CHARACTERIZATION OF THE LIMIT
OF SOME HIGHER DIMENSIONAL
THIN DOMAIN PROBLEMS

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ABSTRACT. A reaction-diffusion equation on a family of three dimensional thin domains, collapsing onto a two dimensional subspace, is considered. In [13] it was proved that, as the thickness of the domains tends to zero, the solutions of the equations converge in a strong sense to the solutions of an abstract semilinear parabolic equation living in a closed subspace of H^1. Also, existence and upper semicontinuity of the attractors was proved. In this work, for a specific class of domains, the limit problem is completely characterized as a system of two-dimensional reaction-diffusion equations, coupled by mean of compatibility and balance boundary conditions.

1. Introduction

Let $\Omega \subset \mathbb{R}^{N+M}$ be an open bounded domain with Lipschitz boundary. Write (x, y) for a generic point of $\mathbb{R}^{N+M}$. For $\varepsilon > 0$, let us consider the "squeezing operator" $T_\varepsilon : \mathbb{R}^{N+M} \to \mathbb{R}^{N+M}$, $(x, y) \mapsto (x, \varepsilon y)$, and define $\Omega_\varepsilon := T_\varepsilon(\Omega)$. Let Γ be a relatively closed portion of $\partial\Omega$ and let $\Gamma_\varepsilon := T_\varepsilon(\Gamma)$. Let us consider the following reaction-diffusion equation

$$(1.1) \quad \begin{cases} u_t = \Delta u + f(u) & \text{for } t > 0, \ (x, y) \in \Omega_\varepsilon, \\ \partial_{\nu_\varepsilon} u = 0 & \text{for } t > 0, \ (x, y) \in \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon, \\ u = 0 & \text{for } t > 0, \ (x, y) \in \Gamma_\varepsilon. \end{cases}$$

Here ν_ε is the exterior normal vector field on $\partial\Omega_\varepsilon$. We assume that f satisfies the following condition:

(H1) $f \in C^1(\mathbb{R} \to \mathbb{R})$ and $|f'(s)| \leq C(|s|^\beta + 1)$ for $s \in \mathbb{R}$, where C and $\beta \in [0, \infty[$ are arbitrary real constants. If $n := M + N > 2$ then in addition, $\beta \leq (p^*/2) - 1$, where $p^* = 2n/(n-2) > 2$.

Let $H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)$ be the closure in $H^1(\Omega_\varepsilon)$ of the space of all $C^1(\overline{\Omega}_\varepsilon)$-functions such that $u = 0$ on Γ_ε. Then it is well known that equation (1.1) generates a semiflow $\widetilde{\pi}_\varepsilon$ on $H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)$. If we suppose in addition that f satisfies the dissipativeness condition

(H2) $\limsup_{|s| \to \infty} f(s)/s \leq -\zeta$ for some $\zeta > 0$,

then the semiflow $\widetilde{\pi}_\varepsilon$ is defined for all $t \geq 0$ and it posseses a compact global attractor $\widetilde{\mathcal{A}}_\varepsilon$.

As $\varepsilon \to 0$ the thin domain Ω_ε degenerates to an N-dimensional domain. Then the question arises, what happens in the limit to the family $(\widetilde{\pi}_\varepsilon)_{\varepsilon>0}$ of semiflows and to the family $(\widetilde{\mathcal{A}}_\varepsilon)_{\varepsilon>0}$ of attractors. Does there exist a limit semiflow and a corresponding limit attractor?

This problem was first considered by Hale and Raugel in [7] for the case when $M = 1$ and the domain Ω is the *ordinate set* of a smooth positive function g defined on an N-dimensional domain ω, i.e.

$$\Omega = \{(x, y) \mid x \in \omega \text{ and } 0 < y < g(x)\},$$

with $\Gamma = \emptyset$ (resp. $\Gamma = \{(x, y) \mid x \in \partial\omega \text{ and } 0 < y < g(x)\}$).

The authors prove that, in this case, there exists a limit semiflow $\widetilde{\pi}_0$, which is defined by the N-dimensional boundary value problem

$$(1.2) \qquad \begin{cases} u_t = \dfrac{1}{g}\mathrm{div}(g\nabla u) + f(u) & \text{for } t > 0, \ x \in \omega, \\[2mm] \dfrac{\partial u}{\partial \nu}u = 0 \ (\text{resp. } u = 0) & \text{for } t > 0, \ x \in \partial\omega. \end{cases}$$

Moreover, $\widetilde{\pi}_0$ has a global attractor $\widetilde{\mathcal{A}}_0$ and, in some sense, the family $(\widetilde{\mathcal{A}}_\varepsilon)_{\varepsilon \geq 0}$ is upper-semicontinuous at $\varepsilon = 0$. See also [16] and the rich bibliography contained therein.

If the domain Ω is not the ordinate set of some function (e.g. if Ω has holes or different horizontal branches) then (1.2) can no longer be a limiting equation for (1.1). Nevertheless, K. Rybakowski and the second author proved in [13] that the family $\widetilde{\pi}_\varepsilon$ still has a limit semiflow. Moreover, there exists a limit global attractor and the upper-semicontinuity result continues to hold.

In order to describe the main results of [13] we first transfer the family (1.1) to boundary value problems on the fixed domain Ω. More explicitly, we use the linear isomorphism $\Phi_\varepsilon \colon H^1(\Omega_\varepsilon) \to H^1(\Omega)$, $u \mapsto u \circ T_\varepsilon$, to transform problem (1.1)

to the equivalent problem

$$(1.3) \quad \begin{cases} u_t = \Delta_x u + \dfrac{1}{\varepsilon^2}\Delta_y u + f(u) & \text{for } t > 0, \ (x,y) \in \Omega, \\[2mm] \nabla_x u \cdot \nu_x + \dfrac{1}{\varepsilon^2}\nabla_y u \cdot \nu_y = 0 & \text{for } t > 0, \ (x,y) \in \partial\Omega \setminus \Gamma, \\[2mm] u = 0 & \text{for } t > 0, \ (x,y) \in \Gamma, \end{cases}$$

on Ω. Here, $\nu = (\nu_x, \nu_y)$ is the exterior normal vector field on $\partial\Omega$.

Let $H^1_\Gamma(\Omega)$ be the closure in $H^1(\Omega)$ of the space of all $C^1(\overline{\Omega})$-functions such that $u = 0$ on Γ. Then equation (1.3) can be written in the abstract form

$$(1.4) \qquad\qquad\qquad \dot{u} + A_\varepsilon u = \widehat{f}(u)$$

where $\widehat{f}\colon H^1_\Gamma(\Omega) \to L^2(\Omega)$ is the Nemytskiĭ operator generated by the function f, and A_ε is the selfadjoint linear operator (with compact resolvent) induced by the following bilinear form

$$a_\varepsilon(u,v) := \int_\Omega \left(\nabla_x u \cdot \nabla_x v + \frac{1}{\varepsilon^2} \nabla_y u \cdot \nabla_y v \right) dx\,dy, \quad u, v \in H^1_\Gamma(\Omega).$$

Equation (1.4) then defines a semiflow π_ε on $H^1_\Gamma(\Omega)$ which is equivalent to $\widetilde{\pi}_\varepsilon$ and has the global attractor $\mathcal{A}_\varepsilon := \Phi_\varepsilon(\widetilde{\mathcal{A}}_\varepsilon)$, consisting of the orbits of all full bounded solutions of (1.4).

Notice that, for every fixed $\varepsilon > 0$ and $u \in H^1_\Gamma(\Omega)$, the formula

$$|u|_\varepsilon = (a_\varepsilon(u,u) + |u|^2_{L^2(\Omega)})^{1/2}$$

defines a norm on $H^1_\Gamma(\Omega)$ which is equivalent to $|\cdot|_{H^1_\Gamma(\Omega)}$. However, $|u|_\varepsilon \to \infty$ as $\varepsilon \to 0^+$ whenever $\nabla_y u \neq 0$ in $L^2(\Omega)$. In fact, we see that for $u \in H^1_\Gamma(\Omega)$

$$\lim_{\varepsilon \to 0^+} a_\varepsilon(u,u) = \begin{cases} \displaystyle\int_\Omega |\nabla_x u|^2 \, dx\,dy & \text{if } \nabla_y u = 0, \\[2mm] \infty & \text{otherwise.} \end{cases}$$

Thus the family $a_\varepsilon(u,u)$, $\varepsilon > 0$, of real numbers has a finite limit (as $\varepsilon \to 0$) if and only if $u \in H^1_{\Gamma,s}(\Omega)$, where we define

$$H^1_{\Gamma,s}(\Omega) := \{ u \in H^1_\Gamma(\Omega) \mid \nabla_y u = 0 \}.$$

This is a closed linear subspace of $H^1_\Gamma(\Omega)$.

The corresponding limit bilinear form is given by the formula:

$$(1.5) \qquad a_0(u,v) := \int_\Omega \nabla_x u \cdot \nabla_x v \, dx\,dy, \quad u, v \in H^1_{\Gamma,s}(\Omega).$$

Assume from now on that $H^1_{\Gamma,s}(\Omega)$ is infinite dimensional. Then the form a_0 uniquely determines a densely defined selfadjoint linear operator $A_0\colon D(A_0) \subset H^1_{\Gamma,s(\Omega)} \to L^2_{\Gamma,s}(\Omega)$ by the usual formula

$$(1.6) \qquad a_0(u,v) = \langle A_0 u, v \rangle_{L^2(\Omega)}, \quad \text{for } u \in D(A_0) \text{ and } v \in H^1_{\Gamma,s(\Omega)}.$$

Notice that A_0 has compact resolvent. Here, $L^2_{\Gamma,s}(\Omega)$ is the closure of $H^1_{\Gamma,s}(\Omega)$ in the L^2-norm, so $L^2_{\Gamma,s}(\Omega)$ is a closed linear subspace of $L^2(\Omega)$.

One can show that the Nemytskiĭ operator $\widehat{f}$ maps the space $H^1_{\Gamma,s}(\Omega)$ into $L^2_{\Gamma,s}(\Omega)$. Consequently the abstract parabolic equation

$$(1.7) \qquad\qquad \dot{u} + A_0 u = \widehat{f}(u)$$

defines a semiflow π_0 on the space $H^1_{\Gamma,s}(\Omega)$. This is the limit semiflow of the family π_ε. The following results are proved in [13]:

THEOREM A. *Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of positive numbers convergent to zero and $(u_n)_{n\in\mathbb{N}}$ be a sequence in $L^2(\Omega)$ converging in the norm of $L^2(\Omega)$ to some $u_0 \in L^2_{\Gamma,s}(\Omega)$. Moreover, let $(t_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of positive numbers converging to some positive number t_0. Then*

$$|e^{-t_n A_{\varepsilon_n}} u_n - e^{-t_0 A_0} u_0|_{\varepsilon_n} \to 0 \quad as\ n \to \infty.$$

If, in addition, $u_n \in H^1(\Omega)$ for every $n \in \mathbb{N}$ and if $u_0 \in H^1_{\Gamma,s}(\Omega)$, then

$$|u_n \pi_{\varepsilon_n} t_n - u_0 \pi_0 t_0|_{\varepsilon_n} \to 0 \quad as\ n \to \infty.$$

The limit semiflow π_0 possesses a global attractor $\mathcal{A}_0$. The upper-semicontinuity result alluded to above reads as follows:

THEOREM B. *The family of attractors $(\mathcal{A}_\varepsilon)_{\varepsilon\in[0,1]}$ is upper-semicontinuous at $\varepsilon = 0$ with respect to the family of norms $|\cdot|_\varepsilon$. This means that*

$$\lim_{\varepsilon\to 0^+} \sup_{u\in\mathcal{A}_\varepsilon} \inf_{v\in\mathcal{A}_0} |u - v|_\varepsilon = 0.$$

In particular, there exists an $\varepsilon_1 > 0$ and an open bounded set U in $H^1(\Omega)$ including all the attractors $\mathcal{A}_\varepsilon$, $\varepsilon \in [0, \varepsilon_1]$.

REMARK. Theorems A and B were actually proved in the case $\Gamma = \emptyset$, but the proof is valid (with only minor changes) also in the general case, as long as $H^1_{\Gamma,s}(\Omega)$ is infinite dimensional.

The definition of the linear operator A_0, as given above, is not very explicit. If $N = M = 1$, however, it was shown in [13] and [14] that there is a large class of the so-called *nicely decomposed* domains on which A_0 can be characterized as a system of one-dimensional second order linear differential operators, coupled to each other by certain compatibility and Kirchhoff type balance conditions. In this case, the abstract limit equation (1.7) is equivalent to a parabolic equation on a finite graph. Roughly speaking, a planar domain Ω admits a *nice decomposition* if, up to a set of measure zero contained in a set Z of finitely many vertical lines, Ω can be decomposed into finitely many domains Ω_k, $k = 1, \ldots, r$ in such a way that at Z the various sets Ω_k and Ω_l "join" in a nice way. Points of $\overline{\Omega} \cap Z$ are, intuitively speaking, those at which connected components of the vertical

sections Ω_x bifurcate (see Figure 3 in [13]). In higher dimensions it is not clear whether it is possible to describe a reasonable, sufficiently large, class of domains for which an explicit characterization of $H^1_{\Gamma,s}(\Omega)$ and of $D(A_0)$ can be carried on. Nevertheless, in some concrete cases, one can go along the same ideas of [13] and give a nice characterization of these spaces. In this paper we concentrate on the case $N = 2$ and $M = 1$ and we illustrate with two examples how this is possible. Our examples deal with a set Ω which is obtained by removing from a cylinder a smaller cylinder contained in the interior of the first. More precisely, take open sets ω, ω_1, ω_2 and ω_3 in $\mathbb{R}^2$ such that

$$\omega \text{ is bounded, connected and has } C^2\text{-boundary,}$$
$$\omega_2 = \omega_3 \Subset \omega \text{ have } C^2\text{-boundary,}$$
$$\omega_1 := \omega \setminus \overline{\omega}_2.$$

Notice that ω_1 is not necessarily connected. Moreover, let h_1, h_2 and h_3 be positive real numbers, with $h_1 > h_2 + h_3$. Then we define

(1.8) $$\Omega := (\omega \times\,]0, h_1[) \setminus \overline{(\omega_2 \times\,]h_3, h_1 - h_2[)}.$$

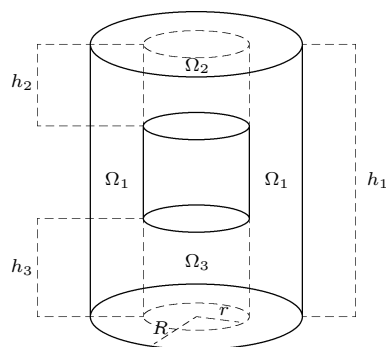Figure 1 represents the domain Ω, when ω and ω_2 are balls centered at 0.

FIGURE 1. The Domain Ω

For later use we need also to define

$$\Omega_1 := \omega_1 \times\,]0, h_1[, \quad \Omega_2 := \omega_2 \times\,]h_1 - h_2, h_1[, \quad \Omega_3 := \omega_3 \times\,]0, h_3[,$$

and

$$\Omega_4 := \omega_1 \times\,]h_3, h_1 - h_2[, \quad \Omega_5 := \omega \times\,]h_1 - h_2, h_1[,$$
$$\Omega_6 := \omega \times\,]0, h_3[, \qquad\qquad \Omega_7 := \mathbb{R}^2 \times\,]h_3, h_1 - h_2[.$$

Finally, we set

$$\Gamma_1 := \partial\omega \times [0, h_1], \quad \Gamma_2 := \partial\omega_2 \times [h_3, h_1 - h_2], \quad \Gamma_L := \Gamma_1 \cup \Gamma_2.$$

We shall consider equation (1.1) on $\Omega_\varepsilon = T_\varepsilon(\Omega)$, where Ω is the domain defined above, with two different sets of boundary conditions, namely with $\Gamma = \emptyset$ and with $\Gamma = \Gamma_L$. We shall see that these different boundary conditions give rise to completely different behaviors as $\varepsilon \to 0$. In fact, when $\Gamma = \emptyset$, i.e. we impose the Neumann boundary condition on the whole $\partial\Omega_\varepsilon$, equation (1.7) is equivalent to the following system of two-dimensional reaction-diffusion equations

$$(1.9) \quad \begin{cases} u_{it} = \Delta u_i + f(u_i) & \text{for } t > 0, \ x \in \omega_i, \ i = 1, 2, 3, \\ u_1(x) = u_2(x) = u_3(x) & \text{for } t > 0, \ x \in \partial\omega_2, \\ \partial_{\nu_1} u_1 = 0 & \text{for } t > 0, \ x \in \partial\omega, \\ \sum_{i=1}^{3} h_i \nabla u_i \cdot \nu_i = 0 & \text{for } t > 0, \ x \in \partial\omega_2. \end{cases}$$

Here ν_i, $i = 1, 2, 3$, is the outward normal vector field on $\partial\omega_i$ for $i = 1, 2, 3$, respectively. Observe that the three equations in (1.9) are coupled by compatibility and Kirchoff type balance conditions on the "interface" $\partial\omega_2$. Figure 2 below illustrates the "limit" of the family (Ω_ε) as $\varepsilon \to 0$ for the domain represented in Figure 1.
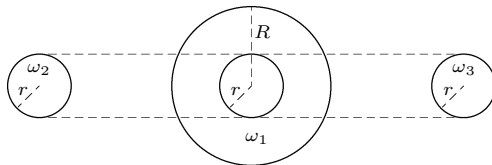

FIGURE 2. The "limit" of the Ω_ε

On the other hand, when $\Gamma = \Gamma_L$, i.e. we impose the Dirichlet boundary condition on the 'lateral' surface Γ_L, equation (1.7) is equivalent to the following system of two-dimensional reaction-diffusion equations

$$(1.10) \quad \begin{cases} u_{it} = \Delta u_i + f(u_i) & \text{for } t > 0, \ x \in \omega_i, \ i = 1, 2, 3, \\ u_i(x) = 0 & \text{for } t > 0, \ x \in \partial\omega_i, \ i = 1, 2, 3. \end{cases}$$

So in this case the "limit" problem is a completely decoupled system of scalar reaction-diffusion equations.

These two examples furnish a prototype for many concrete situations that may occur in practice. In particular, we point out that the core of the problem consists in proving regularity of the solutions of the linear equation

$$A_0 u = w \quad \text{with } w \in L^2_{\Gamma, s}(\Omega).$$

Once the spaces $L^2_{\Gamma,s}(\Omega)$, $H^1_{\Gamma,s}(\Omega)$ and $D(A_0)$ have been characterized, one can easily show that (1.7) is equivalent to a system of concrete reaction-diffusion equations of type (1.9) or (1.10).

Finally, as we shall explain in Section 3, the characterization of A_0 and of its domain can be exploited to compute the eigenvalues of A_0 in some specific situations, like the one represented in Figure 1. Of course, informations on the location and on the multiplicity of the eigenvalues of A_0 are very important in the study of local bifurcations of (1.7).

2. Characterization of $H^1_{\Gamma,s}(\Omega)$

We begin by recalling a general notion introduced in [13]: we say that an open set $\Omega \in \mathbb{R}^{N+M}$ has *connected vertical sections* if for every $x \in \mathbb{R}^N$ the x-section Ω_x is connected. Of course, this section is nonempty if and only if $x \in P(\Omega)$, where $P:\mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$, $(x,y) \mapsto x$ is the projection onto the first N components. The following proposition was proved in [13]:

PROPOSITION 2.1. *Suppose Ω has connected vertical sections. Let $J := P(\Omega)$ and define the function $p: J \to]0,\infty[$ by $x \mapsto \mu_M(\Omega_x)$. If $u \in L^2(\Omega)$ satisfies $\nabla_y u = 0$ in the distributional sense, then there is a null set S in $\mathbb{R}^{N+M}$ and a function $v \in L^1_{\mathrm{loc}}(J)$ such that $u(x,y) = v(x)$ for every $(x,y) \in \Omega \setminus S$. Moreover, $p^{1/2}v \in L^2(J)$. If $u \in H^1(\Omega)$ then $\partial_{x_i}v \in L^1_{\mathrm{loc}}(J)$ for $i = 1,\ldots,N$ and we can choose the null set S so that $u(x,y) = v(x)$ and $\partial_{x_i}u(x,y) = \partial_{x_i}v(x)$ for every $i = 1, \ldots, N$ and $(x,y) \in \Omega \setminus S$. Moreover, $p^{1/2}\partial_{x_i}v \in L^2(J)$ for every $i = 1, \ldots, N$.*

Now we come back to the domain Ω defined by (1.8). In what follows, we may assume indifferently that $\Gamma = \Gamma_L$ or $\Gamma = \emptyset$. For $k = 1, \ldots, 7$ let us define

$$H^1_s(\Omega_k) := \{u \in H^1(\Omega_k) \mid \nabla_y u = 0 \text{ a.e.}\}.$$

Moreover, let us define $L^2_s(\Omega_k)$ as the closure of $H^1_s(\Omega_k)$ in $L^2(\Omega_k)$.

LEMMA 2.2. *For $k = 1,\ldots,6$, the following properties hold:*

(a) *whenever $u \in L^2_{\Gamma,s}(\Omega)$, then $u|_{\Omega_k} \in L^2_s(\Omega_k)$,*
(b) *whenever $u \in H^1_{\Gamma,s}(\Omega)$, then $u|_{\Omega_k} \in H^1_s(\Omega_k)$.*

PROOF. Part (b) is obvious and part (a) follows directly from part (b) and from the definition of $L^2_{\Gamma,s}(\Omega)$ and $L^2_s(\Omega_k)$. □

For $k = 1,2,3$, let us define the spaces $L_k := L^2(\omega_k)$ and $H_k := H^1(\omega_k)$. Define on L_k and H_k the scalar products

$$\langle u,v \rangle_{L_k} := \int_{\omega_k} h_k u(x)v(x)\,dx,$$

$$\langle u, v \rangle_{H_k} := \int_{\omega_k} h_k u(x) v(x) \, dx + \int_{\omega_k} h_k \nabla u(x) \cdot \nabla v(x) \, dx,$$

respectively. Moreover, for $k = 1, 2, 3$, let us define the mapping

$$\imath_k \colon L_s^2(\Omega_k) \to L_k, \quad u \mapsto v,$$

where v is the function given by proposition 2.1. It turns out that $\imath_k$ is an isometry of $L_s^2(\Omega_k)$ onto L_k for $k = 1, 2, 3$. Moreover, $\imath_k$ restricts to an isometry of $H_s^1(\Omega_k)$ onto H_k for $k = 1, 2, 3$. Let us define the product spaces

$$L_\oplus := L_1 \oplus L_2 \oplus L_3 := \{[u] = (u_1, u_2, u_3) \mid u_k \in L_k, k = 1, 2, 3\},$$
$$H_\oplus := H_1 \oplus H_2 \oplus H_3 := \{[u] = (u_1, u_2, u_3) \mid u_k \in H_k, k = 1, 2, 3\}$$

with the scalar products

$$\langle [u], [v] \rangle_{L_\oplus} := \langle u_1, v_1 \rangle_{L_1} + \langle u_2, v_2 \rangle_{L_2} + \langle u_3, v_3 \rangle_{L_3},$$
$$\langle [u], [v] \rangle_{H_\oplus} := \langle u_1, v_1 \rangle_{H_1} + \langle u_2, v_2 \rangle_{H_2} + \langle u_3, v_3 \rangle_{H_3},$$

respectively. It is easy to check that $L_\oplus$ and $H_\oplus$ are Hilbert spaces. Besides, let us define the map

$$\imath_\oplus \colon L_{\Gamma,s}^2(\Omega) \to L_\oplus, \quad \imath_\oplus u := (\imath_1(u|_{\Omega_1}), \imath_2(u|_{\Omega_2}), \imath_3(u|_{\Omega_3})).$$

Observe that

$$\langle u, v \rangle_{L^2(\Omega)} = \langle \imath_\oplus u, \imath_\oplus v \rangle_{L_\oplus} \quad \text{for } u \text{ and } v \in L_{\Gamma,s}^2(\Omega),$$
$$\langle u, v \rangle_{L^2(\Omega)} + a_0(u, v) = \langle \imath_\oplus u, \imath_\oplus v \rangle_{H_\oplus} \quad \text{for } u \text{ and } v \in H_{\Gamma,s}^1(\Omega).$$

It follows by Lemma 2.2 that $\imath_\oplus$ is an isometry of $L_{\Gamma,s}^2(\Omega)$ into $L_\oplus$ and that $\imath_\oplus$ restricts to an isometry of $H_{\Gamma,s}^1(\Omega)$ into $H_\oplus$. Finally, let us define

$$H_\oplus^0 := \{[u] \in H_\oplus \mid u_k \in H_0^1(\omega_k) \text{ for } k = 1, 2, 3\},$$
$$H_\oplus^C := \{[u] \in H_\oplus \mid {}^\tau u_1(x) = {}^\tau u_2(x) = {}^\tau u_3(x) \ \mathcal{H}^1\text{-a.e. on } \partial\omega_2\},$$

where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure in $\mathbb{R}^2$ and ${}^\tau u_k$ is the trace of u_k on $\partial\omega_k$ for $k = 1, 2, 3$. We call

$$(2.1) \qquad\qquad {}^\tau u_1(x) = {}^\tau u_2(x) = {}^\tau u_3(x) \quad \mathcal{H}^1\text{-a.e. on } \partial\omega_2$$

the *compatibility condition* on $\partial\omega_2$.

Now we are able to characterize the spaces $H_{\Gamma,s}^1(\Omega)$ and $L_{\Gamma,s}^2(\Omega)$:

THEOREM 2.3. *The following properties hold:*

(a) $\imath_\oplus(L_{\Gamma,s}^2(\Omega)) = L_\oplus$,
(b) $\imath_\oplus(H_{\Gamma,s}^1(\Omega)) = H_\oplus^C$ *if* $\Gamma = \emptyset$ *and* $\imath_\oplus(H_{\Gamma,s}^1(\Omega)) = H_\oplus^0$ *if* $\Gamma = \Gamma_L$.

PROOF. We begin by proving (b). Let $\Gamma = \Gamma_L$ or $\Gamma = \emptyset$ and let $u \in H^1_{\Gamma,s}(\Omega)$. Let $\imath_\oplus u := [v] = (v_1, v_2, v_3)$. We shall prove that

$$(2.2) \qquad {}^\tau v_1(x) = {}^\tau v_2(x) = {}^\tau v_3(x) \quad \mathcal{H}^1\text{-a.e. on } \partial\omega_2.$$

By the definition of $\imath_\oplus$ and by Proposition 2.1, there exists a null set $S \subset \mathbb{R}^3$ such that

$$u(x,y) = v_k(x) \quad \text{for all } (x,y) \in \Omega_k \setminus S \text{ and for } k = 1, 2, 3.$$

On the other hand, again by Proposition 2.1, we can find two functions v_5 and $v_6 \in H^1(\omega)$ and we can choose the set S in such a way that

$$u(x,y) = v_l(x) \quad \text{for all } (x,y) \in \Omega_l \setminus S \text{ and for } l = 5, 6.$$

It follows that

$$\begin{aligned}
v_1(x) &= v_5(x) = v_6(x) &&\text{a.e. in } \omega_1, \\
v_2(x) &= v_5(x) &&\text{a.e. in } \omega_2, \\
v_3(x) &= v_6(x) &&\text{a.e. in } \omega_3.
\end{aligned}$$

Define the functions $\widetilde{v}_5$ and $\widetilde{v}_6 \colon \omega \to \mathbb{R}$ by

$$\widetilde{v}_5(x) := \begin{cases} v_1(x) & \text{if } x \in \omega_1, \\ v_2(x) & \text{if } x \in \omega_2, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \widetilde{v}_6(x) := \begin{cases} v_1(x) & \text{if } x \in \omega_1, \\ v_3(x) & \text{if } x \in \omega_3, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\widetilde{v}_5 = v_5$ and $\widetilde{v}_6 = v_6$ almost everywhere in ω and hence $\widetilde{v}_5$ and $\widetilde{v}_6 \in H^1(\omega)$. This in turns implies (2.2) (see [1, Lemma A 5.10, p. 195]). This proves that $\imath_\oplus H^1_{\Gamma,s}(\Omega) \subset H^C_\oplus$. Assume now that $\Gamma = \Gamma_L$. We shall show that in this case $v_1 \in H^1_0(\omega_1)$. Let us define the function $\widetilde{u} \colon \Omega_7 \to \mathbb{R}$ by

$$\widetilde{u}(x,y) := \begin{cases} u(x,y) & \text{if } x \in \Omega_4, \\ 0 & \text{otherwise.} \end{cases}$$

Since ${}^\tau u(x,y) = 0$ $\mathcal{H}^2$-a.e. on Γ_L, it follows that $\widetilde{u} \in H^1_s(\Omega_7)$ (here $\mathcal{H}^2$ is the two-dimensional Hausdorff measure in $\mathbb{R}^3$ and ${}^\tau u$ is the trace of u on $\partial\Omega$). By Proposition 2.1, there exist a null set $S \subset \mathbb{R}^3$ and a function $v_7 \in H^1(\mathbb{R}^2)$ such that

$$\widetilde{u}(x,y) = v_7(x) \quad \text{for all } (x,y) \in \Omega_7 \setminus S.$$

Observe that $v_7 = 0$ a.e. in $\mathbb{R}^2 \setminus \omega_1$. On the other hand, again by Proposition 2.1, we can find a function $v_4 \in H^1(\omega_1)$ and we can choose the set S in such a way that

$$u(x,y) = v_4(x) \quad \text{for all } (x,y) \in \Omega_4 \setminus S.$$

It follows that $v_1 = v_4 = v_7$ almost everywhere in ω_1. This in turn implies that $^\tau v_1(x) = 0$ $\mathcal{H}^1$-a.e. on $\partial\omega_1$ (see again [1]), i.e. $v_1 \in H_0^1(\omega_1)$. So far, we have proved that $\imath_\oplus(H_{\Gamma,s}^1(\Omega)) \subset H_\oplus^C$ and, if $\Gamma = \Gamma_L$, $\imath_\oplus(H_{\Gamma,s}^1(\Omega)) \subset H_\oplus^0$.

Assume now that $[v] \in H_\oplus^C$. We shall prove that there exists a function $u \in H_{\Gamma,s}^1(\Omega)$, with $\Gamma = \emptyset$, such that $\imath_\oplus u = [v]$. Let us define a function u on Ω in the following way:

$$u(x,y) := \begin{cases} v_k(x) & \text{if } (x,y) \in \Omega_k, \ k = 1,2,3, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $u|_{\Omega_1} \in H^1(\Omega_1)$. Moreover, $u|_{\Omega_5} \in H^1(\Omega_5)$. In fact, the function $\widetilde{v}_5 : \omega \to \mathbb{R}$ defined by

$$\widetilde{v}_5(x) := \begin{cases} v_1(x) & \text{if } x \in \omega_1, \\ v_2(x) & \text{if } x \in \omega_2, \\ 0 & \text{otherwise,} \end{cases}$$

is in $H^1(\omega)$, since $^\tau v_1(x) = {}^\tau v_2(x)$ $\mathcal{H}^1$-a.e. on $\partial\omega_2$ (see again [1]). Analogously, $u|_{\Omega_6} \in H^1(\Omega_6)$. Now since $(\Omega_l)_{l=1,5,6}$ is an open covering of Ω, it follows immediately that $u \in H^1(\Omega)$. It is easily verified that $\nabla_y u = 0$ almost everywhere, so $u \in H_{\Gamma,s}^1(\Omega)$. By construction, $\imath_\oplus u = [v]$.

Assume now that $[v] \in H_\oplus^0$. We shall prove that there exists a function $u \in H_{\Gamma,s}^1(\Omega)$, with $\Gamma = \Gamma_L$, such that $\imath_\oplus u = [v]$. As before, let us define a function u on Ω in the following way:

$$u(x,y) := \begin{cases} v_k(x) & \text{if } (x,y) \in \Omega_k, \ k = 1,2,3, \\ 0 & \text{otherwise.} \end{cases}$$

By the same arguments as above, it follows easily that $u \in H^1(\Omega)$ and that $\nabla_y u = 0$ almost everywhere. We shall show that $^\tau u = 0$ on Γ_L. To this end, let us choose sequences $(v_k^n)_{n \in \mathbb{N}}$, $v_k^n \in C_0^1(\omega_k)$, $v_k^n \to v_k$ in $H^1(\omega_k)$ as $n \to \infty$, $k = 1,2,3$, and let us define

$$u^n(x,y) := \begin{cases} v_k^n(x) & \text{if } (x,y) \in \Omega_k, \ k = 1,2,3, \\ 0 & \text{otherwise,} \end{cases}$$

for $n \in \mathbb{N}$. Then $u^n \in C^1(\overline{\Omega})$ and $u^n(x) = 0$ on Γ_L for all $n \in \mathbb{N}$. Moreover, it is easy to verify that $u_n \to u$ in $H^1(\Omega)$, so we deduce that $u \in H_{\Gamma,s}^1(\Omega)$. By construction we have that $\imath_\oplus u = [v]$. This concludes the proof of part (b).

Now we prove (a). Let $[v] \in L_\oplus$. We shall prove that there exists $v \in L_{\Gamma,s}^2(\Omega)$ such that $\imath_\oplus u = [v]$. Again, we define a function u on Ω in the following way:

$$u(x,y) := \begin{cases} v_k(x) & \text{if } (x,y) \in \Omega_k, \ k = 1,2,3, \\ 0 & \text{otherwise.} \end{cases}$$

Then $u \in L^2(\Omega)$. We claim that $u \in L^2_{\Gamma,s}(\Omega)$, both with $\Gamma = \Gamma_L$ and with $\Gamma = \emptyset$. This means that u can be approximated in the L^2-norm by functions of $H^1_{\Gamma,s}(\Omega)$. To this end, let us choose sequences $(v_k^n)_{n \in \mathbb{N}}$, $v_k^n \in C^1_0(\omega_k)$, $v_k^n \to v_k$ in $L^2(\omega_k)$ as $n \to \infty$, $k = 1, 2, 3$, and let us define

$$u^n(x, y) := \begin{cases} v_k^n(x) & \text{if } (x, y) \in \Omega_k, \ k = 1, 2, 3, \\ 0 & \text{otherwise.} \end{cases}$$

for $n \in \mathbb{N}$. Then, as in the proof of part (a), $u^n \in H^1_{\Gamma,s}(\Omega)$ for all $n \in \mathbb{N}$, both with $\Gamma = \Gamma_L$ and with $\Gamma = \emptyset$. Moreover, it is easy to verify that $u_n \to u$ in $L^2(\Omega)$, so we deduce that $u \in L^2_{\Gamma,s}(\Omega)$. By construction we have that $\imath_\oplus u = [v]$ and the proof is complete. $\square$

COROLLARY 2.4. *The space $H^1_{\Gamma,s}(\Omega)$ is infinite dimensional, both with $\Gamma = \emptyset$ and with $\Gamma = \Gamma_L$.*

3. H^2-regularity and characterization of $D(A_0)$

Let us define the bilinear forms

$$a_k(u, v) := \int_{\omega_k} h_k \nabla u(x) \cdot \nabla v(x) \, dx, \quad u, v \in H_k$$

on $H_k \times H_k$, $k = 1, 2, 3$, and the bilinear form

$$a_\oplus([u], [v]) := a_1(u_1, v_1) + a_2(u_2, v_2) + a_3(u_3, v_3), \quad [u], [v] \in H_\oplus$$

on $H_\oplus \times H_\oplus$. Let us indicate by $a_\oplus^C$ and $a_\oplus^0$ the restrictions of $a_\oplus$ to $H_\oplus^C \times H_\oplus^C$ and $H_\oplus^0 \times H_\oplus^0$, respectively. Let $A_\oplus^C$ (resp. $A_\oplus^0$) be the self-adjoint operator generated by $a_\oplus^C$ (resp. $a_\oplus^0$) in $H_\oplus^C$ (resp. $H_\oplus^0$). Finally, let us indicate simply by a the bilinear form a_0 on $H^1_{\Gamma,s}(\Omega) \times H^1_{\Gamma,s}(\Omega)$ defined in (1.5), and by A the corresponding self-adjoint operator A_0 defined in (1.6). Observe that

$$a(u, v) = a_\oplus(\imath_\oplus u, \imath_\oplus v) \quad \text{for } u \text{ and } v \in H^1_{\Gamma,s}(\Omega).$$

Assume that $\Gamma = \emptyset$. If $u \in D(A)$, then, for all $v \in H^1_{\Gamma,s}(\Omega)$, we have

$$\langle Au, v \rangle_{L^2(\Omega)} = a(u, v) = a_\oplus^C(\imath_\oplus u, \imath_\oplus v).$$

On the other hand, $\langle Au, v \rangle_{L^2(\Omega)} = \langle \imath_\oplus Au, \imath_\oplus v \rangle_{L_\oplus}$. It follows that

$$a_\oplus^C(\imath_\oplus u, \imath_\oplus v) = \langle \imath_\oplus Au, \imath_\oplus v \rangle_{L_\oplus}$$

for all $v \in H^1_{\Gamma,s}(\Omega)$, so $\imath_\oplus u \in D(A_\oplus^C)$ and $A_\oplus^C \imath_\oplus u = \imath_\oplus Au$. Similarly, one can prove that, whenever $[u] \in D(A_\oplus^C)$, then $\imath_\oplus^{-1}[u] \in D(A)$ and $A\imath_\oplus^{-1}[u] = \imath_\oplus^{-1} A_\oplus^C[u]$. This means that $\imath_\oplus$ restricts to an isometry of $D(A)$ onto $D(A_\oplus^C)$ and that $A = \imath_\oplus^{-1} A_\oplus^C \imath_\oplus$.

In the same way we can prove that, if $\Gamma = \Gamma_L$, then $\imath_\oplus$ restricts to an isometry of $D(A)$ onto $D(A_\oplus^0)$ and that $A = \imath_\oplus^{-1} A_\oplus^0 \imath_\oplus$.

So the problem of characterizing $D(A)$ reduces to the problem of characterizing $D(A_\oplus^C)$ and $D(A_\oplus^0)$.

We need the following regularity result:

PROPOSITION 3.1. *Let $[u] \in H_\oplus$ and $[w] \in L_\oplus$. Assume that one of the following properties holds:*

(a) $[u] \in H_\oplus^C$ *and* $a_\oplus([u], [v]) = \langle [w], [v] \rangle_{L_\oplus}$ *for all* $[v] \in H_\oplus^C$,
(b) $[u] \in H_\oplus^0$ *and* $a_\oplus([u], [v]) = \langle [w], [v] \rangle_{L_\oplus}$ *for all* $[v] \in H_\oplus^0$.

Then $u_k \in H^2(\omega_k)$ for $k = 1, 2, 3$.

PROOF. See the Appendix.   □

For $k = 1, 2, 3$ let us define the spaces

$$Z_k := H^2(\omega_k) \quad \text{and} \quad Z_k^0 := H^2(\omega_k) \cap H_0^1(\omega_k).$$

Moreover, let us define the spaces

$$Z_\oplus := Z_1 \oplus Z_2 \oplus Z_3 \quad \text{and} \quad Z_\oplus^0 := Z_1^0 \oplus Z_2^0 \oplus Z_3^0.$$

Then we have the following characterization of $D(A_\oplus^C)$ and $D(A_\oplus^0)$:

THEOREM 3.2. *The following properties hold*

(a) $D(A_\oplus^0) = Z_\oplus^0$ *and* $A_\oplus^0[u] = (-\Delta u_1, -\Delta u_2, -\Delta u_3)$ *for* $[u] \in Z_\oplus^0$,
(b) $D(A_\oplus^C) = Z_\oplus^C$ *and* $A_\oplus^C[u] = (-\Delta u_1, -\Delta u_2, -\Delta u_3)$ *for* $[u] \in Z_\oplus^C$, *where $Z_\oplus^C$ is the subspace of $Z_\oplus$ consisting of all $[u] = (u_1, u_2, u_3)$ satisfying*

$$\begin{aligned} {}^\tau u_1(x) = {}^\tau u_2(x) &= {}^\tau u_3(x) \quad \mathcal{H}^1\text{-a.e. on } \partial\omega_2, \\ \partial_{\nu_1} u_1(x) &= 0 \qquad \mathcal{H}^1\text{-a.e. on } \partial\omega, \end{aligned}$$

and

$$(3.1) \qquad h_1 \nabla u_1 \cdot \nu_1 + h_2 \nabla u_2 \cdot \nu_2 + h_3 \nabla u_3 \cdot \nu_3 = 0 \quad \mathcal{H}^1\text{-a.e. on } \partial\omega_2,$$

where ν_k is the outward normal vector field on $\partial\omega_k$ for $k = 1, 2, 3$. We call (3.1) the (Kirchoff type) balance condition on $\partial\omega_2$.

PROOF. First we prove (a). Let $[u] \in D(A_\oplus^0)$. Then by definition there exists $[w] \in L_\oplus$ such that

$$a_\oplus([u], [v]) = \langle [w], [v] \rangle_{L_\oplus} \quad \text{for all } [v] \in H_\oplus^0.$$

Since by Proposition 3.1 $u_k \in H^2(\omega_k) \cap H_0^1(\omega_k)$ for $k = 1, 2, 3$, we obtain immediately that $[u] \in Z_\oplus^0$. Moreover, a simple integration by parts yields

$$-\sum_{k=1}^3 \int_{\omega_k} h_k v_k(x) \Delta u_k(x)\, dx = \sum_{k=1}^3 \int_{\omega_k} h_k \nabla v_k(x) \cdot \nabla u_k(x)\, dx$$

$$= \sum_{k=1}^3 \int_{\omega_k} h_k v_k(x) w_k(x)\, dx$$

for all $[v] \in H_\oplus^0$. Choose $[v] = (v_1, 0, 0)$, with $v_1 \in H_0^1(\omega_1)$ arbitrary. Then by definition $[v] \in H_\oplus^0$. With this choice, we obtain

$$-\int_{\omega_1} h_1 v_1(x) \Delta u_1(x)\, dx = \int_{\omega_1} h_1 v_1(x) w_1(x)\, dx$$

for all $v_1 \in H_0^1(\omega_1)$. This implies that $w_1 = -\Delta u_1$. In the same way, we obtain that $w_k = -\Delta u_k$ for $k = 1, 2, 3$, i.e. $A_\oplus^0[u] = (-\Delta u_1, -\Delta u_2, -\Delta u_3)$.

Assume conversely that $[u] \in Z_\oplus^0$. Then integration by parts implies that

$$\sum_{k=1}^3 \int_{\omega_k} h_k \nabla v_k(x) \cdot \nabla u_k(x)\, dx = -\sum_{k=1}^3 \int_{\omega_k} h_k v_k(x) \Delta u_k(x)\, dx$$

for all $[v] \in H_\oplus^0$. Since $(-\Delta u_1, -\Delta u_2, -\Delta u_3) \in L_\oplus$, it follows that $[u] \in D(A_\oplus^0)$ and the proof of part (a) is complete.

Part (b) is a little more involved. Let $[u] \in D(A_\oplus^C)$. Then by definition there exists $[w] \in L_\oplus$ such that

$$a_\oplus([u], [v]) = \langle [w], [v] \rangle_{L_\oplus} \quad \text{for all } [v] \in H_\oplus^C.$$

By Proposition 3.1, $u_k \in H^2(\omega_k)$ for $k = 1, 2, 3$, so we obtain immediately that $[u] \in Z_\oplus$. Moreover, since $[u] \in H_\oplus^C$, we have of course ${}^\tau u_1(x) = {}^\tau u_2(x) = {}^\tau u_3(x)$ $\mathcal{H}^1$-a.e. on $\partial \omega_2$. A simple integration by parts yields

$$-\sum_{k=1}^3 \int_{\omega_k} h_k v_k(x) \Delta u_k(x)\, dx + \sum_{k=1}^3 \int_{\partial\omega_k} h_k v_k(x) \nabla u_k(x) \cdot \nu_k(x)\, d\mathcal{H}^1 x$$

$$= \sum_{k=1}^3 \int_{\omega_k} h_k \nabla v_k(x) \cdot \nabla u_k(x)\, dx = \sum_{k=1}^3 \int_{\omega_k} h_k v_k(x) w_k(x)\, dx$$

for all $[v] \in H_\oplus^C$. Choose $[v] = (v_1, 0, 0)$, with $v_1 \in H_0^1(\omega_1)$ arbitrary. Then $[v] \in H_\oplus^C$. With this choice, we obtain

$$-\int_{\omega_1} h_1 v_1(x) \Delta u_1(x)\, dx = \int_{\omega_1} h_1 v_1(x) w_1(x)\, dx \quad \text{for all } v_1 \in H_0^1(\omega_1).$$

Since $H_0^1(\omega_1)$ is dense in $L^2(\omega_1)$, we obtain that $w_1 = -\Delta u_1$. In the same way, we obtain that $w_k = -\Delta u_k$ for $k = 1, 2, 3$, i.e. $A_\oplus^C[u] = (-\Delta u_1, -\Delta u_2, -\Delta u_3)$.

Now choose $[v] = (v_1, 0, 0)$ with ${}^\tau v_1 = 0$ $\mathcal{H}^1$-a.e. on $\partial\omega_2$. Then $[v] \in H_\oplus^C$ and we obtain

$$-\int_{\omega_1} h_1 v_1(x) \Delta u_1(x) \, dx + \int_{\partial\omega} h_1 v_1(x) \nabla u_1(x) \cdot \nu_1(x) \, d\mathcal{H}^1 x$$

$$= \int_{\omega_1} h_1 \nabla v_1(x) \cdot \nabla u_1(x) \, dx = a_\oplus^C([v], [u])$$

$$= \langle [v], A_\oplus^C[u] \rangle_{L_\oplus} = -\int_{\omega_1} h_1 v_1(x) \Delta u_1(x) \, dx.$$

It follows that

$$\int_{\partial\omega} h_1 v_1(x) \nabla u_1(x) \cdot \nu_1(x) \, d\mathcal{H}^1 x = 0.$$

Since ${}^\tau v_1$ can be chosen arbitrarily in a dense subspace of $L^2(\partial\omega)$, we obtain that $\partial_{\nu_1} u_1(x) = 0$ $\mathcal{H}^1$-a.e. on $\partial\omega$. Finally, choose $[v]$ in such a way that ${}^\tau v_1(x) = 0$ $\mathcal{H}^1$-a.e. on $\partial\omega$. Then we have

$$-\sum_{k=1}^{3} \int_{\omega_k} h_k v_k(x) \Delta u_k(x) \, dx + \sum_{k=1}^{3} \int_{\partial\omega_2} h_k v_k(x) \nabla u_k(x) \cdot \nu_k(x) \, d\mathcal{H}^1 x$$

$$= \sum_{k=1}^{3} \int_{\omega_k} h_k \nabla v_k(x) \cdot \nabla u_k(x) \, dx = a_\oplus^C([v], [u]) = \langle [v], A_\oplus^C[u] \rangle_{L_\oplus}$$

$$= -\sum_{k=1}^{3} \int_{\omega_k} h_k v_k(x) \Delta u_k(x) \, dx.$$

It follows that

$$\sum_{k=1}^{3} \int_{\partial\omega_2} h_k v_k(x) \nabla u_k(x) \cdot \nu_k(x) \, d\mathcal{H}^1 x = 0.$$

Since $[v] \in H_\oplus^C$, we have ${}^\tau v_1(x) = {}^\tau v_2(x) = {}^\tau v_3(x)$ $\mathcal{H}^1$-a.e. on $\partial\omega_2$. Finally, since ${}^\tau v_1$ can be chosen arbitrarily in a dense subspace of $L^2(\partial\omega_2)$, we obtain that $h_1 \nabla u_1 \cdot \nu_1 + h_2 \nabla u_2 \cdot \nu_2 + h_3 \nabla u_3 \cdot \nu_3 = 0$ $\mathcal{H}^1$-a.e. on $\partial\omega_2$, and hence $[u] \in Z_\oplus^C$.

Assume conversely that $[u] \in Z_\oplus^C$. Then integration by parts implies that

$$\sum_{k=1}^{3} \int_{\omega_k} h_k \nabla v_k(x) \cdot \nabla u_k(x) \, dx$$

$$= -\sum_{k=1}^{3} \int_{\omega_k} h_k v_k(x) \Delta u_k(x) \, dx + \sum_{k=1}^{3} \int_{\partial\omega_k} h_k v_k(x) \nabla u_k(x) \cdot \nu_k(x) \, d\mathcal{H}^1 x$$

for all $[v] \in H_\oplus^C$.

Since $[v] \in H_\oplus^C$, we have ${}^\tau v_1(x) = {}^\tau v_2(x) = {}^\tau v_3(x)$ $\mathcal{H}^1$-a.e. on $\partial\omega_2$. Moreover, since $[u] \in Z_\oplus^C$, we have $\partial_{\nu_1} u_1(x) = 0$ $\mathcal{H}^1$-a.e. on $\partial\omega$ and $h_1 \nabla u_1 \cdot \nu_1 + h_2 \nabla u_2 \cdot$

$\nu_2 + h_3 \nabla u_3 \cdot \nu_3 = 0$ $\mathcal{H}^1$-a.e. on $\partial \omega_2$. This implies immediately that

$$\sum_{k=1}^{3} \int_{\partial \omega_k} h_k v_k(x) \nabla u_k(x) \cdot \nu_k(x) \, d\mathcal{H}^1 x = 0.$$

Since $(-\Delta u_1, -\Delta u_2, -\Delta u_3) \in L_\oplus$, it follows that $[u] \in D(A_\oplus^0)$ and the proof is complete. $\qquad\square$

REMARK. Thanks to Theorem 3.2, one can easily prove that the semiflow generated by equation (1.7) in $H^1_{\Gamma,s}(\Omega)$ with $\Gamma = \emptyset$ (resp. $\Gamma = \Gamma_L$) and the semiflow generated by equation (1.9) (resp. (1.10)) are conjugate by mean of the isometry $\imath_\oplus$.

4. An application: computation of the eigenvalues

In this section we shall explain how the characterization of A_0 and of its domain, obtained in Section 3, can be exploited, in some specific situations, to compute the eigenvalues of A_0. We shall consider the domain Ω described in Figure 1: we choose two real numbers r and R, $0 < r < R$, and we define

$$\omega := \{x \in \mathbb{R}^2 \mid 0 \le |x|^2 < R^2\}, \quad \omega_2 = \omega_3 := \{x \in \mathbb{R}^2 \mid 0 \le |x|^2 < r^2\}.$$

First, we observe that, thanks to Theorem 3.2, in the case $\Gamma = \Gamma_L$ the abstract eigenvalue problem

$$A_0 u = \lambda u$$

is equivalent to the system

(4.1) $$\begin{cases} -\Delta u_j = \lambda u_j & \text{for } x \in \omega_j, \ j = 1, 2, 3, \\ u_j = 0, & \text{for } x \in \partial \omega_j, \ j = 1, 2, 3. \end{cases}$$

The equations in this system are completely decoupled, so in this case the sequence of the eigenvalues of A_0 is just the union of the sequences of eigenvalues of the three Dirichlet problems considered separately. These problems can be easily treated in the standard way by writing the equations in polar coordinates and then using separation of variables. This is a classical result and we don't discuss it here.

The case $\Gamma = \emptyset$ is more interesting. Thanks to Theorem 3.2, the abstract eigenvalue problem

$$A_0 u = \lambda u$$

is equivalent to the system

(4.2) $$\begin{cases} -\Delta u_j = \lambda u_j, & x \in \omega_j, \ j = 1, 2, 3, \\ u_1(x) = u_2(x) = u_3(x), & |x| = r, \\ \partial_{\nu_1} u_1 = 0, & |x| = R, \\ \sum_{j=1}^{3} h_j \nabla u_j \cdot \nu_j = 0, & |x| = r. \end{cases}$$

Also in this case the computation exploits polar coordinates and separation of variables, but we have to be a little careful because of the coupling at the "interface" $\{|x| = r\}$. Let us write for simplicity $A := A_\oplus^C$ and let us indicate by $A^{\mathbb{C}}$ the complexification of A. Then $A^{\mathbb{C}}$ is a self-adjoint operator in the complex Hilbert space $L_\oplus^{\mathbb{C}} := L_\oplus + iL_\oplus$ with domain $D(A^{\mathbb{C}}) = D(A) + iD(A)$. The action of $A^{\mathbb{C}}$ is defined in the obvious way by $A^{\mathbb{C}}([u] + i[v]) := A[u] + iA[v]$. The operators A and $A^{\mathbb{C}}$ have the same eigenvalues with the same multiplicity. Let $\Phi:]0, \infty[\times]0, 2\pi[\to \mathbb{R}^2$, $\Phi(\rho, \theta) \mapsto (\rho\cos\theta, \rho\sin\theta)$ be the system of polar coordinates on $\mathbb{R}^2 \setminus (\mathbb{R}_+ \times \{0\})$.

Set $I_1 :=]r, R[$ and $I_j :=]0, r[$ for $j = 2, 3$. We look for eigenvalue-eigenvector pairs $(\lambda, [u])$, where $\lambda \geq 0$ and $[u]$ has the form

$$[u] = (u_1, u_2, u_3)$$
(4.3)   with $(u_j \circ \Phi)(\rho, \theta) = v_j(\rho)e^{in\theta}$, $(\rho, \theta) \in I_j \times]0, 2\pi[$, $j = 1, 2, 3$.

Here $n \in \mathbb{Z}$ and $v_j: I_j \to \mathbb{R}$ for $j = 1, 2, 3$. Let us recall that the Laplacian in two-dimensional polar coordinates assumes the form

$$\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}.$$

Let us fix $n \in \mathbb{Z}$. Then an eigenvalue-eigenvector pair of the form (4.3) must satisfy

(4.4)
$$\begin{cases} -\left(v_j'' + \dfrac{1}{\rho}v_j' - \dfrac{n^2}{\rho^2}v_j\right) = \lambda v_j, & \rho \in I_j, \ j = 1, 2, 3, \\ v_2 \text{ and } v_3 \text{ regular at } 0, \\ v_1(r) = v_2(r) = v_3(r), \\ v_1'(R) = 0, \\ h_1 v_1'(r) = h_2 v_2'(r) + h_3 v_3'(r). \end{cases}$$

If $\lambda = 0$ and $n = 0$, the space of solutions of (4.4) is one-dimensional, and is generated by $(v_1, v_2, v_3) = (1, 1, 1)$. In fact a fundamental system of solutions for the equation

$$v_j'' + \frac{1}{\rho}v_j' = 0$$

is given by 1 and $\log\rho$. If $\lambda = 0$ and $n \neq 0$, then (4.4) has no non-trivial solutions. In fact, a fundamental system of solutions for the equation

$$v_j'' + \frac{1}{\rho}v_j' - \frac{n^2}{\rho^2}v_j = 0$$

is given by ρ^n and ρ^{-n}.

Assume now that $\lambda \neq 0$. Setting $\widetilde{v}_j(\xi) := v_j(\xi/\sqrt{\lambda})$, we transform the equations

$$(4.5) \qquad -\left(v_j'' + \frac{1}{\rho}v_j' - \frac{n^2}{\rho^2}v_j\right) = \lambda v_j, \quad j = 1, 2, 3$$

to

$$(4.6) \qquad \widetilde{v}_j'' + \frac{1}{\xi}\widetilde{v}_j' + \left(1 - \frac{n^2}{\xi^2}\right)\widetilde{v}_j = 0, \quad j = 1, 2, 3.$$

The latter are Bessel equations of order $|n|$ and, for $j = 1, 2, 3$, a corresponding fundamental system of solutions is given by $J_{|n|}(\xi)$ and $Y_{|n|}(\xi)$, where $J_{|n|}$ and $Y_{|n|}$ are the first and the second Bessel function of order $|n|$ (see e.g. [19]). It follows that a fundamental system of solutions for the equations (4.5) for $j = 1, 2, 3$ is given by

$$J_{|n|}(\sqrt{\lambda}\rho), \quad Y_{|n|}(\sqrt{\lambda}\rho).$$

It is well known that $Y_{|n|}$ is singular at 0. It follows that, for a given positive λ, (4.4) admits nontrivial solutions if and only if we can find real constants c_i, $i = 1, \ldots, 4$, with $(c_1, c_2, c_3, c_4) \neq (0, 0, 0, 0)$, such that

$$(4.7) \qquad \begin{cases} c_1 J_{|n|}'(\sqrt{\lambda}R) + c_4 Y_{|n|}'(\sqrt{\lambda}R) = 0, \\ c_1 J_{|n|}(\sqrt{\lambda}r) + c_4 Y_{|n|}(\sqrt{\lambda}r) = c_2 J_{|n|}(\sqrt{\lambda}r), \\ c_2 J_{|n|}(\sqrt{\lambda}r) = c_3 J_{|n|}(\sqrt{\lambda}r), \\ c_1 h_1 J_{|n|}'(\sqrt{\lambda}r) + c_4 h_1 Y_{|n|}'(\sqrt{\lambda}r) = c_2 h_2 J_{|n|}'(\sqrt{\lambda}r) + c_3 h_3 J_{|n|}'(\sqrt{\lambda}r). \end{cases}$$

This is possible if and only if $\det M(n, \lambda, r, R) = 0$, where

$$M(n, \lambda, r, R) = \begin{pmatrix} J_{|n|}'(\sqrt{\lambda}R) & 0 & 0 & Y_{|n|}'(\sqrt{\lambda}R) \\ J_{|n|}(\sqrt{\lambda}r) & -J_{|n|}(\sqrt{\lambda}r) & 0 & Y_{|n|}(\sqrt{\lambda}r) \\ 0 & J_{|n|}(\sqrt{\lambda}r) & -J_{|n|}(\sqrt{\lambda}r) & 0 \\ h_1 J_{|n|}'(\sqrt{\lambda}r) & -h_2 J_{|n|}'(\sqrt{\lambda}r) & -h_3 J_{|n|}'(\sqrt{\lambda}r) & h_1 Y_{|n|}'(\sqrt{\lambda}r) \end{pmatrix}.$$

Observe that $\det M(n, \lambda, r, R)$ is an analytic function of $\lambda > 0$. It follows that, for every $n \in \mathbb{Z}$, the zeroes of $\det M(n, \lambda, r, R)$ in $\mathbb{R}_+$ form a sequence λ_{nm}, $m = 1, 2, \ldots$ of eigenvalues of $A^{\mathbb{C}}$ and hence of A. Thus we obtain that the set

$$\{\lambda_{nm} \mid n \in \mathbb{Z}, \ m = 1, 2, \ldots\} \cup \{0\}$$

is contained in the sequence of the eigenvalues of $A^{\mathbb{C}}$ and hence of A. The corresponding eigenfunctions can be computed by solving the system (4.7) with $\lambda = \lambda_{nm}$. If (c_1, c_2, c_3, c_4) is a nontrivial real solution of (4.7), then

$$((c_1 J_{|n|}(\sqrt{\lambda_{nm}}\rho) + c_4 Y_{|n|}(\sqrt{\lambda_{nm}}\rho))e^{in\theta}, c_2 J_{|n|}(\sqrt{\lambda_{nm}}\rho)e^{in\theta}, c_3 J_{|n|}(\sqrt{\lambda_{nm}}\rho)e^{in\theta})$$

is an eigenfunction for the eigenvalue λ_{nm}, expressed in polar coordinates. Thus, for $n \in \mathbb{Z}$ and $m = 1, 2, \dots$ fixed, we obtain a finite set of orthonormal eigenfunctions

$$\{[u]_{nm}^{\ell} \mid \ell = 1, \dots, p(n, m)\}$$

for the eigenvalue λ_{nm}. Notice that $p(n, m) \leq 4$. However, the multiplicity of λ_{nm} can be larger than $p(n, m)$, since we can have $\lambda_{\overline{n}\,\overline{m}} = \lambda_{nm}$ for some $(\overline{n}, \overline{m}) \neq (n, m)$.

Finally, we claim that all eigenvalues and eigenfunctions of $A^{\mathbb{C}}$ can be obtained in this way. To this end, for $n \in \mathbb{Z}$ let us first define the space

$$(L_{\oplus}^{\mathbb{C}})_n := \{[u] \in L_{\oplus}^{\mathbb{C}} \mid (u_j \circ \Phi)(\rho, \theta) = v_j(\rho)e^{\mathrm{i}n\theta},$$
$$v_j \colon I_j \to \mathbb{C}, \ (\rho, \theta) \in I_j \times]0, 2\pi[, \ j = 1, 2, 3\}.$$

Observe that a triple of functions (u_1, u_2, u_3), $u_j \colon \omega_j \to \mathbb{C}$, $j = 1, 2, 3$, satisfying $(u_j \circ \Phi)(\rho, \theta) = v_j(\rho)e^{\mathrm{i}n\theta}$ for some $v_j \colon I_j \to \mathbb{C}$, $(\rho, \theta) \in I_j \times]0, 2\pi[$, $j = 1, 2, 3$, belongs to $(L_{\oplus}^{\mathbb{C}})_n$ if and only if

$$\int_{I_j} \rho |v_j(\rho)|^2 \, d\rho < \infty \quad \text{for } j = 1, 2, 3.$$

In fact, $\rho = J\Phi(\rho, \theta)$ for $(\rho, \theta) \in]0, \infty[\times]0, 2\pi[$. It is also easy to check that

$$(L_{\oplus}^{\mathbb{C}})_n \perp (L_{\oplus}^{\mathbb{C}})_{\overline{n}} \quad \text{for } n \neq \overline{n}.$$

Moreover,

$$\overline{\bigoplus_{n \in \mathbb{Z}} (L_{\oplus}^{\mathbb{C}})_n} = L_{\oplus}^{\mathbb{C}}$$

since $\{e^{\mathrm{i}n\theta} \mid n \in \mathbb{Z}\}$ is a complete orthonormal system in $L^2(]0, 2\pi[, \mathbb{C})$.

Write

$$[u]_{00} := \left(\sum_{j=1}^{3} h_j |\omega_j| \right)^{-1/2} (1, 1, 1).$$

If we show that, for a fixed $n \in \mathbb{Z}$, $n \neq 0$, the set

$$\{[u]_{nm}^{\ell} \mid \ell = 1, \dots, p(n, m), \ m = 1, 2, \dots\}$$

is a complete orthonormal system in $(L_{\oplus}^{\mathbb{C}})_n$ and that the set

$$\{[u]_{0m}^{\ell} \mid \ell = 1, \dots, p(0, m), \ m = 1, 2, \dots\} \cup \{[u]_{00}\}$$

is a complete orthonormal system in $(L_{\oplus}^{\mathbb{C}})_0$, we are done.

Let us define the Hilbert space

$$\mathbb{L}_{\oplus} := \{[v] = (v_1, v_2, v_3) \mid \rho^{1/2} v_j(\rho) \in L^2(I_j, \mathbb{R}), \ j = 1, 2, 3\}$$

equipped with the scalar product

$$\{[v], [\nu]\}_{\oplus} := \sum_{j=1}^{3} \int_{I_j} h_j \rho v_j(\rho) \nu_j(\rho) \, d\rho, \quad [v], [\nu] \in \mathbb{L}_{\oplus}.$$

Set $\mathbb{L}_{\oplus}^{\mathbb{C}} = \mathbb{L}_{\oplus} + \mathrm{i} \, \mathbb{L}_{\oplus}$, i.e.

$$\mathbb{L}_{\oplus}^{\mathbb{C}} := \{[v] = (v_1, v_2, v_3) \mid \rho^{1/2} v_j(\rho) \in L^2(I_j, \mathbb{C}), \ j = 1, 2, 3\}.$$

Moreover, let us define the isometry $\jmath \colon \mathbb{L}_{\oplus}^{\mathbb{C}} \to (L_{\oplus}^{\mathbb{C}})_n$ by

$$(v_1, v_2, v_3) \mapsto (2\pi)^{-1/2}(w_1 \circ \Phi^{-1}, w_3 \circ \Phi^{-1}, w_3 \circ \Phi^{-1}),$$

where $w_j(\rho, \phi) := v_j(\rho)e^{\mathrm{i}n\theta}$, $(\rho, \theta) \in I_j \times {]0, 2\pi[}$, $j = 1, 2, 3$. It is enough to prove that the sets

$$\begin{aligned}
\mathcal{B}_n &:= \{\jmath^{-1}[u]_{nm}^{\ell} \mid \ell = 1, \ldots, p(n, m), \ m = 1, 2, \ldots\}, \quad n \in \mathbb{Z} \setminus \{0\}, \\
\mathcal{B}_0 &:= \{\jmath^{-1}[u]_{0m}^{\ell} \mid \ell = 1, \ldots, p(0, m), \ m = 1, 2, 3, \ldots\} \cup \{\jmath^{-1}[u]_{00}\}
\end{aligned}$$

are complete orthonormal systems in $\mathbb{L}_{\oplus}^{\mathbb{C}}$. Actually, since

$$\begin{aligned}
\jmath^{-1}[u]_{nm}^{\ell} &= (v_{nm,1}^{\ell}, v_{nm,2}^{\ell}, v_{nm,3}^{\ell}) \in \mathbb{L}_{\oplus} \\
&\quad \text{for } \ell = 1, \ldots, p(n, m), \ m = 1, 2, 3, \ldots \text{ and for all } n \in \mathbb{Z},
\end{aligned}$$

it is enough to prove that $\mathcal{B}_n$ and $\mathcal{B}_0$ are complete orthonormal systems in $\mathbb{L}_{\oplus}$.

Set $\lambda_{nm}^{\ell} := \lambda_{nm}$ for $\ell = 1, \ldots, p(n, m)$, $m = 1, 2, \ldots$, $n \in \mathbb{Z}$. For $n \neq 0$, the set

$$\mathcal{E}_n := \{(\lambda_{nm}^{\ell}, \jmath^{-1}[u]_{nm}^{\ell}) \mid \ell = 1, \ldots, p(n, m), \ m = 1, 2, \ldots\}$$

is by construction the set of eigenvalue-eigenvector pairs of the system

$$(4.8) \qquad \begin{cases}
-\left(v_j'' + \dfrac{1}{\rho}v_j' - \dfrac{n^2}{\rho^2}v_j\right) = \lambda v_j, \quad \rho \in I_j, \ j = 1, 2, 3, \\[2mm]
v_2 \text{ and } v_3 \text{ regular at } 0, \\[1mm]
v_1(r) = v_2(r) = v_3(r), \\[1mm]
v_1'(R) = 0, \\[1mm]
h_1 v_1'(r) = h_2 v_2'(r) + h_3 v_3'(r).
\end{cases}$$

For $n = 0$, the set

$$\mathcal{E}_0 := \{(\lambda_{0m}^{\ell}, \jmath^{-1}[u]_{0m}^{\ell}) \mid \ell = 1, \ldots, p(0, m), \ m = 1, 2, 3, \ldots\} \cup \{(0, \jmath^{-1}[u]_{00})\}$$

is by construction the set of eigenvalue-eigenvector pairs of the system

$$(4.9) \quad \begin{cases} -\left(v_j'' + \dfrac{1}{\rho}v_j'\right) = \lambda v_j, & \rho \in I_j, \ j = 1,2,3, \\ v_2 \text{ and } v_3 \text{ regular at } 0, \\ v_1(r) = v_2(r) = v_3(r), \\ v_1'(R) = 0, \\ h_1 v_1'(r) = h_2 v_2'(r) + h_3 v_3'(r). \end{cases}$$

Let us define the spaces

$$\mathbb{H}_\oplus^0 := \{[v] \in \mathbb{L}_\oplus \mid v_j \in H^1_{\mathrm{loc}}(I_j),$$
$$\rho^{1/2} v_j'(\rho) \in L^2(I_j), \ j = 1,2,3, \ v_1(r) = v_2(r) = v_3(r)\}$$

and, for $n \in \mathbb{Z} \setminus \{0\}$,

$$\mathbb{H}_\oplus^n := \{[v] \in \mathbb{H}_\oplus^0 \mid \rho^{-1/2} v_j(\rho) \in L^2(I_j), \ j = 1,2,3\},$$

equipped with the scalar products

$$\{\{[v],[\nu]\}\}_\oplus^0 := \sum_{j=1}^3 \int_{I_j} h_j \rho v_j'(\rho) \nu_j'(\rho) \, d\rho + \sum_{j=1}^3 \int_{I_j} h_j \rho v_j(\rho) \nu_j(\rho) \, d\rho,$$

where $[v],[\nu] \in \mathbb{H}_\oplus^0$, and

$$\{\{[v],[\nu]\}\}_\oplus^n := \sum_{j=1}^3 \int_{I_j} h_j \rho v_j'(\rho) \nu_j'(\rho) \, d\rho$$
$$+ \sum_{j=1}^3 \int_{I_j} h_j \frac{n^2}{\rho} v_j(\rho) \nu_j(\rho) \, d\rho + \sum_{j=1}^3 \int_{I_j} h_j \rho v_j(\rho) \nu_j(\rho) \, d\rho,$$

where $[v],[\nu] \in \mathbb{H}_\oplus^n$, respectively. Then one can show that $\mathbb{H}_\oplus^0$ and $\mathbb{H}_\oplus^n$, $n \in \mathbb{Z} \setminus \{0\}$, are densely and compactly imbedded in $\mathbb{L}_\oplus$.

Let us define the bilinear forms

$$a\{[v],[\nu]\}_\oplus^0 := \sum_{j=1}^3 \int_{I_j} h_j \rho v_j'(\rho) \nu_j'(\rho) \, d\rho,$$

for $[v],[\nu] \in \mathbb{H}_\oplus^0$ and

$$a\{[v],[\nu]\}_\oplus^n := \sum_{j=1}^3 \int_{I_j} h_j \rho v_j'(\rho) \nu_j'(\rho) \, d\rho + \sum_{j=1}^3 \int_{I_j} h_j \frac{n^2}{\rho} v_j(\rho) \nu_j(\rho) \, d\rho$$

for $[v],[\nu] \in \mathbb{H}_\oplus^n$ on $\mathbb{H}_\oplus^0$ and $\mathbb{H}_\oplus^n$, respectively. Then we have that the set $\mathcal{E}_0$ is the complete set of "proper value – proper vector" pairs of

$$(4.10) \quad \begin{cases} [v] \in \mathbb{H}_\oplus^0, \\ a\{[v],[\nu]\}_\oplus^0 = \lambda\{[v],[\nu]\}_\oplus & \text{for all } [\nu] \in \mathbb{H}_\oplus^0. \end{cases}$$

Analogously, for all $n \in \mathbb{Z} \setminus \{0\}$, the set $\mathcal{E}_n$ is the complete set of "proper value – proper vector" pairs of

(4.11)
$$\begin{cases} [v] \in \mathbb{H}_\oplus^n, \\ a\{[v], [\nu]\}_\oplus^n = \lambda\{[v], [\nu]\}_\oplus \quad \text{for all } [\nu] \in \mathbb{H}_\oplus^n. \end{cases}$$

Actually, (4.10) (resp. (4.11)) can be considered as the "weak formulation" of (4.9) (resp. (4.8)).

By the abstract theory of proper values for couples of bilinear forms (see e.g. [17] or [20]), we finally obtain that $\mathcal{B}_0$ and $\mathcal{B}_n$, $n \in \mathbb{Z} \setminus \{0\}$, are complete orthonormal systems in $\mathbb{L}_\oplus$.

5. Appendix

In this appendix we give the

PROOF OF PROPOSITION 3.1. Assume first that (a) holds and remind that $u_k \in H_0^1(\omega_k)$ for $k = 1, 2, 3$. Choose $[v] = (v_1, 0, 0)$, with $v_1 \in H_0^1(\omega_1)$ arbitrary. Then by definition $[v] \in H_\oplus^0$. With this choice, we obtain

$$\int_{\omega_1} h_1 \nabla u_1(x) \cdot \nabla v_1(x)\, dx = \int_{\omega_1} h_1 w_1(x) v_1(x)\, dx \quad \text{for all } v_1 \in H_0^1(\omega_1).$$

Since $\partial\omega_1$ is of class C^2, the classical regularity results for the Dirichlet problem apply to the present situation and we get without any further effort that $u_1 \in H^2(\omega_1)$. In the same way, we obtain that $u_k \in H^2(\omega_k)$ for $k = 1, 2, 3$.

If (a) holds, the situation is much more complicated: we cannot apply directly the classical regularity results for elliptic equations, because of the coupling at the "interface" $\partial\omega_2$. We shall use a partition of unity on $\overline{\omega}$ in order to isolate the regions where no coupling occurs: within these regions we can again apply the classical results. On the other hand, the partition of unity allows us to "localize" the analysis on the interface. The main difficulty consists in the fact that we have to handle with the three functions u_1, u_2 and u_3 simultaneously. Fortunately, the compatibility condition (2.1), in local coordinates, is invariant under "horizontal" translations. Then we shall exploit the well known method of translations due to L. Nirenberg and obtain at once H^2 regularity of u_k, $k = 1, 2, 3$.

We start by carefully choosing an open covering of $\overline{\omega}$. Since $\partial\omega_2$ is of class C^2, we can cover it by a finite number of open sets $U_1, \ldots, U_m$, in such a way that, for $i = 1, \ldots, m$, there exists a C^2 diffeomorphism $\Phi_i :]-1, 1[\times]-1, 1[\to U_i$ with the property that

    (1) $\partial\omega_2 \cap U_i = \Phi_i(]-1, 1[\times \{0\})$,
    (2) $\omega_2 \cap U_i = \Phi_i(]-1, 1[\times]-1, 0[)$,
    (3) $\omega_1 \cap U_i = \Phi_i(]-1, 1[\times]0, 1[)$.

Notice that $U_i \cap \partial\omega = \emptyset$. Moreover, we take $U_0 \Subset \omega_2$ in such a way that $U_0, \ldots, U_m$ form an open covering of $\overline{\omega}_2$. Notice that $U_0 \cap \overline{\omega}_1 = \emptyset$. Finally, we take $U_{m+1} \Subset \mathbb{R}^2$ in such a way that $U_{m+1} \cap \overline{\omega}_2 = \emptyset$ and $U_1, \ldots, U_{m+1}$ form an open covering of $\overline{\omega}_1$. Then $U_0, \ldots, U_{m+1}$ is an open covering of $\overline{\omega}$.

For $i = 0, \ldots, m + 1$, let $\theta_i \in C_0^\infty(\mathbb{R}^2)$, with $\operatorname{supp}\theta_i \subset U_i$, be a partition of unity on $\overline{\omega}$, i.e. $\sum_{i=0}^{m+1} \theta_i \equiv 1$ on $\overline{\omega}$. Let us observe that $\sum_{i=0}^{m} \theta_i \equiv 1$ on $\overline{\omega}_2 = \overline{\omega}_3$ and $\sum_{i=1}^{m+1} \theta_i \equiv 1$ on $\overline{\omega}_1$. Then we have

$$u_1 = \sum_{i=1}^{m+1} \theta_i u_1, \quad u_2 = \sum_{i=0}^{m} \theta_i u_2, \quad \text{and} \quad u_3 = \sum_{i=0}^{m} \theta_i u_3.$$

So it is sufficient to show that

$$u_{i,1} := \theta_i u_1 \in H^2(\omega_1) \quad \text{for } i = 1, \ldots, m + 1,$$
$$u_{i,2} := \theta_i u_2 \in H^2(\omega_2) \quad \text{for } i = 0, \ldots, m,$$
$$u_{i,3} := \theta_i u_3 \in H^2(\omega_3) \quad \text{for } i = 0, \ldots, m.$$

Let us observe that $\operatorname{supp} u_{m+1,1} \subset U_{m+1} \cap \overline{\omega}_1$, $\operatorname{supp} u_{0,2}$ and $\operatorname{supp} u_{0,3} \subset U_0$, and $\operatorname{supp} u_{i,j} \subset U_i \cap \overline{\omega}_j$ for $i = 1, \ldots, m$ and $j = 1, 2, 3$.

We prove first that $u_{0,2} \in H^2(\omega_2)$ and $u_{0,3} \in H^2(\omega_3)$, the simplest case. Let $v_2 \in H_0^1(\omega_2)$. We have

$$\int_{\omega_2} \nabla u_{0,2}(x) \cdot \nabla v_2(x)\, dx$$
$$= \int_{\omega_2} u_2(x) \nabla \theta_0(x) \cdot \nabla v_2(x)\, dx + \int_{\omega_2} \theta_0(x) \nabla u_2(x) \cdot \nabla v_2(x)\, dx$$
$$= \int_{\omega_2} u_2(x) \nabla \theta_0(x) \cdot \nabla v_2(x)\, dx + \int_{\omega_2} \nabla u_2(x) \cdot \nabla(\theta_0 v_2)(x)\, dx$$
$$\quad - \int_{\omega_2} v_2(x) \nabla u_2(x) \cdot \nabla \theta_0(x)\, dx.$$

Since $(0, \theta_0 v_2, 0) \in H_\oplus^C$, we have

$$\int_{\omega_2} \nabla u_2(x) \cdot \nabla(\theta_0 v_2)(x)\, dx = \int_{\omega_2} w_2(x) \theta_0(x) v_2(x)\, dx.$$

Moreover, since $u_2 \in H^1(\omega_2)$ and $v_2 \in H_0^1(\omega_2)$,

$$\int_{\omega_2} u_2(x) \nabla \theta_0(x) \cdot \nabla v_2(x)\, dx = -\int_{\omega_2} \operatorname{div}(u_2 \nabla \theta_0)(x) v_2(x)\, dx.$$

Let us write $\widetilde{w}_2 := -\operatorname{div}(u_2 \nabla \theta_0) + w_2 \theta_0 - \nabla u_2 \cdot \nabla \theta_0$. Then $\widetilde{w}_2 \in L^2(\omega_2)$ and

$$\int_{\omega_2} \nabla u_{0,2}(x) \cdot \nabla v_2(x)\, dx = \int_{\omega_2} \widetilde{w}_2(x) v_2(x)\, dx.$$

Since $v_2 \in H_0^1(\omega_2)$ is arbitrary, we obtain that $u_{0,2} \in H_0^1(\omega_2)$ is a weak solution of $-\Delta u = \widetilde{w}_2$ on ω_2, $u = 0$ on $\partial\omega_2$. Then by the standard regularity results for

the Dirichlet problem we obtain that $u_{0,2} \in H^2(\omega_2)$. In the same way we can prove that $u_{0,3} \in H^2(\omega_2)$.

Next, we consider $u_{m+1,1}$. As we have already mentioned,

$$\operatorname{supp} u_{m+1,1} \subset U_{m+1} \cap \overline{\omega}_1 = (U_{m+1} \cap \omega_1) \cup \partial\omega.$$

This implies that $^\tau u_{m+1,1} = 0$ on $\partial\omega_2$. Let $v_1 \in H^1(\omega_1)$, $^\tau v_1 = 0$ on $\partial\omega_2$. Then we have

$$\int_{\omega_1} \nabla u_{m+1,1}(x) \cdot \nabla v_1(x)\, dx$$

$$= \int_{\omega_1} u_1(x)\nabla\theta_{m+1}(x) \cdot \nabla v_1(x)\, dx + \int_{\omega_1} \theta_{m+1}(x)\nabla u_1(x) \cdot \nabla v_1(x)\, dx$$

$$= \int_{\omega_1} u_1(x)\nabla\theta_{m+1}(x) \cdot \nabla v_1(x)\, dx + \int_{\omega_1} \nabla u_1(x) \cdot \nabla(\theta_{m+1}v_1)(x)\, dx$$

$$- \int_{\omega_1} v_1(x)\nabla u_1(x) \cdot \nabla\theta_{m+1}(x)\, dx.$$

Since $(\theta_{m+1}v_1, 0, 0) \in H_\oplus^C$, we have

$$\int_{\omega_1} \nabla u_1(x) \cdot \nabla(\theta_{m+1}v_1)(x)\, dx = \int_{\omega_1} w_1(x)\theta_{m+1}(x)v_1(x)\, dx.$$

Let us write $\widetilde{w}_1 := w_1\theta_{m+1} - \nabla u_1 \cdot \nabla\theta_{m+1}$ and $\widetilde{W}_1 := u_1\nabla\theta_{m+1}$. Then $\widetilde{w}_1 \in L^2(\omega_1)$ and $\widetilde{W}_1 \in H^1(\omega_1, \mathbb{R}^2)$ and we have

$$\int_{\omega_1} \nabla u_{m+1,1}(x) \cdot \nabla v_1(x)\, dx = \int_{\omega_1} \widetilde{w}_1(x)v_1(x)\, dx + \int_{\omega_1} \widetilde{W}_1(x) \cdot \nabla v_1(x)\, dx$$

for all $v_1 \in H^1(\omega_1)$ with $^\tau v_1 = 0$ on $\partial\omega_2$. Then we can apply the classical regularity results for elliptic equations with mixed boundary conditions (see e.g. [18]). Observe that $\partial\omega_1 = \partial\omega \cup \partial\omega_2$ and that the Dirichlet condition is imposed on the whole $\partial\omega_2$, whereas no a-priori condition is imposed on $\partial\omega$. Since $\partial\omega_2$ and $\partial\omega$ are smooth and both closed and open in $\partial\omega_1$, all the hypotheses of Theorem 2.24 in [18] are satisfied. So we obtain that $u_{m+1,1} \in H^2(\omega_1)$.

Finally, we shall prove that $u_{i,j} \in H^2(U_i \cap \omega_j)$ for $j = 1, 2, 3$ and $i = 1, \ldots, m$. Let us fix $i = 1, \ldots, m$, and let us take $(v_1, v_2, v_3) \in H_\oplus^C$ with $\operatorname{supp} v_j \subset U_i \cap \overline{\omega}_j$ for $j = 1, 2, 3$. Then we have

$$\sum_{j=1}^{3} \int_{U_i \cap \omega_j} h_j \nabla u_{i,j}(x) \cdot \nabla v_j(x)\, dx$$

$$= \sum_{j=1}^{3} \int_{U_i \cap \omega_j} h_j u_j(x)\nabla\theta_i(x) \cdot \nabla v_j(x)\, dx + \sum_{j=1}^{3} \int_{U_i \cap \omega_j} h_j \theta_i(x)\nabla u_j(x) \cdot \nabla v_j(x)\, dx$$

$$= \sum_{j=1}^{3} \int_{U_i \cap \omega_j} h_j u_j(x) \nabla \theta_i(x) \cdot \nabla v_j(x) \, dx + \sum_{j=1}^{3} \int_{U_i \cap \omega_j} h_j \nabla u_j(x) \cdot \nabla(\theta_i v_j)(x) \, dx$$

$$- \sum_{j=1}^{3} \int_{U_i \cap \omega_j} h_j v_j(x) \nabla u_j(x) \cdot \nabla \theta_i(x) \, dx.$$

Now observe that $(\theta_i v_1, \theta_i v_2, \theta_i v_3) \in H_\oplus^C$, so

$$\sum_{j=1}^{3} \int_{U_i \cap \omega_j} h_j \nabla u_j(x) \cdot \nabla(\theta_i v_j)(x) \, dx = \sum_{j=1}^{3} \int_{\omega_j} h_j \nabla u_j(x) \cdot \nabla(\theta_i v_j)(x) \, dx$$

$$= \sum_{j=1}^{3} \int_{\omega_j} h_j w_j(x) \theta_i(x) v_j(x) \, dx = \sum_{j=1}^{3} \int_{U_i \cap \omega_j} h_j w_j(x) \theta_i(x) v_j(x) \, dx.$$

Let us write $\widetilde{w}_j := w_j \theta_i - \nabla u_j \cdot \nabla \theta_i$ and $\widetilde{W}_j := u_j \nabla \theta_i$ for $j = 1, 2, 3$. Then $\widetilde{w}_j \in L^2(\omega_j)$ and $\widetilde{W}_j \in H^1(\omega_j, \mathbb{R}^2)$ for $j = 1, 2, 3$, and we have

$$(5.1) \quad \sum_{j=1}^{3} \int_{U_i \cap \omega_j} h_j \nabla u_{i,j}(x) \cdot \nabla v_j(x) \, dx$$

$$= \sum_{j=1}^{3} \int_{U_i \cap \omega_j} h_j \widetilde{w}_j(x) v_j(x) \, dx + \sum_{j=1}^{3} \int_{U_i \cap \omega_j} h_j \widetilde{W}_j(x) \cdot \nabla v_j(x) \, dx$$

for all $[v] \in H_\oplus^C$ with $\operatorname{supp} v_j \subset U_i \cap \overline{\omega}_j$ for $j = 1, 2, 3$.

Set $Q_i := \,]{-}1, 1[\, \times \,]{-}1, 1[$, $Q_{i,j} := \Phi_i^{-1}(U_i \cap \omega_j)$ for $j = 1, 2, 3$, i.e. $Q_{i,1} = \,]{-}1, 1[\, \times \,]0, 1[$, $Q_{i,2} = Q_{i,3} = \,]{-}1, 1[\, \times \,]{-}1, 0[$, and $\overline{u}_{i,j}(\xi) := u_{i,j}(\Phi(\xi))$, $\overline{v}_j(\xi) := v_j(\Phi_i(\xi))$ for $\xi \in Q_{i,j}$, $j = 1, 2, 3$. Then $\overline{u}_{i,j}$ and $\overline{v}_j \in H^1(Q_{i,j})$. Moreover, $\operatorname{supp} \overline{u}_{i,j}$ and $\operatorname{supp} \overline{v}_j$ are contained in $Q_{i,j} \cup (\,]{-}1, 1[\, \times \{0\})$. Besides, ${}^\tau \overline{u}_{i,1} = {}^\tau \overline{u}_{i,2} = {}^\tau \overline{u}_{i,3}$ and ${}^\tau \overline{v}_1 = {}^\tau \overline{v}_2 = {}^\tau \overline{v}_3$ $\mathcal{H}^1$-almost everywhere on $\,]{-}1, 1[\, \times \{0\}$. Then, changing coordinates in (5.1), we have

$$\sum_{j=1}^{3} \int_{Q_{i,j}} h_j J\Phi_i(\xi) D\Phi_i^{-1}(\Phi_i(\xi)) D\Phi_i^{-1}(\Phi_i(\xi))^T \nabla \overline{u}_{i,j}(\xi) \cdot \nabla \overline{v}_j(\xi) \, d\xi$$

$$= \sum_{j=1}^{3} \int_{Q_{i,j}} h_j J\Phi_i(\xi) \overline{w}_j(\xi) \overline{v}_j(\xi) \, d\xi$$

$$+ \sum_{j=1}^{3} \int_{Q_{i,j}} h_j J\Phi_i(\xi) D\Phi_i^{-1}(\Phi_i(\xi)) \overline{W}_j(\xi) \cdot \nabla \overline{v}_j(\xi) \, d\xi,$$

where $J\Phi_i(\xi)$ is the Jacobian determinant of $D\Phi_i(\xi)$, $\overline{w}_j(\xi) := \widetilde{w}_j(\Phi_i(\xi)) \in L^2(Q_{i,j})$ and $\overline{W}_j(\xi) := \widetilde{W}_j(\Phi_i(\xi)) \in H^1(Q_{i,j}, \mathbb{R}^2)$ for $j = 1, 2, 3$. Write

$$J\Phi_i D\Phi_i^{-1}(\Phi_i) D\Phi_i^{-1}(\Phi_i)^T =: (g_i^{\mu\nu})_{\mu\nu} \in C^1(Q_i, \mathcal{M}(2 \times 2)),$$

$$J\Phi_i \overline{w}_j =: \alpha_j \in L^2(Q_{i,j}) \quad \text{and} \quad J\Phi_i D\Phi_i^{-1} \overline{W}_j =: \beta_j \in H^1(Q_{i,j}, \mathbb{R}^2).$$

Observe also that the matrix $(g_i^{\mu\nu})_{\mu\nu}$ is symmetric and uniformly strongly elliptic on Q_i, i.e. there exists a positive constant K such that

$$\sum_{\mu,\nu=1}^{2} g_i^{\mu\nu}(\xi)h_\mu h_\nu \geq K|h|^2 \quad \text{for all } \xi \in \overline{Q}_i \text{ and all } h \in \mathbb{R}^2.$$

Then we have

$$(5.2) \quad \sum_{j=1}^{3} \int_{Q_{i,j}} h_j \sum_{\mu,\nu=1}^{2} g_i^{\mu\nu}(\xi)\partial_\mu \overline{u}_{i,j}(\xi)\partial_\nu \overline{v}_j(\xi)\, d\xi$$
$$= \sum_{j=1}^{3} \int_{Q_{i,j}} h_j \alpha_j(\xi)\overline{v}_j(\xi)\, d\xi + \sum_{j=1}^{3} \int_{Q_{i,j}} h_j \beta_j(\xi) \cdot \nabla \overline{v}_j(\xi)\, d\xi$$

for all $(\overline{v}_1, \overline{v}_2, \overline{v}_3) \in H_\oplus^C(Q_i)$, where $H_\oplus^C(Q_i)$ is the set of all triples $(\overline{v}_1, \overline{v}_2, \overline{v}_3) \in H^1(Q_{i,1}) \times H^1(Q_{i,2}) \times H^1(Q_{i,3})$ with $\operatorname{supp} \overline{v}_j \subset Q_{i,j} \cup (]{-1},1[\times \{0\})$ and ${}^\tau \overline{v}_1 = {}^\tau \overline{v}_2 = {}^\tau \overline{v}_3$ $\mathcal{H}^1$-almost everywhere on $]{-1},1[\times \{0\}$.

Now we are in a position to use the method of translations of Nirenberg. First, let us recall that for $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$ one defines

$$\tau_h u(z) := u(z+h) \quad \text{and} \quad \delta_h u(z) := \frac{\tau_h u(z) - u(z)}{h}, \quad \text{for } z \in \mathbb{R}^n.$$

We shall use "horizontal" translations: let $h := (\chi, 0) \in \mathbb{R}^2$ with

$$|h| < (1/2)\operatorname{dist}(\operatorname{supp} \overline{u}_{i,j}, \{-1, 1\} \times \mathbb{R}) \quad \text{for } j = 1, 2, 3.$$

Then it is very easy to see that $(\tau_h \overline{u}_{i,1}, \tau_h \overline{u}_{i,2}, \tau_h \overline{u}_{i,3})$, $(\delta_h \overline{u}_{i,1}, \delta_h \overline{u}_{i,2}, \delta_h \overline{u}_{i,3})$ and $(\delta_{-h}\delta_h \overline{u}_{i,1}, \delta_{-h}\delta_h \overline{u}_{i,2}, \delta_{-h}\delta_h \overline{u}_{i,3}) \in H_\oplus^C(Q_i)$. So we can use

$$(\overline{v}_1, \overline{v}_2, \overline{v}_3) := (\delta_{-h}\delta_h \overline{u}_{i,1}, \delta_{-h}\delta_h \overline{u}_{i,2}, \delta_{-h}\delta_h \overline{u}_{i,3})$$

as a test function in (5.2). A simple change of variable yields

$$\sum_{j=1}^{3} \int_{Q_{i,j}} h_j \sum_{\mu,\nu=1}^{2} \delta_h(g_i^{\mu\nu}\partial_\mu \overline{u}_{i,j})(\xi)\partial_\nu(\delta_h \overline{u}_{i,j})(\xi)\, d\xi$$
$$= -\sum_{j=1}^{3} \int_{Q_{i,j}} h_j \alpha_j(\xi)\delta_{-h}\delta_h \overline{u}_{i,j}(\xi)\, d\xi + \sum_{j=1}^{3} \int_{Q_{i,j}} h_j(\delta_h \beta_j)(\xi) \cdot \nabla(\delta_h \overline{u}_{i,j})(\xi)\, d\xi.$$

Since

$$\delta_h(g_i^{\mu\nu}\partial_\mu \overline{u}_{i,j}) = \tau_h(g_i^{\mu\nu})\partial_\mu(\delta_h \overline{u}_{i,j}) + \delta_h(g_i^{\mu\nu})\partial_\mu \overline{u}_{i,j},$$

we obtain

$$\sum_{j=1}^{3} \int_{Q_{i,j}} h_j \sum_{\mu,\nu=1}^{2} (\tau_h g_i^{\mu\nu})(\xi) \partial_\mu(\delta_h \overline{u}_{i,j})(\xi) \partial_\nu(\delta_h \overline{u}_{i,j})(\xi) \, d\xi$$

$$= - \sum_{j=1}^{3} \int_{Q_{i,j}} h_j \sum_{\mu,\nu=1}^{2} (\delta_h g_i^{\mu\nu})(\xi) \partial_\mu \overline{u}_{i,j}(\xi) \partial_\nu(\delta_h \overline{u}_{i,j})(\xi) \, d\xi$$

$$- \sum_{j=1}^{3} \int_{Q_{i,j}} h_j \alpha_j(\xi) \delta_{-h} \delta_h \overline{u}_{i,j}(\xi) \, d\xi + \sum_{j=1}^{3} \int_{Q_{i,j}} h_j (\delta_h \beta_j)(\xi) \cdot \nabla(\delta_h \overline{u}_{i,j})(\xi) \, d\xi.$$

Now let us recall that

$$|\delta_{-h} \delta_h \overline{u}_{i,j}|_{L^2(Q_{i,j})} \le |\nabla(\delta_h \overline{u}_{ij})|_{L^2(Q_{i,j},\mathbb{R}^2)},$$
$$|\delta_h \beta_j|_{L^2(Q_{i,j},\mathbb{R}^2)} \le |D\beta|_{L^2(Q_{i,j},\mathcal{M}(2\times 2))}.$$

So we get

$$\overline{K} \sum_{j=1}^{3} \int_{Q_{i,j}} |\nabla(\delta_h \overline{u}_{i,j})|^2 \, d\xi$$

$$\le \sum_{j=1}^{3} |(g_i^{\mu\nu})|_{C^1(\overline{Q}_i)} |\nabla \overline{u}_{i,j}|_{L^2(Q_{i,j},\mathbb{R}^2)} |\nabla(\delta_h \overline{u}_{i,j})|_{L^2(Q_{i,j},\mathbb{R}^2)}$$

$$+ \sum_{j=1}^{3} |\alpha_j|_{L^2(Q_{i,j})} |\nabla(\delta_h \overline{u}_{i,j})|_{L^2(Q_{i,j},\mathbb{R}^2)}$$

$$+ \sum_{j=1}^{3} |D\beta|_{L^2(Q_{i,j},\mathcal{M}(2\times 2))} |\nabla(\delta_h \overline{u}_{i,j})|_{L^2(Q_{i,j},\mathbb{R}^2)},$$

for some positive constant $\overline{K}$. This in turn implies that there exists a constant $C > 0$ such that

$$\sum_{j=1}^{3} |\nabla(\delta_h \overline{u}_{i,j})|_{L^2(Q_{i,j},\mathbb{R}^2)}^2 \le C \sum_{j=1}^{3} |\nabla(\delta_h \overline{u}_{i,j})|_{L^2(Q_{i,j},\mathbb{R}^2)}$$

and hence

$$\left(\sum_{j=1}^{3} |\nabla(\delta_h \overline{u}_{i,j})|_{L^2(Q_{i,j},\mathbb{R}^2)} \right)^2 \le 3C \sum_{j=1}^{3} |\nabla(\delta_h \overline{u}_{i,j})|_{L^2(Q_{i,j},\mathbb{R}^2)}.$$

So, for all sufficiently small $h = (\chi, 0)$, we have obtained that

$$(5.3) \qquad\qquad |\delta_h(\nabla \overline{u}_{i,j})|_{L^2(Q_{i,j},\mathbb{R}^2)} \le 3C \quad \text{for } j = 1, 2, 3.$$

It is well known that estimates (5.3) hold if and only if

$$\partial_1 \partial_\nu \overline{u}_{i,j} \in L^2(Q_{i,j}) \quad \text{for } \nu = 1, 2 \text{ and } j = 1, 2, 3$$

So, in order to complete the proof, we only need to show that $\partial_2^2 \overline{u}_{i,j} \in L^2(Q_{i,j})$ for $j = 1, 2, 3$. This can be easily done by mean of straightforward manipulations of the distributional identities

$$-\sum_{\mu,\nu=1}^{2} \partial_\nu (g_i^{\mu\nu} \partial_\mu \overline{u}_{i,j}) = \alpha_j - \sum_{\nu=1}^{2} \partial_\nu \beta_j^\nu, \quad j = 1, 2, 3,$$

like in the classical proof of regularity for elliptic equations. $\square$

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