde{X}$ is a compact subset such that $\widetilde{X} = \bigcup_{j\in\mathbb{Z}}\tau^j K$ and class z can be realized by a cycle c in K. Then for any integer N, class z may be realized by the cycle

$$\sum_{j=N}^{N+\ell-1} b_j\tau^j c \quad \text{lying in} \quad \bigcup_{N\leq j<N+\ell}\tau^j K.$$

Hence z is movable to both ends $\pm\infty$ of $\widetilde{X}$. $\square$

6. Cohomological lower bound for $\operatorname{cat}(X,\xi)$

In this section we will give cohomological lower bounds on $\operatorname{cat}(X,\xi)$. For simplicity we assume that ξ is integral, i.e. $\xi\in H^1(X;\mathbb{Z})$.

6.1. Statement of the result. Let $\mathbf{k}$ be a field. Given a finite CW-complex X and an integral cohomology class $\xi \in H^1(X;\mathbb{Z})$. For any nonzero $a \in \mathbf{k}$ there is a local system over X with fiber $\mathbf{k}$ such that the monodromy along any loop γ in X is a multiplication by $a^{\langle \xi,\gamma\rangle}$

$$a^{\langle \xi,\gamma\rangle}\colon \mathbf{k} \to \mathbf{k}, \quad q \mapsto a^{\langle \xi,\gamma\rangle} q, \quad \text{for } q \in \mathbf{k}.$$

Here $\langle \xi,\gamma\rangle \in \mathbb{Z}$ denotes the value of the class ξ on the loop γ. This local system will be denoted a^ξ. The cohomology of this local system $H^q(X;a^\xi)$ is a vector space over $\mathbf{k}$ of finite dimension. Note that for $a = 1$ the local system a^ξ is the constant local system $\mathbf{k}$. If $a, b \in \mathbf{k}^*$ are two nonzero numbers, there is an isomorphism of local systems

$$a^\xi \otimes b^\xi \simeq (ab)^\xi.$$

Hence we have a well-defined cup-product pairing

$$\cup\colon \quad H^q(X;a^\xi) \otimes H^{q'}(X;b^\xi) \to H^{q+q'}(X;(ab)^\xi). \tag{6.1}$$

The following Theorem is one of the main results of this section.

THEOREM 6.1. *Assume that there exist cohomology classes*

$$u \in H^q(X;a^\xi), \quad v \in H^{q'}(X;b^\xi), \quad w_j \in H^{d_j}(X;\mathbf{k}), \quad \textit{where } j = 1,\dots,r,$$

such that

(i) $d_1 > 0, \dots, d_r > 0$,
(ii) *the cup product*

$$u \cup v \cup w_1 \cup w_2 \cup \ldots \cup w_r \in H^d(X;(ab)^\xi), \tag{6.2}$$

is nontrivial, where $d = q + q' + d_1 + \ldots + d_r$,
(iii) *the numbers* $a, b \in \mathbf{k}^*$ *do not belong to a finite subset* $\mathrm{Supp}(X,\xi) \subset \mathbf{k}^*$ *depending on the pair* (X,ξ)*, see below.*

Then $\mathrm{cat}\,(X,\xi) > r$.

Property (iii) may also be expressed by saying that the numbers $a \in \mathbf{k}^*$ and $b \in \mathbf{k}^*$ are *generic*.

For $\xi \neq 0$ and $a \neq 1$ the bundle a^ξ admits no globally defined flat sections and so the zero-dimensional cohomology $H^0(X;a^\xi) = 0$ vanishes. Hence for $\xi \neq 0$ the classes u and v in Theorem 6.1 must automatically have positive degrees, i.e. $q > 0$ and $q' > 0$.

6.2. The set $\mathrm{Supp}(X,\xi)$**.** For $\xi = 0$ we will define $\mathrm{Supp}(X,\xi) \subset \mathbf{k}^*$ to be the empty set. Now we will define the set $\mathrm{Supp}(X,\xi) \subset \mathbf{k}^*$ assuming that $\xi \in H^1(X;\mathbb{Z})$ is nonzero and indivisible. Let $p\colon \widetilde{X} \to X$ be the infinite cyclic covering corresponding to ξ. Fix a generator $\tau\colon \widetilde{X} \to \widetilde{X}$ of the group of covering translations such that $f(\tau x) = f(x) + 1$ for all $x \in \widetilde{X}$, where $f\colon \widetilde{X} \to \mathbb{R}$ is a continuous function satisfying $df = p^*\omega$. Here ω is a closed 1-form on X in the cohomology class ξ. Since the translate $\tau\colon \widetilde{X} \to \widetilde{X}$ acts on the homology, we obtain that $H_*(\widetilde{X};\mathbf{k})$ is a graded module over the ring $\Lambda = \mathbf{k}[\tau,\tau^{-1}]$ of Laurent polynomials. The ring Λ is Noetherian, hence $H_*(\widetilde{X};\mathbf{k})$ is finitely generated over Λ. Its Λ-torsion submodule $T = \mathrm{Tor}_\Lambda(H_*(\widetilde{X};\mathbf{k}))$ (which, according to the previous section, coincides with the set of homology classes movable to $\pm\infty$) is a finite-dimensional $\mathbf{k}$-vector space. Multiplication by τ is an invertible $\mathbf{k}$-linear operator $\tau\colon T \to T$. We will define the set $\mathrm{Supp}(X,\xi) \subset \mathbf{k}^*$ as the set of eigenvalues of $\tau^{-1}\colon T \to T$.

In the case when $\xi = \lambda\eta$, where $\eta \in H^1(X;\mathbb{Z})$ is indivisible and $\lambda \in \mathbb{Z}$, $\lambda > 0$, we will define the set $\mathrm{Supp}(X,\xi)$ as $\{a \in \mathbf{k}^*; a^\lambda \in \mathrm{Supp}(X,\eta)\}$.

Note that for $\mathbf{k} = \mathbb{C}$ the set $\mathrm{Supp}(X,\xi) \subset \mathbb{C}$ consists of finitely many algebraic numbers. Hence for $\mathbf{k} = \mathbb{C}$ in Theorem 6.1 one may always take for $a, b \in \mathbb{C}^*$ arbitrary transcendental numbers.

6.3. Lifting property. The proof of Theorem 6.1 is based on the following lifting property of cohomology classes:

Theorem 6.2. *Let X be a finite* CW-*complex, $\xi \in H^1(X;\mathbb{Z})$, $a \in \mathbf{k}^*$, $a \notin \mathrm{Supp}(X,\xi)$. Fix a continuous closed* 1-*form ω in cohomology class ξ and a compact $K \subset X$, such that $\xi|_K = 0$. Then for any open subset $F \subset K$ satisfying condition* (b) *of Definition* 3.1 *with respect to closed* 1-*form ω with integer $N > 0$ large enough, the homomorphism*

$$H^q(X,F;a^\xi) \to H^q(X;a^\xi) \tag{6.3}$$

is an epimorphism.

Proof. We have already observed in Example 3.2 that for $\xi = 0$ condition (b) of Definition 3.1 may be satisfied for large N only if $F = \emptyset$. Therefore Theorem 6.2 is true for $\xi = 0$.

We will assume below that $\xi \neq 0$. Moreover, without loss of generality, we may assume that class $\xi \in H^1(X;\mathbb{Z})$ is indivisible.

Let $f\colon \widetilde{X} \to \mathbb{R}$ be a continuous function (unique up to a constant) satisfying $df = p^*\omega$. Then $f(\tau x) = f(x) + 1$ for all $x \in \widetilde{X}$, where $\tau\colon \widetilde{X} \to \widetilde{X}$ a generator of the group of translations. Since $\xi|_K = 0$, the compact $K \subset X$ may be lifted to $\widetilde{X}$. Fix such a lift $K \subset \widetilde{X}$. We will assume that the function $f|_K$ assumes values in $[0,c] \subset \mathbb{R}$ for some integer $c > 0$; this may always be achieved by adding a constant to f.

Let $N' > 0$ be the number given by Lemma 5.2 applied to a lift of K to $\widetilde{X}$. Then any cycle in $K \subset \widetilde{X}$, which is homologous in $\widetilde{X}$ to a cycle in $\bigcup_{j \le -N'} \tau^j(K)$, is movable to $-\infty$, and hence (according to Lemma 5.3) represents a Λ-torsion homology class in $H_*(\widetilde{X};\mathbf{k})$.

It follows that for any subset $F \subset K$ satisfying condition (b') of Subsection 3.2 with $N > N'$, the homomorphism $H_*(\widetilde{F};\mathbf{k}) \to H_*(\widetilde{X};\mathbf{k})$ induced by the inclusion $\widetilde{F} \to \widetilde{X}$, takes values in the Λ-torsion submodule T of $H_*(\widetilde{X};\mathbf{k})$. The set $\widetilde{F}$ is a disjoint union of infinitely many copies of F and hence the homology of $\widetilde{F}$ is $H_*(\widetilde{F};\mathbf{k}) \simeq H_*(F;\mathbf{k}) \otimes_{\mathbf{k}} \Lambda$.

The claim that (6.3) is an epimorphism is equivalent to the claim that $H^*(X;a^\xi) \to H^*(F;a^\xi)$ is the zero map. Using duality between homology and cohomology we see that the latter is equivalent to the statement that the homomorphism $H_*(F;a^{-\xi}) \to H_*(X;a^{-\xi})$, induced by the inclusion $F \to X$ on the homology of the dual local system $a^{-\xi}$, is zero. Since $\xi|_F = 0$, the local system $a^{-\xi}|_F \simeq \mathbf{k}$ is trivial. Hence we want to show that the homomorphism $H_*(F;\mathbf{k}) \to H_*(\widetilde{X};a^{-\xi})$ is zero. The inclusion $F \to X$ equals the composition $F \xrightarrow{\subset} K \longrightarrow \widetilde{X} \xrightarrow{p} X$, and we know that $H_*(F;\mathbf{k}) \to H_*(\widetilde{X};\mathbf{k})$ takes values in the Λ-torsion submodule T.

To complete the proof it is enough to show that $p_*(T) = 0$ holds for $a \notin \mathrm{Supp}(X,\xi)$, where the homomorphism $p_*\colon H_*(\widetilde{X};\mathbf{k}) \to H_*(X;a^{-\xi})$ is induced by the covering projection $p\colon \widetilde{X} \to X$. Consider the following well-known exact sequence

$$\cdots \to H_i(\widetilde{X};\mathbf{k}) \xrightarrow{\tau - b} H_i(\widetilde{X};\mathbf{k}) \xrightarrow{p_*} \mathbf{H}_i(X;b^\xi) \to \cdots$$

where $b = a^{-1}$. The linear map $\tau - b\colon T \to T$ is an isomorphism (here we use the assumption that $a \notin \mathrm{Supp}(X,\xi)$) and hence the submodule $T \subset H_*(\widetilde{X};\mathbf{k})$ is contained in the image of $\tau - b$ and therefore $p_*(T) = 0$. $\square$

6.4. Proof of Theorem. If $\xi = 0$ under the conditions of Theorem 6.1 the classical Lusternik–Schnirelman theory gives $\mathrm{cat}\,(X) > r+2$; hence Theorem 6.1 holds for $\xi = 0$.

We will assume below that the class $\xi \neq 0$ is nonzero and indivisible. Suppose that $\mathrm{cat}\,(X,\xi) \leq r$. Let us show that any cup-product (6.2) satisfying conditions (i)–(iii) of Theorem 6.1 must vanish. Since ξ is an integral class $\xi \in H^1(X;\mathbb{Z})$, we may cover X by two open subsets $X = U \cup W$, such that $\xi|_{\overline{U}} = 0$, $\xi|_{\overline{W}} = 0$. Choose a continuous closed 1-form ω in cohomology class ξ. Our assumption $\mathrm{cat}\,(X,\xi) \leq r$ implies that for any $N > 0$ there is an open cover $X = F \cup F_1 \cup \ldots \cup F_r$ satisfying properties (a) and (b) of Definition 3.1.

Each cohomology class $w_j \in H^{d_j}(X;\mathbf{k})$ may be lifted to a relative cohomology class $\widetilde{w}_j \in H^{d_j}(X,F_j;\mathbf{k})$, where $j = 1,\ldots,r$. This follows from the cohomological exact sequence

$$H^{d_j}(X,F_j;\mathbf{k}) \to H^{d_j}(X;\mathbf{k}) \to H^{d_j}(F_j;\mathbf{k}),$$

since $d_j > 0$ and the second map vanishes as a consequence of the fact that the inclusion $F_j \to X$ is null-homotopic.

Assuming that $N > 0$ is large enough we may use the lifting property of Theorem 6.2, applied to compact $K = \overline{U}$, to find a lift of the cohomology class $u \in H^q(X; a^\xi)$ to a relative cohomology class $\widetilde{u} \in H^q(X, F \cap U; a^\xi)$. Similarly, by Theorem 6.2 applied to the compact $\overline{W}$, we may lift the class $v \in H^{q'}(X; b^\xi)$ to a relative cohomology class $\widetilde{v} \in H^{q'}(X, F \cap W; a^\xi)$. Therefore the product (6.2) is obtained from the product

$$\widetilde{u} \cup \widetilde{v} \cup \widetilde{w}_1 \cup \ldots \cup \widetilde{w}_r \in H^*(X, X; (ab)^\xi) = 0$$

(lying in the trivial group) by restricting onto X. Hence any cup-product (6.2) must vanish. □

REMARK. Lemma 6.6 in [7] is incorrect as stated. As a consequence Proposition 6.5 and Corollary 6.7 of [7] are correct only with some extra conditions. The lifting property of Theorem 6.2 of the present paper replaces Corollary 6.7 of [7].

6.5. Cohomological estimate using Massey products. Let X be a finite CW-complex and let $\xi \in H^1(X; \mathbb{Z})$.

DEFINITION 6.3. A cohomology class $v \in H^q(X; \mathbf{k})$ is called a *ξ-survivor* if vanishes the cup-product $v \cup \xi = 0$ and vanish all higher Massey products of the form

$$\langle v, \underbrace{\xi, \ldots, \xi}_{r \text{ times}} \rangle \in H^{q+1}(X; \mathbf{k})$$

for any $r > 1$.

We refer to Sections 5 and 9 of paper [7], where these Massey products are described in detail.

THEOREM 6.4. *Let X be a finite* CW*-complex and let $\xi \in H^1(X; \mathbb{Z})$ be indivisible. Assume that there exist cohomology classes $w_j \in H^{d_j}(X; \mathbf{k})$ of positive degree $d_j > 0$, where $j = 1, \ldots, r$, having a nonzero cup-product*

$$0 \neq w_1 \cup w_2 \cup \ldots \cup w_r \in H^*(X; \mathbf{k}).$$

If among the classes $w_1, \ldots, w_r$ at least two are ξ-survivors, then

$$\mathrm{cat}\,(X, \xi) \geq r - 1.$$

PROOF. We may assume in the proof that the coefficient field $\mathbf{k}$ is algebraically closed; otherwise we could replace $\mathbf{k}$ by its algebraic closure.

Suppose that the first two cohomology classes w_1, w_2 are ξ-survivors. It is shown in Section 9.5 of [7], that one may deform w_1 and w_2 to cohomology classes $w_1' \in H^{d_1}(X; a^\xi)$ and $w_2' \in H^{d_2}(X; a^{-\xi})$, where $a \in \mathbf{k}$ is generic, such that the cup-product

$$0 \neq w_1' \cup w_2' \cup w_3 \cup \ldots \cup w_r \in H^*(X; \mathbf{k})$$

is still nonzero. Now Theorem 6.4 follows from Theorem 6.1. □

EXAMPLE 6.5. Let $T^n = S^1 \times \ldots \times S^1$ be an n-dimensional torus. Fix a point $x = (x_1, \ldots, x_n) \in T^n$ and for $i = 1, \ldots, n$ consider the $(n-1)$-dimensional subtorus $T_i^{n-1} \subset T^n$ consisting of points with i-th coordinate equal to x_i. Let $X = T^n \# (S^1 \times S^{n-1})$ be obtained from T^n by adding a handle of index 1. In other words, we remove from T^n interiors of two small disjoint disks D_1 and D_2 and connect their boundaries by a tube $S^{n-1} \times [0,1]$. We will assume that the disks D_1 and D_2 do not meet the subtori T_i^{n-1} for $i = 1, \ldots, n$. Let $\xi \in H^1(X; \mathbb{Z})$ be the cohomology class, Poincaré dual to ∂D_1.

Each torus $T_i \subset X$ has a trivial normal bundle and hence determines a cohomology class $w_i \in H^1(X; \mathbb{Z})$. It is clear that the cup-product

$$w_1 \cup \ldots \cup w_n \in H^n(X; \mathbb{Z})$$

is nonzero since $T_1, \ldots, T_n$ are mutually transversal and their intersection consists of one point. As in Section 5 of [7] one may show that the classes w_i are all ξ-survivors. Theorem 6.4 applies and gives $\operatorname{cat}(X, \xi) \geq n - 1$. By Theorem 4.1 any smooth closed 1-form in class ξ has at least $n - 1$ geometrically distinct zeros.

This estimate is sharp and cannot be improved. Indeed, since $\dim X = n$ Theorem 3.2 of [7] claims that there always exists a closed 1-form ω in class ξ with $n - 1$ zeros.

EXAMPLE 6.6. Let X be a bouquet $X = Y \vee S^1$ as in Example 3.5. We will assume that the cohomology class $\xi \in H^1(X; \mathbb{Z})$ is such that $\xi|_Y = 0$ and $\xi|_{S^1} \neq 0$ is the generator. We want to find the estimate given by Theorem 6.4 in this example. We have $H^q(X; \mathbf{k}) = H^q(Y; \mathbf{k})$ for $q > 1$ and $H^1(X; \mathbf{k}) = H^1(Y; \mathbf{k}) \oplus H^1(S^1; \mathbf{k})$. The last summand is generated by class ξ. It has trivial cup-products and Massey products with all other classes. If cohomology classes $w_j \in H^{d_j}(Y; \mathbf{k})$, where $j = 1, \ldots, r$, are such that $d_j > 0$ and

$$0 \neq w_1 \cup \ldots \cup w_r \in H^*(Y; \mathbf{k}),$$

then Theorem 6.4 applies and gives $\operatorname{cat}(X, \xi) \geq r - 1$. In Example 3.5 we have shown that $\operatorname{cat}(X, \xi) = \operatorname{cat}(Y) - 1$. Hence we obtain $\operatorname{cat}(Y) \geq r$, which is weaker by 1 than the well-known inequality claiming that $\operatorname{cat}(Y)$ is greater than the cup-length of Y.

This example shows that our cohomological estimates may be slightly improved.

Appendix A. Closed 1-forms on topological spaces

Differential 1-forms are defined only for smooth manifolds. *Closed* 1-forms may be defined for general topological spaces, as we show in this appendix.

A *continuous closed* 1*-form* ω *on a topological space* X is defined as a collection $\{f_U\}_{U\in\mathcal{U}}$ of continuous real-valued functions $f_U: U \to \mathbb{R}$, where $\mathcal{U} = \{U\}$ is an open cover of X, such that for any pair $U, V \in \mathcal{U}$ the difference

$$f_U|_{U\cap V} - f_V|_{U\cap V}: U \cap V \to \mathbb{R}$$

is a locally constant function. Another such collection $\{g_V\}_{V\in\mathcal{V}}$ (where $\mathcal{V}$ is another open cover of X) defines *an equivalent* closed 1-form if for any point $x \in X$ there is an open neighbourhood W such that for some open sets $U \in \mathcal{U}$ and $V \in \mathcal{V}$ containing W the difference $f_U|_W - g_V|_W$ is locally constant. The set of all continuous closed 1-forms on X is a real vector space.

As an example consider an open cover $\mathcal{U} = \{X\}$ consisting of the whole space X. Then any continuous function $f: X \to \mathbb{R}$ defines a closed 1-form on X, which is denoted by df.

For two continuous functions $f, g: X \to \mathbb{R}$ holds $df = dg$ if and only if the difference $f - g: X \to \mathbb{R}$ is locally constant.

One may integrate continuous closed 1-forms along continuous paths. Let ω be a continuous closed 1-form on X given by a collection of continuous real-valued functions $\{f_U\}_{U\in\mathcal{U}}$ with respect to an open cover $\mathcal{U}$ of X. Let $\gamma: [0,1] \to X$ be a continuous path. The line integral $\int_\gamma \omega$ is defined as follows. Find a subdivision $t_0 = 0 < t_1 < \ldots < t_N = 1$ of the interval $[0,1]$ such that for any i the image $\gamma[t_i, t_{i+1}]$ is contained in a single open set $U_i \in \mathcal{U}$. Then we define

$$\int_\gamma \omega = \sum_{i=0}^{N-1} [f_{U_i}(\gamma(t_{i+1})) - f_{U_i}(\gamma(t_i))]. \tag{A.1}$$

The standard argument shows that the integral (A.1) does not depend on the choice of the subdivision and the open cover $\mathcal{U}$.

LEMMA A.1. *For any pair of continuous paths* $\gamma, \gamma': [0,1] \to X$ *with common beginning* $\gamma(0) = \gamma'(0)$ *and common end points* $\gamma(1) = \gamma'(1)$*, holds*

$$\int_\gamma \omega = \int_{\gamma'} \omega,$$

provided that γ *and* γ' *are homotopic relative to the boundary.*

PROOF. It is standard. ☐

Any closed 1-form defines the *homomorphism of periods*

$$\pi_1(X, x_0) \to \mathbb{R}, \quad [\gamma] \mapsto \int_\gamma \omega \in \mathbb{R} \tag{A.2}$$

given by integration of 1-form ω along closed loops $\gamma\colon[0,1]\to X$ with $\gamma(0)=x_0=\gamma(1)$.

LEMMA A.2. *The homomorphism of periods* (A.2) *is a group homomorphism.*

LEMMA A.3. *Let X be a path-connected topological space. A continuous closed* 1*-form ω on X equals df for a continuous function $f\colon X\to\mathbb{R}$ if and only if ω defines a trivial homomorphism of periods* (A.2).

PROOF. If $\omega = df$ then $\int_\gamma \omega = f(q) - f(p)$ holds for any path γ in X, where $q=\gamma(1)$, $p=\gamma(0)$. Hence $\int_\gamma\omega = 0$ if γ is a closed loop.

Conversely, assume that the homomorphism of periods (A.2) is trivial. One defines a continuous function $f\colon X\to\mathbb{R}$ by

$$f(x) = \int_{x_0}^{x}\omega.$$

Here the integration is taken over an arbitrary path connecting x_0 to x. Assume that ω is given by a collection of continuous functions $f_U\colon U\to\mathbb{R}$ with respect to an open cover $\{U\}$ of X. Then for any two points x,y lying in the same path-connected component of U,

$$f(y)-f(x) = \int_x^y \omega = f_U(y) - f_U(x).$$

This shows that the function $f - f_U$ is locally constant on U. Hence $df = \omega$. $\square$

Any continuous closed 1-form ω on a topological space X defines a (singular) *cohomology class* $[\omega]\in H^1(X;\mathbb{R})$. It is defined by the homomorphism of periods (A.2) viewed as an element of $\mathrm{Hom}(H_1(X);\mathbb{R}) = H^1(X;\mathbb{R})$. As follows from the above lemma, *two continuous closed* 1*-forms ω and ω' on X have the same cohomology class $[\omega]=[\omega']$ if and only if their difference $\omega-\omega'$ equals df, where $f\colon X\to\mathbb{R}$ is a continuous function.*

Recall that a topological space X is *homologically locally connected* if for every point $x\in X$ and a neighbourhood U of x there exists a neighbourhood V of x in U such that $\widetilde{H}_q(V)\to\widetilde{H}_q(U)$ is trivial for all q.

LEMMA A.4. *Let X be a paracompact Hausdorff homologically locally connected topological space. Then any singular cohomology class $\xi\in H^1(X;\mathbb{R})$ may be realized by a continuous closed* 1*-form on X.*

PROOF. Consider the following exact sequence of sheaves over X

$$0\to\mathbb{R}_X\to C_X\to B_X\to 0. \tag{A.3}$$

Here $\mathbb{R}_X$ denotes the sheaf of locally constant functions, C_X denotes the sheaf of real-valued continuous functions, and B_X denotes the sheaf of germs of continuous functions modulo locally constant. More precisely, B_X is the sheaf corresponding to the presheaf $U \mapsto C_X(U)/\mathbb{R}_X(U)$. Comparing this with our definition of a continuous closed 1-form, we find that *the space $H^0(X; B_X)$ of global sections of B_X coincides with the space of continuous closed 1-forms on X.*

From (A.3), using that C_X is a fine sheaf, we obtain an exact sequence

$$0 \to H^0(X;\mathbb{R}) \to H^0(X;C_X) \xrightarrow{d} H^0(X;B_X) \xrightarrow{[\ ]} H^1(X;\mathbb{R}_X) \to 0.$$

Here $H^0(X;C_X) = C(X)$ is the set of all continuous functions on X, and the map d acts by assigning to a continuous function $f\colon X \to \mathbb{R}$ the closed 1-form $df \in H^0(X;B_X)$. The group $H^1(X;\mathbb{R}_X)$ is the Čech cohomology $\check{H}^1(X;\mathbb{R})$ and the map $[\ ]$ assigns to a closed 1-form ω its Čech cohomology class $[\omega] \in \check{H}^1(X;\mathbb{R})$. The natural map $\check{H}^1(X;\mathbb{R}) \to H^1(X;\mathbb{R})$ is an isomorphism assuming that X is paracompact, Hausdorff, and homologically locally connected, cf. [17, Chapter 6, Section 9]. This implies our statement. $\square$

As we have shown in the proof, the space of continuous closed 1-forms $H^0(X;B_X)$ on a connected topological space X can be described by the following short exact sequence

$$0 \to C(X)/\mathbb{R} \xrightarrow{d} H^0(X;B_X) \xrightarrow{[\ ]} \check{H}^1(X;\mathbb{R}) \to 0.$$

Appendix B. Gradient-convex neighbourhoods

Let M be a smooth manifold and let $f\colon M \to \mathbb{R}$ be a C^2-smooth function with isolated critical points. Let v be a gradient-like vector field for f. This means that $v(f) > 0$ on the complement of the set of critical points, and v coincides with the gradient of f on an open neighbourhood of the critical points with respect to a Riemannian metric.

We will denote by $M \times \mathbb{R} \to M$, $(m,t) \mapsto m \cdot t$, the flow of the field v; we will assume that it is defined for all $t \in \mathbb{R}$.

Lemma B.1. *Any open neighbourhood $V \subset M$ of a critical point $p \in M$, $df_p = 0$, contains a compact neighbourhood U of p, such that*

(1) *for any point $m \in M$ the set $J_m = \{t \in \mathbb{R} : m \cdot t \in U\} \subset \mathbb{R}$ is either empty, or a closed interval $[a_m, b_m]$, possibly degenerated to a point, i.e. with $a_m = b_m$,*
(2) *the function $\{m \in M : J_m \neq \emptyset\} \to \mathbb{R}$, where $m \mapsto a_m \in \mathbb{R}$, is continuous.*

The proof below was essentially suggested by P. Milman.

Proof. We will assume that $f(p) = 0$, $\overline{V}$ is compact, and point p is the only critical point of function f in $\overline{V}$. Let V_0 be an open neighbourhood of p with compact closure $\overline{V}_0 \subset V$ and such that in V_0 there exist local coordinates $x_1, \dots, x_n$ and $v|_{V_0}$ is the gradient of f with respect to a Riemannian metric g_{ij}. Fix a smooth function $\psi\colon M \to [0,1]$, such that $\psi|_{V_0} \equiv 0$ and $\psi|_{(M-V)} \equiv 1$.

We want to show that there is a constant $\lambda > 0$ such that the derivative of the function

$$\Psi = \langle \operatorname{grad} f, \operatorname{grad} f\rangle\colon M \to \mathbb{R}_+$$

along the gradient flow of f satisfies in V_0 the inequalities:

$$\text{(B.1)} \qquad -\lambda\Psi \le \frac{d\Psi}{dt} \le \lambda\Psi.$$

In the local coordinates,

$$\Psi = \sum_{ij} g^{ij} \cdot \frac{\partial f}{\partial x_i} \cdot \frac{\partial f}{\partial x_j}$$

and

$$\frac{d\Psi}{dt} = \sum_{ij} \langle \operatorname{grad} g^{ij}, \operatorname{grad} f\rangle \cdot \frac{\partial f}{\partial x_i} \cdot \frac{\partial f}{\partial x_j} + 2\Big\langle \operatorname{grad} f, \sum_{ij} g^{ij} \cdot \frac{\partial f}{\partial x_j} \cdot \operatorname{grad} \frac{\partial f}{\partial x_i}\Big\rangle.$$

Both the first and second terms in this sum can be viewed as symmetric bilinear forms in the partial derivatives of f with continuous coefficients, and hence our statement (B.1) follows.

We will define now two smooth functions $F_+, F_-\colon M \to \mathbb{R}$ by

$$\text{(B.2)} \qquad F_\pm = \pm 2f + \lambda^{-1} \cdot \Psi + \psi.$$

In $V_0 - \{p\}$ using (B.1) we have

$$\begin{aligned} \frac{dF_+}{dt} &= 2\Psi + \lambda^{-1} \cdot \frac{d\Psi}{dt} \ge \Psi > 0, \\ \frac{dF_-}{dt} &= -2\Psi + \lambda^{-1} \cdot \frac{d\Psi}{dt} \le -\Psi < 0. \end{aligned}$$

Hence we conclude that F_+ increases and F_- decreases along the gradient flow of f in $V_0 - \{p\}$.

We set $g = \max\{F_+, F_-\}\colon M \to \mathbb{R}_+$. It is a continuous function with $g(p) = 0$. For any number $0 < c$ small enough, the set $U_c = \{x \in M; g(x) < c\}$ is an open neighbourhood of p, contained in V_0.

Let V_1 be an open neighbourhood of p with $\overline{V}_1 \subset V_0$. Let $\varepsilon > 0$ be such that any trajectory $\gamma(t) \in M$ of the flow of v with $\gamma(t_1) \in \overline{V}_1$, and $\gamma(t_2) \in \overline{M - V_0}$, where $t_1 < t_2$, satisfies

$$\text{(B.3)} \qquad f(\gamma(t_2)) - f(\gamma(t_1)) > \varepsilon.$$

We will show that $\overline{U}_c$ satisfies the conditions of the lemma assuming that $0 < c < \varepsilon$ and c is small enough, so that $U_c \subset V_1$. Since $\partial\overline{U}_c = g^{-1}(c)$, a trajectory $\gamma(t) = m \cdot t$ enters the set $\overline{U}_c$ at $t = a$ if and only if $F_-(\gamma(a)) = c$ and $F_+(\gamma(a)) \leq c$. Moreover, if $F_-(\gamma(a)) = c$ and $F_+(\gamma(a)) = c$, the trajectory leaves $\overline{U}_c$ immediately (i.e. the trajectory $\gamma(t)$ is tangent to the boundary ∂U_c), and if $F_-(\gamma(a)) < c$, the trajectory penetrates the interior of $\overline{U}_c$.

Similarly, if $t = b$ is such that $F_-(\gamma(b)) \leq c$ and $F_+(\gamma(b)) = c$, the trajectory $\gamma(t)$ leaves the set $\overline{U}_c$ for $t > b$. We know that while $\gamma(t)$ stays in V_1, the function F_+ increases and so the trajectory remains away from $\overline{U}_c$. Can this trajectory return to $\overline{U}_c$ for some large time $t = c > b$? If this happens, then $f(\gamma(c)) > \varepsilon$ (because of (B.3) and using $f(\gamma(b)) \geq 0$) and hence $g(\gamma(c)) > \varepsilon$.

This proves that under our assumptions on c the set $\{t; \gamma(t) \in \overline{U}_c\}$ coincides with the interval $[a, b]$.

Now we are left to prove statement (2). Let $A \subset M$ denote the set of points $m \in M$, such that $m \cdot t$ belongs to V_1 for some t. Since $dF_-/dt < 0$, the equation $F_-(m \cdot a_m) = c$ defines a continuous function of $m \in A$. Similarly, the equation $F_+(m \cdot b_m) = c$ defines a continuous function $A \to \mathbb{R}$, where $m \mapsto b_m$. The set $\{m \in M; J_m \neq \emptyset\}$ equals $\{m \in A; a_m \leq b_m\}$. This implies our claim (2). □

References

[1] C. Conley, *Isolated invariant sets and the Morse index* (1976), Amer. Math. Soc.

[2] B. Dubrovin, S. Novikov and A. Fomenko, *Modern Geometry; Methods of the homology theory*, 1984.

[3] Y. Eliashberg and M. Gromov, *Lagrangian intersection theory; finite-dimensional approach*, Geometry of Differential Equations (A. Khovanskiĭ, A. Varchenko and V. Vassiliev, eds.), vol. 186, Amer. Math. Soc., 1998.

[4] H. Fan and J. Jost, *Novikov–Morse theory for Dynamical Systems*, Preprint Math. DS/0010295.

[5] M. Farber, *Massey products and critical points*, Rothenberg Festschrift, Contemporary Mathematics, vol. 231, Amer. Math. Soc., 1999, pp. 67–71.

[6] ———, *Lusternik–Schnirelman theory for closed* 1*-forms*, Comment. Math. Helv. **75** (2000), 156–170.

[7] ———, *Topology of closed* 1*-forms and their critical points*, Topology **40** (2001), 235–258.

[8] S. Frolov and L. Elsgoltz, *Limite infèrieure pour le nombre des valeurs critiques d'une fonction, donnée sur variété*, Mat. Sb. **42** (1935), 637–643.

[9] H. Hofer and D. Salamon, *Floer homology and Novikov rings*, Floer memorial volume, Birkhäuser, 1995, pp. 483–524.

[10] H. V. Le and K. Ono, *Symplectic fixed points, the Calabi invariant and Novikov homology*, Topology **34** (1995), 155–176.

[11] J. Milnor, *Lectures on the h-Cobordism Theorem*, Princeton, New Jersey, 1965.

[12] S. P. Novikov, *The Hamiltonian formalism and a multivalued analogue of Morse theory*, Russian Math. Surveys **37** (1982), 1–56.

[13] ———, *Bloch homology, critical points of functions and closed* 1*-forms*, Soviet Math. Dokl. **33** (1986), 551–555.

[14] J.-C. Sikorav, *Thèse*.

[15] ———, *Un problem de disjonction par isotopic symplectique dans un fibré cotangent*, Ann. Sci. École Norm. Sup. (4) **19** (1986), 543–552.

[16] S. Smale, *Stable manifolds for differential equations and diffeomorphisms*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **17** (1963), 97–116.

[17] E. H. Spanier, *Algebraic Topology*, 1966.

[18] F. Takens, *The minimal number of critical points of a function on a compact manifold and the Lusternik–Schnirelman category*, Invent. Math. **6** (1968), 197–244.

Manuscript received June 22, 2001

Michael Farber
School of Mathematical Sciences
Tel Aviv University
Ramat Aviv 69978, ISRAEL

E-mail address: mfarber@tau.ac.il