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DIFFERENTIAL INCLUSIONS WITH CONSTRAINTS IN BANACH SPACES

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Abstract. The paper provides topological characterization for solution sets of differential inclusions with (not necessarily smooth) functional constraints in Banach spaces. The corresponding compactness and tangency conditions for the right hand-side are expressed in terms of the measure of noncompactness and the Clarke generalized gradient, respectively. The consequences of the obtained result generalize the known theorems about the structure of viable solution set for differential inclusions.

1. Introduction

In this paper we shall be concerned with the topological characterization of the solution set to the following differential inclusion subject to functional constraints

$$(\mathrm{P})_f \qquad \begin{cases} u'(t) \in F(t, u(t)) & \text{a.e. on } I, \\ f(u(t)) \leq 0, \\ u(0) = x_0, \end{cases}$$

where $F: I \times U \multimap E$ ($I := [0,1]$) is an upper semicontinuous closed convex valued map satisfying some compactness condition and U is an open subset of a Banach space E. We shall prove an Aronszajn type result saying that the set of

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all solutions is a R_δ set in $C(I,E)$. Throughout the paper, by a solution of $(\mathrm{P})_f$ we mean an *absolutely continuous* function $u\colon I\to E$ such that for every $t\in I$

$$u(t)=x_0+\int_0^t v(s)\,ds$$

where $v\colon I\to E$ is a Bochner measurable selection of $F(\,\cdot\,,u(\,\cdot\,))$.

Clearly, if $u\colon I\to E$ is a solution of $(\mathrm{P})_f$ then it is a viable trajectory in the set $K:=\{x\in U\mid f(x)\le 0\}$, that is a solution to the problem

$$(\mathrm{P})_K \qquad \begin{cases} u'(t)\in F(t,u(t)) & \text{a.e. on } I,\\ u(t)\in K, \\ u(0)=x_0\in K.\end{cases}$$

If one expects $(\mathrm{P})_K$ to have any solutions (in K) the directions of F should be adjusted to the "shape" of K. More precisely, the Haddad theorem says (see [14]) that a viable trajectory starts from each point of $x\in K$ provided the following tangency condition holds

$$F(t,x)\cap T_K^{\mathcal{B}}(x)\neq\emptyset \quad \text{for all } (t,x)\in I\times K \tag{1}$$

where $T_K^{\mathcal{B}}(x)$ denotes the Bouligand tangent cone (to K) at point $x\in K$ (in the autonomous case even the converse is true). Nevertheless, the tangency (1) is still not sufficient for the problem $(\mathrm{P})_K$ (or $(\mathrm{P})_f$) to have the mentioned structure of the solution set.

Example 1.1. Let $f\colon\mathbb{R}^2\to\mathbb{R}$ be given by

$$f(x,y):=|x^2+y^2-2x||x^2+y^2+2x|.$$

Define $F\colon\mathbb{R}^2\to\mathbb{R}^2$ by

$$F(x,y):=\begin{cases} (y,1-x) & \text{if } x^2+y^2-2x\le 0,\\ (-y,1+x) & \text{if } x^2+y^2+2x\le 0,\\ \left(\dfrac{2xy}{x^2+y^2},\dfrac{y^2-x^2}{x^2+y^2}\right) & \text{otherwise.}\end{cases}$$

The problem $(\mathrm{P})_f$ with $u_0=(0,0)$ in this case has exactly two solutions $u_1(t)=(1-\cos t,\sin t)$ and $u_2(t)=(-1+\cos t,\sin t)$.

Example 1.2[1]. Define

$$\begin{aligned} K &:=\{(x,y,z)\in\mathbb{R}^3\mid x^2+y^2+z^2\le 2,\ \sqrt{x^2+y^2}\ge z\},\\ R &:=\{(x,y,1)\mid x^2+y^2=1\},\\ Q &:=\{(x,y,1)\mid x^2+y^2\le 1\}.\end{aligned}$$

[1] The example was suggested by Wojcieh Kryszewski.

Let $F: K \multimap \mathbb{R}^3$ be given by

$$F(x,y,z) := \begin{cases} Q & \text{if } (x,y,z) \in K \setminus R, \\ \operatorname{conv}[Q \cup \{(-y,x,0)\}] & \text{if } (x,y,z) \in R. \end{cases}$$

One may check that the tangency condition (1) is satisfied. The set of solutions for $(\mathrm{P})_K$ starting from $(0,0,0)$ is hemeomorphic to the circle S^1, hence it is not R_δ.

Thus, the first example shows that even the connectedness of the solution set depends on K. And the second one indicates that even for regular sets (in the sense of Definition 4.4, cf. [9]) the condition (1) is not sufficient for the solution set to be acyclic. For that reason the set K has to be regular and the tangency assumptions have to be expressed in terms of Clarke's tangent cones.

The problem of the structure of solution sets for $(\mathrm{P})_K$ has been addressed by many authors: by Bothe ([7]), Hu and Papageorgiou ([15], [16]) for closed convex subsets of Banach spaces, Plaskacz ([17], [18]) for proximate retracts in finite dimensional spaces, Górniewicz, Nistri and Obukhovski ([13]) for proximate retracts in Hilbert spaces and by Bader and Kryszewski ([4]) for a wide class of regular subsets of a finite dimensional space.

Section 2 contains basic definitions and preliminary results. In Section 3 we shall study the solution set structure for seemingly less general (than $(\mathrm{P})_K$) problem $(\mathrm{P})_f$ and in Section 4 the obtained result is applied to $(\mathrm{P})_K$, which gives a generalization of theorems by the mentioned authors in the case the right hand-side is upper semicontinuous.

2. Preliminaries

By E we denote a (possibly infinite dimensional) *Banach space*; $\|\cdot\|$ stands for its norm. Given $x \in E$ and $\varepsilon > 0$, $B(x,\varepsilon) := \{y \in E \mid \|x-y\| < \varepsilon\}$, $D(x,\varepsilon) := \{y \in E \mid \|x-y\| \leq \varepsilon\}$ and, in particular, $B := B(0,1)$ and $D := D(0,1)$. The *closure*, the *interior*, the *boundary* and the *convex envelope* of $A \subset E$ are denoted by $\operatorname{cl} A$, $\operatorname{int} A$, $\operatorname{bd} A$ and $\operatorname{conv} A$, respectively, and $B(A,\varepsilon) := \{g \in E \mid \text{exists } x \in A \text{ such that } \|x-y\| < \varepsilon\}$.

By E^* we denote the topological dual of E and put $D^* := D(0,1)$ in E^*. $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the duality pairing: $\langle p,x\rangle := p(x)$, for $p \in E^*$ and $x \in E$.

Let $\varphi: X \multimap E$ be a set-valued map defined on a metric space X. We say that φ is ε-δ *upper semicontinuous* if, for any $x \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $\varphi(x') \subset B(\varphi(x),\varepsilon)$ if $d(x,x') < \delta$. Obviously, if φ is upper semicontinous then it is ε-δ upper semicontinuous. The converse statement holds, if φ has compact values.

Let $f\colon U \to \mathbb{R}$ be a locally Lipschitz function defined on an open subset of a Banach space E. For each $a \in \mathbb{R}$, the sublevel set is defined as

$$f^a := \{x \in U \mid f(x) \le a\}.$$

The Clarke *directional derivative* of f at a point $x \in U$ in the direction $u \in E$ is given by

$$f^\circ(x;u) := \limsup_{y\to x,\, h\to 0^+} \frac{f(y+hu)-f(y)}{h}.$$

The generalized gradient is a set

$$\partial f(x) := \{p \in E^* \mid \langle p,u\rangle \le f^\circ(x;u) \text{ for all } u \in E\}.$$

One may show that

$$f^\circ(x;u) = \sup_{p\in\partial f(x)} \langle p,u\rangle \qquad \text{for any } u \in E, \tag{2}$$

$$\inf_{p\in\partial f(x)} \langle p,u\rangle = -\inf_{u\in D} f^\circ(x;u). \tag{3}$$

By the polar cone to the gradient $\partial f(x)$ ($x \in U$) we mean

$$\partial f(x)^\circ := \{u \in E \mid f^\circ(x;u) \le 0\}$$

and by the normal cone

$$\partial f(x)^{\circ\circ} := \{p \in E^* \mid \sup_{u\ \in\partial f(x)^\circ} \langle p,u\rangle \le 0\}, \tag{4}$$

which is the weak-$*$ closure of $\bigcup_{\lambda>0} \lambda\partial f(x)$. If $0 \notin \partial f(x)$, then $\partial f(x)^{\circ\circ} = \bigcup_{\lambda>0} \lambda\partial f(x)$. If $K \subset E$ is closed, $x \in K$, then the *Clarke tangent cone* $T_K(x)$ is given by

$$T_K(x) := \left\{u \in E \;\middle|\; \lim_{y\to x,\, y\in K,\, h\to 0^+} \frac{d_K(y+hu)}{h} = 0\right\}$$

or, equivalently, $T_K(x) = \partial d_K(x)^\circ$ where $d_K(x) = d(x,K) := \inf_{y\in K} \|y-x\|$ (see e.g. [2], [3] or [8] for details).

Lemma 2.1. *If $W\colon X \multimap E$ defined on a metric space X is a set-valued map with convex values of nonempty interior such that for any $u \in E$ the set $\{y \in X \mid u \in \operatorname{int} W(y)\}$ is open, then W is lower semicontinuous.*

Proof. Take any $x \in X$, $u \in W(x)$ and $\varepsilon > 0$. Since $W(x)$ is convex, one can choose $v \in B$ such that $u + \varepsilon v \in \operatorname{int} W(x)$. By assumption, the set $\{y \in X \mid u + \varepsilon v \in \operatorname{int} W(y)\}$ is an open neighbourhood of x. So we obtain that for any $\varepsilon > 0$ there is $\delta > 0$ such that, for each $y \in B(x,\delta)$, $B(u,\varepsilon) \cap W(y) \ne \emptyset$. $\square$

Further we shall need a version of Lemma 5.1 in [5][2].

[2] The proof is exactly the same as that in [5].

Lemma 2.2. *Let Y be a subset of a metric space X and $F\colon Y \multimap E$ an ε-δ upper semicontinuous map with closed convex values and $W\colon X \multimap E$ a lower semicontinuous map with convex values such that*

$$F(x) \cap W(x) \neq \emptyset \quad \text{for } x \in Y.$$

Then for any $\varepsilon > 0$ there exists a locally Lipschitz $h\colon U \to E$ defined on an open neigborhood U of Y such that

$$\begin{aligned} h(x) &\in F(B(x,\varepsilon) \cap Y) + \varepsilon B && \text{for } x \in Y, \\ h(x) &\in W(x) + \varepsilon B && \text{for } x \in U. \end{aligned}$$

Lemma 2.3. *Let X be a metric space and $W\colon X \multimap E$ a convex valued map such that*

$$\operatorname{int} W(x) \neq \emptyset \quad \text{for all } x \in X, \tag{5}$$

$$\{y \in X \mid u \in \operatorname{int} W(y)\} \text{ is open for any } u \in E. \tag{6}$$

Given $\varepsilon > 0$ and continuous $h\colon X \to E$ with

$$h(x) \in W(x) + (\varepsilon/2)B \quad \text{for } x \in X, \tag{7}$$

there exists a locally Lipschitz $g\colon X \to E$ such that

(i) $\|g(x) - h(x)\| < \varepsilon$ *for* $x \in X$,
(ii) $g(x) \in \operatorname{int} W(x)$ *for* $x \in X$.

Proof. Let $x \in X$. In view of (7) there is $u_x \in B$ with $h(x) + (\varepsilon/2)u_x \in W(x)$. By assumption (5) and the convexity of $W(x)$, there exists $\overline{u}_x \in B$ such that

$$v_x := h(x) + (\varepsilon/2)u_x + (\varepsilon/2)\overline{u}_x \in \operatorname{int} W(x)$$

and, by (6) and the continuity of h, the set

$$V_x := \{y \in U \mid v_x \in \operatorname{int} W(y),\ \|v_x - h(y)\| < \varepsilon\}$$

is open. Let $\{\lambda_s\}_{s \in S}$ be a locally Lipschitz partition of unity inscribed into the open covering $\{V_x\}_{x \in X}$ of X (see [6]). Put

$$g(x) := \sum_{s \in S} \lambda_s(x) v_{x_s}$$

where x_s are chosen so that the support of λ_s is contained in V_{x_s} (for $s \in S$). One can easily verify that g has the required properties. □

The following result based on the existence and uniqueness theorem for differential equations will be crucial for showing the existence of viable trajectories.[3]

[3] Usually, the existence of viable trajectories is shown by proximal aiming construction.

Proposition 2.4. *Suppose that a map $g: I \times U \to E$ is locally Lipschitz in the second variable and continuous in time and that $f: U \to \mathbb{R}$ is locally Lipschitz with $f^0 = \{x \in U \mid f(x) \le 0\}$ closed (in E). If*

$$f^\circ(x; g(t,x)) < 0 \quad \text{for all } t \in I \text{ and } x \in f^{-1}(0),$$

then the Cauchy problem

$$\begin{cases} u'(t) = g(t, u(t)) & \text{on } I, \\ u(0) = x_0 \in f^0 \end{cases} \tag{8}$$

admits a unique solution contained in f^0.

Proof. The local existence theorem implies that there is a nonempty maximal interval $I_0 \subset I$ on which the unique solution $u: I_0 \to U$ of (8) exists. We shall prove that $u(I_0) \subset f^0$. It follows then that $I_0 = I$. Indeed, if $I_0 \ne I$, then $\overline{u}$ could be extended beyond I_0 as f^0 is closed, a contradiction.

Suppose to the contrary that there exists $t \in I_0$ with $f(u(t)) > 0$ and put

$$\overline{t} := \inf\{t \in I_0 \mid f(u(t)) > 0\}.$$

By the continuity $f(u(\overline{t})) = 0$. Clearly

$$\limsup_{h \to 0^+} \frac{f(u(\overline{t}+h)) - f(u(\overline{t}))}{h} \le f^\circ(u(\overline{t}); u'(\overline{t})) = f^\circ(u(\overline{t}); g(\overline{t}, u(\overline{t}))) < 0.$$

Hence, there is $\delta > 0$ such that for $h \in (0,\delta)$, $f(u(\overline{t}+h)) - f(u(\overline{t})) < 0$, and consequently $f(u(\overline{t}+h)) < 0$, which contradicts the definition of $\overline{t}$. □

3. Differential inclusions with functional constraints

Suppose f is a locally Lipschitz function $f: U \to \mathbb{R}$ such that

(H1) $0 \notin \partial f(x)$ for $x \in U \setminus f^0$,
(H2) $f^a := \{x \in U \mid f(x) \le a\}$ is closed for some $a > 0$.

Consider the problem

$$\begin{cases} u'(t) \in F(t, u(t)) & \text{a.e. on } I, \\ f(u(t)) \le 0, \\ u(0) = u_0, \end{cases} \tag{P$_f$}$$

where $F: I \times U \multimap E$ is an ε-δ upper semicontinuous map with closed convex values. By a *tangency module* at $(t,x) \in I \times (U \setminus f^0)$ we mean

$$\omega(t,x) = \omega_{F,f}(t,x) := \inf_{u \in F(t,x)} d(u, \partial f(x)^\circ).$$

We shall assume further that

(H3) $\sup_{I \times (f^\varepsilon \setminus f^0)} \omega \to 0$ as $\varepsilon \to 0^+$, (H4) there exist $k_0, k_1 \in L^1(I;\mathbb{R})$ and $p\colon U \to \mathbb{R}$ with $\sup_{f^\varepsilon} p \to 0$ as $\varepsilon \to 0^+$ such that for any bounded $\Gamma \subset U$

$$\lim_{\tau\to 0^+} \alpha(F(I(t,\tau)\times\Gamma) \le k_0(t)\alpha(\Gamma) + k_1(t)\sup_\Gamma p$$

where $I(t,\tau) := [t-\tau, t+\tau]\cap I$ and α is the Kuratowski measure of noncompactness,[4]

(H5) there exists $c \in L^1(I;\mathbb{R})$ such that for a.e. $t \in I$ and each $x \in U$

$$\sup_{u\in F(t,x)} \|u\| \le c(t)(1+\|x\|).$$

REMARK 3.1. (a) If f satisfies (H1) and (H2) then for $x \in U \setminus f^0$

$$d(u,\partial f(x)^\circ) = \max\left\{0, \sup_{p\in\partial f(x)} \frac{\langle p,u\rangle}{\|p\|}\right\}. \tag{9}$$

Indeed, by use of the Sion lemma ([19]) one has

$$\begin{aligned} d(u,\partial f(x)^\circ) &= \inf_{v\in\partial f(x)^\circ} \|u-v\| = \inf_{v\in\partial f(x)^\circ} \sup_{p\in D^*} \langle p, u-v\rangle \\ &= -\sup_{v\in\partial f(x)^\circ} \inf_{p\in D^*} \langle p, u-v\rangle = -\inf_{p\in D^*} \sup_{v\in\partial f(x)^\circ} \langle p,u-v\rangle \\ &= \sup_{p\in D^*} (\langle p,u\rangle - \sup_{v\in\partial f(x)^\circ} \langle p,v\rangle) = \sup_{p\in D^*\cap\partial f(x)^{\circ\circ}} \langle p,u\rangle. \end{aligned}$$

The latter equality follows from the fact that $\partial f(x)^\circ$ is a cone, which implies that either $\sup_{v\in\partial f(x)^\circ}\langle p,v\rangle = \infty$ or $\sup_{v\in\partial f(x)^\circ}\langle p,v\rangle = 0$ and then $p \in \partial f(x)^{\circ\circ}$ (see (4)). By (H1), if $p \in D^*\cap\partial f(x)^{\circ\circ}$, then $p = \lambda q/\|q\|$ for some $q \in \partial f(x)$ and $\lambda \in [0,1]$, which gives (9).

Hence, in view of (2),

$$\begin{aligned} \omega_{F,f}(t,x) &= \inf_{u\in F(t,x)} \max\left\{0, \sup_{p\in\partial f(x)} \frac{\langle p,u\rangle}{\|p\|}\right\} \\ &\le \frac{1}{d(0,\partial f(x))} \inf_{u\in F(t,x)} \max\{0, f^\circ(x;u)\}. \end{aligned}$$

If additionally there is a constant $m_f > 0$ such that

$$d(0,\partial f(x)) \ge m_f \quad \text{on } U\setminus f^0,$$

then, for $x \in U\setminus f^0$ and $t \in I$,

$$\omega_{F,f}(t,x) \le \frac{1}{m_f} \inf_{u\in F(t,x)} \max\{0, f^\circ(x;u)\},$$

which simplifies considerably the verification of (H3). For instance, if $F(t,x)\cap \partial f(x)^\circ \ne \emptyset$, then $\omega_{F,f}(t,x) = 0$.

[4] See e.g. [11].

(b) The condition (H4) is slightly weaker than

$$\lim_{\tau\to 0^+} \alpha\left(F(I(t,\tau)\times\Gamma)\right) \le k(t)\alpha(\Gamma), \tag{10}$$

but it appears useful in the next section where a mapping satisfying (H4) but not (10) is involved.

(c) Notice that under the above assumptions for $x \in U\setminus f^0$ the values $F(t,x)$ are not necessarily compact, which is of importance in the next section. However, if $x \in f^0$, then the compactness of $F(t,x)$ is implied directly by (H4).

Under these assumptions we are able to prove the main result.

THEOREM 3.2. *If a locally Lipschitz $f\colon U \to \mathbb{R}$ and an ε-δ upper semicontinuous closed convex valued map $F\colon I\times U \multimap E$ satisfy* (H1)–(H5), *then the solution set for* $(\mathrm{P})_f$ *is* R_δ.

PROOF. Take any $\varepsilon > 0$ and define $F_\varepsilon\colon I\times f^\varepsilon \multimap E$ by

$$F_\varepsilon(t,x) := \begin{cases} F(t,x) & \text{if } f(x) < 0,\\ \mathrm{cl}\,[F(t,x)+\omega_\varepsilon D] & \text{if } f(x)\ge 0,\end{cases}$$

where $\omega_\varepsilon = \varepsilon + \sup_{I\times(f^\varepsilon\setminus f^0)}\omega$, and $W_\varepsilon\colon U\multimap E$ by

$$W_\varepsilon(x) := \begin{cases} E & \text{if } f(x) < \varepsilon/2,\\ \partial f(x)^\circ & \text{if } f(x)\ge \varepsilon/2.\end{cases}$$

The mappings F_ε and W_ε have the following properties:

- (i) F_ε is an ε-δ upper semicontinuous map with closed convex values,
- (ii) $\{y\in U \mid u \in \mathrm{int}\, W_\varepsilon(y)\}$ is open for any $u\in E$,
- (iii) W_ε is lower semicontinuous,
- (iv) for any $t\in I$ and $x \in f^\varepsilon$, $F_\varepsilon(t,x)\cap W_\varepsilon(x)\ne\emptyset$.

(i) is a direct consequence of the ε-δ upper semicontinuity of F and (H3). To prove (ii) take any $u\in E$ and notice that, if $0\notin\partial f(x)$ then $\mathrm{int}\,[\partial f(x)^\circ] = \{v\in E \mid f^\circ(x;v) < 0\}$. Hence the set $\{y\in U \mid u\in \mathrm{int}\, W_\varepsilon(y)\} = \{y \in U \mid f(y) < \varepsilon/2\}\cup\{y\in U \mid f^\circ(y;u) < 0,\ f(y)\ge\varepsilon/2\}$. The latter set in the sum is not open, but if x belongs to it, then $f^\circ(y;u)<0$ for y in a neighbourhood of x, so the sum must be open. (iii) follows from (ii) and Lemma 2.1. To see (iv) take any $x\in f^\varepsilon$ with $f(x)\ge\varepsilon/2$. There is $\overline{u}\in F(t,x)$ such that $d(\overline{u},\partial f(x)^\circ) < \varepsilon + \inf_{u\in F(t,x)} d(u,\partial f(x)^\circ)\le\omega_\varepsilon$. So there is $\tilde{u}\in\partial f(x)^\circ$ such that $\|\tilde{u}-\overline{u}\| < \omega_\varepsilon$. Clearly, $\tilde{u} = \overline{u} + (\tilde{u}-\overline{u}) \in F(t,x)+\omega_\varepsilon D\subset F_\varepsilon(t,x)$.

Thus, we can apply Lemma 2.2 to F_ε and W_ε and obtain a continuous $h_\varepsilon\colon I\times U_\varepsilon\to E$, where U_ε is open and $f^\varepsilon\subset U_\varepsilon\subset U$, such that

$$\begin{aligned} h_\varepsilon(t,x) &\in F_\varepsilon\left(I(t,\varepsilon)\times(B(x,\varepsilon)\cap f^\varepsilon)\right)+\varepsilon B && \text{for } (t,x)\in I\times f^\varepsilon,\\ h_\varepsilon(t,x) &\in W_\varepsilon(x) + (\varepsilon/2)B && \text{for } (t,x)\in I\times U_\varepsilon. \end{aligned}$$

Applying Lemma 2.3 to $h_\varepsilon\colon I \times U_\varepsilon \to E$ (putting $X := I \times U_\varepsilon$) one gets a locally Lipschitz $g_\varepsilon\colon I \times U_\varepsilon \to E$ such that

$$g_\varepsilon(t,x) \in F_\varepsilon\,(I(t,\varepsilon) \times (B(x,\varepsilon) \cap f^\varepsilon)) + 2\varepsilon B,$$
$$g_\varepsilon(t,x) \in \operatorname{int} W_\varepsilon(x).$$

Now consider the initial value problem

$$(\mathrm{P})_g \qquad \begin{cases} u'(t) = g_\varepsilon(t,u(t)), \\ u(0) = x_0. \end{cases}$$

In view of Proposition 2.4, $(\mathrm{P})_g$ has the unique solution $\overline{u}$ (in f^ε) being also a solution to the following problem

$$(\mathrm{P})_\varepsilon \qquad \begin{cases} u'(t) \in \widehat{F}_\varepsilon(t,u(t)), \\ f(u(t)) \le \varepsilon, \\ u(0) = x_0, \end{cases}$$

where

$$\widehat{F}_\varepsilon(t,x) := \operatorname{cl}\operatorname{conv}[F(I(t,\varepsilon) \times (D(x,\varepsilon) \cap f^\varepsilon)) + (2\varepsilon + \omega_\varepsilon)D]. \tag{11}$$

Denote the set of solutions of $(\mathrm{P})_\varepsilon$ by S_ε. For any $\varepsilon > 0$, $S \subset S_\varepsilon$. We shall show that each squence (u_n) with $u_n \in S_{\varepsilon_n}$, where $\varepsilon_n := 1/n$, has a subsequence convergent to some $u \in S$. Then S is nonempty and compact, and $S = \bigcap_{n\ge1} \operatorname{cl} S_{\varepsilon_n}$. By the compactness of S

$$\alpha(S_{\varepsilon_n}) \le \sup_{v \in S_{\varepsilon_n}} d(v,S),$$

which gives $\alpha(S_{\varepsilon_n}) \to 0$ as $n \to \infty$. Moreover, if we show that $\operatorname{cl} S_\varepsilon$ are contractible then the proof will be completed in view of the following characterization of R_δ sets:

Let S be a nonempty subset of a metric space. The following statements are equivalent

(i) *S is compact and of R_δ type,*
(ii) *$S = \bigcap_{n\ge1} S_n$ where $S_{n+1} \subset S_n$ (for $n \ge 1$), S_n are closed contractible and $\alpha(S_n) \to 0$.*

Let $\varphi\colon I \to \mathbb{R}$ be given by $\varphi(t) := \alpha(\{u_n(t)\}_{n=1}^\infty)$ where $u_n \in S_{\varepsilon_n}$, $n \ge 1$. In view of the growth condition (H5) and the Gronwall inequality, the sequence (u_n) is bounded and φ is well-defined. Clearly, $\varphi(0) = 0$ and φ is Lipschitz as (u_n) is uniformly Lipschitz (by (H5)), so it is a.e. differentiable. It is shown in [11, pp. 115–116] that, for a.e. $t \in I$,

$$\varphi'(t) \le 2\alpha(\{u_n'(t)\}_{n=1}^\infty).$$

Further, for any $k \geq l \geq 1$, one gets

$$\begin{aligned}\varphi'(t) &\leq 2\alpha\bigg(\bigcup_{n=k}^{\infty} \widehat{F}_{\varepsilon_n}(t, u_n(t))\bigg)\\ &\leq 2\alpha\bigg(\bigcup_{n=k}^{\infty} F(I(t,\varepsilon_n) \times D(u_n(t),\varepsilon_n))\bigg) + 4\varepsilon_l + 2\omega_{\varepsilon_l}\\ &\leq 2\alpha(F(I(t,\varepsilon_k) \times D(\{u_n(t)\}_{n=l}^{\infty},\varepsilon_l))) + 4\varepsilon_l + 2\omega_{\varepsilon_l}.\end{aligned}$$

In view of (H4), for any $l \geq 1$, there is $k_l \geq l$ such that

$$\begin{aligned}&\alpha(F(I(t,\varepsilon_{k_l}) \times D(\{u_n(t)\}_{n=l}^{\infty},\varepsilon_l)))\\ &\qquad \leq k_0(t)\alpha(D(\{u_n(t)\}_{n=l}^{\infty},\varepsilon_l)) + k_1(t)\sup p(D(\{u_n(t)\}_{n=1}^{\infty},\varepsilon_l)) + \varepsilon_l.\end{aligned}$$

Hence

$$\varphi'(t) \leq 2k_0(t)(\varphi(t) + \varepsilon_l) + 2k_1(t)\sup_{f^{\varepsilon_l}} p + 6\varepsilon_l + 2\omega_{\varepsilon_l}.$$

Passing with $l \to \infty$, in view of (H3) and (H4), one gets $\varphi'(t) \leq 2k_0(t)\varphi(t)$ and, by the Gronwall inequality, φ is constant and $\varphi(0) = 0$ on I. Hence, by the generalized Ascoli–Arzela theorem (u_n) contains a subsequence convergent to some $u \in C(I,E)$.

Now one has to verify that $u \in S$. To this end we show that (u_n') is weakly compact. In fact we adapt the idea from the proof of Theorem 9.1 in [11]. Put $M := \operatorname{cl}\operatorname{conv} F(I \times u(I))$. Clearly, M is compact. Let $r\colon E \to M$ be a retraction such that for each $x \in M$

$$\|r(x) - x\| \leq 2d_M(x),$$

(see [6]). We use the following weak compactness criterion (see [12]):

If M is a weakly compact convex subset of a Banach space E, then the set $\{u \in L^1(I,E) \mid u(t) \in M$ for a.e. $t \in I\}$ is weakly compact in $L^1(I,E)$.[5]

Applying it we infer that $(r \circ u_n')$ converges weakly in $L^1(I,E)$ to some $v \in L^1(I,E)$ (over a subsequence). By (11), for a.e. $t \in I$,

$$u_n'(t) \in \operatorname{cl}\operatorname{conv}[F(I(t,\varepsilon_n) \times D(u(t),\varepsilon_n + \|u_n - u\|)) + (2\varepsilon_n + \omega_{\varepsilon_n})D].$$

Take any $\eta > 0$. Let $N_0 \geq 1$ be such that $2\varepsilon_n + \omega_{\varepsilon_n} < \eta/2$ for all $n \geq N_0$. Fix $t \in I$ and use the ε-δ upper semicontinuity of F to get $n_t \geq n_0$ and $\delta_t > 0$ so that, for $s \in I(t,\delta_t)$ and $n \geq n_t$,

$$F(I(s,\varepsilon_n) \times D(u(s),\varepsilon_n + \|u_n - u\|)) \subset F(t,u(t)) + (\eta/2)B.$$

Consequently for $n \geq \max\{n_t, N_0\}$ and a.e. $s \in I(t,\delta_t)$

$$u_n'(s) \in \operatorname{cl}[F(t,u(t)) + \eta B].$$

[5] Recall that the measurability and integrability in the Bochner sense is considered.

Hence, by the compactness of I, there is $N \geq N_0$, such that for $n \geq N$ and a.e. $t \in I$

$$u_n'(t) \in M + \eta B.$$

And this along with $\|r(u_n'(t)) - u_n'(t)\| \leq 2d_M(u_n'(t))$ (a.e. on I) gives $u_n' \rightharpoonup v$ (weakly in $L^1(I,E)$). Finally, by use of the so-called convergence theorem ([1, Theorem 1, p. 60]) we infer $v(t) \in F(t,u(t))$ a.e. on I and since $v = u'$ (a.e. on I), it is clear that $u \in S$.

It remains to prove the contractibility of $\operatorname{cl} S_\varepsilon$ for any $\varepsilon > 0$. Let $y \in f^\varepsilon$, $s \in I$ and $\overline{u}(\,\cdot\,; s, y)\colon [s,1] \to E$ be the unique solution of

$$\begin{cases} u' = g_\varepsilon(t,u), \\ u(s) = y, \end{cases}$$

(existing in view of Proposition 2.4). Define $h\colon I \times C(I, f^\varepsilon) \to C(I, f^\epsilon)$ by the formula

$$[h(s,u)](t) := \begin{cases} u(t) & \text{if } t \in [0,s), \\ \overline{u}(t; s, u(s)) & \text{if } t \in [s,1]. \end{cases}$$

Notice that if $u \in S_\varepsilon$ then $h(s,u) \in S_\varepsilon$ too. So $h(s, \operatorname{cl} S_\varepsilon) \subseteq \operatorname{cl} S_\varepsilon$ provided h is continuous. To prove the continuity choose $(s_n) \subset I$ and $(u_n) \subset \operatorname{cl} S_\varepsilon$ with $s_n \to s$ and $u_n \to u$. Then for large n, either

$$\|[h(s,u)](t) - [h(s_n,u_n)](t)\| \leq \|u(t) - u_n(t)\|$$

or

$$\|[h(s,u)](t) - [h(s_n,u_n)](t)\| \leq \|\overline{u}(t; s, u(s)) - \overline{u}(t; s_n, u_n(s_n))\|.$$

It follows from the local lipschitzianity of g_ε that there are $L > 0$ and $\theta > 0$ such that, for $x, x' \in B(u(I), \theta)$ and all $t \in I$, $\|g_\varepsilon(t,x) - g_\varepsilon(t,x')\| \leq L\|x - x'\|$. If $s_n < s$, then for large n

$$\begin{aligned} \|\overline{u}(t; s, u(s)) - \overline{u}(t; s_n, u_n(s_n))\| &\leq \|u(s) - u_n(s_n)\| + C(s - s_n) \\ &\quad + L \int_s^t \|\overline{u}(\tau; s, u(s)) - \overline{u}(\tau; s_n, u_n(s_n))\|\, d\tau, \end{aligned}$$

for some constant $C > 0$. Now the Gronwall inequality implies $h(s_n,u_n) \to h(s,u)$ in $C(I,E)$. The similar argument goes for $s \leq s_n$. Hence, h is continuous and $\operatorname{cl} S_\varepsilon$ is contractible to $\overline{u}(\,\cdot\,; 0, x_0)$. In view of the earlier remarks the proof is completed. □

4. The structure of viable solution sets

In this section we shall deal with the viability problem

$$(\mathrm{P})_K \qquad \begin{cases} u'(t) \in F(t,u(t)) & \text{a.e. on } I, \\ u(t) \in K, \\ u(0) = x_0, \end{cases}$$

where $F: I \times K \multimap E$ and $K \subset E$ is a closed set.

Inclusions on proximate retracts. Recall that a closed $K \subset E$ is said to be a *proximate retract* (see [17]) provided there exists a continuous function $r: U \to K$ defined on an open neighbourhood $U := B(K, \theta)$ of K satisfying

$$d_K(x) = \|x - r(x)\| \quad \text{for } x \in U. \tag{12}$$

Suppose $K \subset E$ is a proximate retract and $F: I \times K \multimap E$ an upper semicontinuous map with compact convex values being a set contraction with linear growth, that is

$$\lim_{\tau \to 0^+} \alpha(F(I(t,\tau) \times \Gamma)) \leq k(t)\alpha(\Gamma) \quad \text{for any bounded } \Gamma \subset K, \tag{13}$$

and there is $c \in L^1(I; \mathbb{R})$ such that

$$\max_{u \in F(t,x)} \|u\| \leq c(t)(1 + \|x\|) \quad \text{for each } (t,x) \in I \times K. \tag{14}$$

Moreover, F is assumed to satisfy the tangency condition

$$F(t,x) \cap T_K(x) \neq \emptyset \quad \text{for } (t,x) \in I \times \operatorname{bd} K. \tag{15}$$

Lemma 4.1. *For any $x \in U$, $T_K(r(x)) \subset \partial d_K(x)^\circ$.*

Proof. Suppose $u \in T_K(r(x))$, that is $\lim_{y \xrightarrow{K} r(x),\, h \to 0^+} d_K(y + hu)/h = 0$. By (12), $d_K(y + hu) - d_K(y) = d_K(y + hu) - \|r(y) - y\| \leq d_K(r(y) + hu)$, which yields $d_K^\circ(x; u) \leq 0$. □

Let $\overline{F}: I \times U \multimap E$ be given by $\overline{F}(t,x) = F(t, r(x))$. By Lemma 4.1 and (15)

$$\overline{F}(t,x) \cap \partial d_K(x)^\circ \neq \emptyset \quad \text{for } (t,x) \in I \times \operatorname{bd} K. \tag{16}$$

Lemma 4.2. *For any $\varepsilon > 0$ and bounded $\Gamma \subset U$*

$$\lim_{\tau \to 0^+} \alpha(\overline{F}(I(t,\tau) \times \Gamma)) \leq k(t)\alpha(\Gamma) + 2k(t) \sup_\Gamma d_K.$$

Proof. By the assumption

$$\lim_{\tau \to 0^+} \overline{F}(I(t,\tau) \times \Gamma) \leq k(t)\alpha(r(\Gamma)). \tag{17}$$

It is clear that since, for each $x, y \in U$, $\|r(x) - r(y)\| \leq d_K(x) + d_K(y) + \|x - y\|$, one has $\alpha(r(\Gamma)) \leq \alpha(\Gamma) + 2\sup_\Gamma d_K$, which along with (17) gives the desired inequality. □

Finally, we obtain an extension of the results from [17], [13] and [15] (for upper semicontinuous fields).

Theorem 4.3. *If K is a proximate retract and an upper semicontinuous map $F\colon I \times K \multimap E$ with compact convex values satisfies* (13)–(15)*, then the solution set of* $(\mathrm{P})_K$ *is of R_δ type.*

Proof. Indeed, by (16) and Lemma 4.2 one is allowed to apply Theorem 3.2 to $\overline{F}$ and $d_K|U$. □

Inclusions on regular sets. We start with the definition of regular set.

Definition 4.4 (cf. [9]). We say that a closed set $K \subset E$ of the form $K = \{x \in \operatorname{cl} U \mid f(x) \le 0\}$, with locally Lipschitz $f\colon \operatorname{cl} U \to \mathbb{R}$ and open $U \subset E$, is said to be *regular* if

(i) $0 \notin \partial f(x)$ for $x \in U \setminus K$,
(ii) for any $\eta > 0$ there exists $\delta > 0$ such that $f^\delta \subset B(K,\eta)$.

Suppose that K represented by f is regular and that $F\colon I \times K \multimap E$ is an upper semicontinuous compact convex valued map satisfying (13) and (14). We require that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall (t,x) \in I \times K \ \forall y \in B(x,\delta) \setminus K \quad [F(t,x) + \varepsilon B] \cap \partial f(y)^\circ \ne \emptyset. \tag{18}$$

Remark 4.5. If F and f satisfy (18) then, in particular, the pointwise tangency holds, that is

$$\lim_{y \to x,\, y \notin K} \inf_{u \in F(t,x)} d(u, \partial f(y)^\circ) = 0,$$

which, by compactness, is equivalent to the following tangency condition (introduced in [4])

$$F(t,x) \cap \liminf_{y \to x,\, y \notin K} \partial f(y)^\circ \ne \emptyset. \tag{19}$$

Hence (18) is a "uniform" version of (19).

Theorem 4.6. *If $K \subset E$ is regular, represented by f, and $F\colon I \times K \multimap E$ satisfies conditions* (13)–(15)*, then the solution set for the problem* $(\mathrm{P})_K$ *is R_δ in $C(I,E)$.*

Proof. By (18), for any $n \ge 1$ there is $\delta_n > 0$ such that for all $t \in I$, $x \in K$ and $y \in B(x,\delta_n) \setminus K$

$$[F(t,x) + (1/n)B] \cap \partial f(y)^\circ \ne \emptyset. \tag{20}$$

One may assume that $\delta_n \to 0^+$ and $\delta_{n+1} < \delta_n$, for $n \ge 1$. Let $\{(U_\omega, a_\omega\}_{\omega\in\Omega}$ be a Dugundji system for U and K (see [6]), i.e.

(i) $\{U_\omega\}_{\omega\in\Omega}$ is a locally finite covering of $U \setminus K$,
(ii) if $x \in U_\omega$, then $\|x - a_\omega\| \le 2d_K(x)$.

There is a locally finite partition of unity $\{\lambda_\omega\}_{\omega\in\Omega}$ subordinated to $\{U_\omega\}_{\omega\in\Omega}$. Further put $V_n := \{x \in U \mid \delta_{n+1} < 2d_K(x) < \delta_{n-1}\}$ for $n \geq 2$ and $U_0 := \{x \in U \mid 2d_K(x) < \delta_1\}$. Clearly, $\{V_n\}_{n=2}^{\infty}$ is a covering of $U_0 \setminus K$ and there is a partition of unity $\{\eta_n\}_{n=2}^{\infty}$ subordinate to $\{V_n\}_{n=2}^{\infty}$. Define $\overline{F}: I \times U_0 \multimap E$ by

$$\overline{F}(t,x) := \begin{cases} F(t,x) & \text{if } x \in K, \\ \operatorname{cl}\operatorname{conv}\Big[\sum_{n=1}^{\infty}\sum_{\omega\in\Omega} \eta_{n+1}(x)\lambda_\omega(x)F_{\omega,n}(t)\Big] & \text{if } x \in U_0 \setminus K, \end{cases}$$

where $F_{\omega,n}(t) := F(t,a_\omega) + (1/n)D$. $\overline{F}$ has the following properties:

(i) The map $\overline{F}$ is ε-δ upper semicontinuous.

(ii) For any bounded $\Gamma \subset U_0$

$$\lim_{\tau\to 0^+} \alpha(\overline{F}(I(t,\tau) \times \Gamma) \leq 2k(t)\alpha(\Gamma) + 2(k(t)+1)\sup_{\Gamma} p$$

with $p: U_0 \to [0,\infty)$ given by

$$p(x) := \begin{cases} 0 & \text{if } x \in K, \\ \max\{\delta_n, 1/n\} & \text{if } \delta_{n+1} \leq 2d_K(x) < \delta_n. \end{cases}$$

(iii) $\overline{F}(t,x) \cap \partial f(x)^\circ \neq \emptyset$ for each $(t,x) \in I \times (U_0 \setminus K)$.

The ε-δ upper semicontinuity of $\overline{F}$ on $I \times (U_0 \setminus K)$ is clear. Suppose that $\overline{F}$ is not ε-δ upper semicontinuous at some $(t,x) \in I \times K$. Then there are sequences $(x_m) \subset U_0 \setminus K$ converging to x and $t_m \to t$ and $\varepsilon > 0$ such that

$$\overline{F}(t_m,x_m) \not\subset F(t,x) + \varepsilon B. \tag{21}$$

Let $n_m \geq 2$ be such that $\delta_{n_m+1} \leq 2d_K(x_m) < \delta_{n_m}$. Then

$$\overline{F}(t_m,x_m) = \operatorname{cl}\operatorname{conv}\Big[\sum_{\omega\in\Omega_{x_m}} (\eta_{n_m+1}(x_m)\lambda_\omega(x_m)F_{\omega,n_m}(t_m) + \eta_{n_m}(x_m)\lambda_\omega(x_m)F_{\omega,n_m-1}(t_m))\Big],$$

where $\Omega_{x_m} := \{\omega \in \Omega \mid \lambda_\omega(x_m) \neq 0\}$. For each $\omega \in \Omega_{x_m}$, $\|x - a_\omega\| \leq \|a_\omega - x_m\| + \|x_m - x\| \leq 2d_K(x_m) + \|x_m - x\|$, therefore for large m and all $\omega \in \Omega_{x_m}$, $F_{\omega,n_m} \cup F_{\omega,n_m-1} \subset F(t,x) + \varepsilon B$ and consequently we get the converse of (21)

$$\overline{F}(t_m,x_m) \subset \operatorname{cl}\operatorname{conv}\Big[\bigcup_{\omega\in\Omega_{x_m}} (F_{\omega,n_m} \cup F_{\omega,n_m-1})\Big] \subset F(t,x) + \varepsilon B,$$

a contradiction.

Now proceed with (ii). Let $\Gamma \subset U$ be bounded and put

$$n(\Gamma) := \min\{n \geq 2 \mid \Gamma \cap V_n \neq \emptyset\}$$

(without loss of generality one may suppose that $\Gamma\cap(U\setminus K)\neq\emptyset$). Then, by (13), the definition of $\overline{F}$ and the property of Dugundji systems,

$$\begin{aligned}&\lim_{\tau\to 0^+}\alpha(\overline{F}(I(t,\tau)\times\Gamma))\\&\le k(t)\alpha(\Gamma\cap K)+\lim_{\tau\to 0^+}\alpha(\operatorname{conv}[F(I(t,\tau)\times D(\Gamma,\delta_{n(\Gamma)})\cap K)+n(\Gamma)^{-1}D])\\&\le 2k(t)\alpha(\Gamma)+k(t)\delta_{n(\Gamma)}+2n(\Gamma)^{-1}\le 2k(t)\alpha(\Gamma)+2(k(t)+1)\sup_{\Gamma}p.\end{aligned}$$

To see (iii) take $(t,x)\in I\times(U_0\setminus K)$. If $\delta_2\le 2d_K(x)<\delta_1$, then $\overline{F}(t,x)=\sum_{\omega\in\Omega_x}\lambda_\omega(x)F_{\omega,1}(t)$. Since $\|a_\omega-x\|\le 2d_K(x)<\delta_1$, for all $\omega\in\Omega_x$, by (20), we infer $F_{\omega,1}(t)\cap\partial f(x)\neq\emptyset$. If $\delta_n\le 2d_K(x)<\delta_{n-1}$, $n>2$ and $\omega\in\Omega_x$, then $\|a_\omega-x\|\le 2d_K(x)<\delta_{n-1}$. Hence, by (20), $F_{\omega,n-1}(t)\cap\partial f(x)^\circ\neq\emptyset$ and $F_{\omega,n-2}(t)\cap\partial f(x)^\circ\neq\emptyset$, which gives the claim.

By condition (ii) of Definition 4.4, the function p has the property

$$\lim_{\varepsilon\to 0^+}\sup_{f^\varepsilon}p=0.$$

Finally, apply Theorem 3.2 for $\overline{F}|U_0$ and $f|U_0$ to complete the proof. □

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