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A GENERIC PROPERTY FOR THE EIGENFUNCTIONS OF THE LAPLACIAN

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Abstract. In this work we show that, generically in the set of $\mathcal{C}^2$ bounded regions of $\mathbb{R}^n$, $n \geq 2$, the inequality $\int_\Omega \phi^3 \neq 0$ holds for any eigenfunction of the Laplacian with either Dirichlet or Neumann boundary conditions.

1. Introduction

Perturbation of the boundary for boundary value problems in PDEs have been investigated by several authors, from many points of view, since the pioneering works of Rayleigh ([8]) and Hadamard ([3]). There is, for example, a extensive literature under the label "shape analysis" or "shape optimization", on which the main issue is to determine conditions for a region to be optimal with respect to some cost functional (see, for example [2], [11] and [10]).

In particular, generic properties for solutions of boundary value problems have been considered by Micheletti ([7]), Uhlenbeck ([12]), Saut and Teman ([9]) and others. Many problems of this kind have also been considered by Henry in [4] where a kind of Differential Calculus with the domain as the independent variable was developed. This approach allows the utilization of standard analytic tools such as Implicit Function Theorems and Lyapunov–Schmidt method. In his

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work, Henry also formulated and proved a generalized form of the Transversality Theorem, which will be the main tool used in our arguments.

We consider here the following question: is it true, generically in the set of $\mathcal{C}^2$ regions in $\mathbb{R}^n$ $n \geq 2$, that

$$\int_\Omega \phi^3 \neq 0 \quad \text{for any eigenfunction of the Laplacian}$$
$$\text{(with either Neumann or Dirichlet boundary condition?)}$$

The result is easily seen to be false for $n = 1$. In fact, in this case, $\int_I \phi^3 = 0$ for any nonconstant eigenfunction in the interval I. We will show, however, that the situation is quite different if $n \geq 2$; the property is indeed generic in a sense to be made precise below.

As pointed out to the first author by Prof. K. Rybakowski, the question above appears in connection with the study of stability for nonconstant equilibria of the reaction-diffusion system

$$\begin{cases} \partial_t u = (D_0 + \mu D_1)\Delta u + g(u) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial N} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $g\colon \mathbb{R}^p \to \mathbb{R}^p \in \mathcal{C}^2$, $g(0) = 0$, $Dg(0) = 0$.

The plan of this paper is as follows. In Section 2, we state some background results needed in the sequel. We prove the result for Dirichlet boundary conditions in Section 3, and for Neumann boundary conditions in Section 4.

The authors wish to dedicate this work to the memory of Professor Dan Henry, whose untimely death is a great loss to the mathematical community. Dan's ideas helped to shape the mathematical thinking of a great number of researchers working in the field of qualitative theory of partial differential equations. The first author also wishes to acknowledge his immense debt to Dan as a teacher and to express continuing admiration both for his exceptional mathematical skills and for his courage in the face of misfortune.

2. Preliminaries

The results in this section were taken from the monograph of Henry [4], where full proofs can be found. The formulas in Section 2.2 can also be found in [10].

2.1. Some notation and geometrical preliminaries. Given a function f defined in a neighbourhood of $x \in \mathbb{R}^n$, its m-derivative at x can be considered as a homogeneous polynomial of degree m

$$h \to D^m f(x) h^m$$

in $\mathbb{R}^n$, with norm

$$|D^m f(x)| = \max_{|h| \leq 1} |D^m f(x) h^m|,$$

or as a m-linear symmetric form, or as the collection of partial derivatives

$$D^m f(x) = \left\{ \left(\frac{\partial}{\partial x}\right)^\alpha \;\middle|\; |\alpha| = m \right\}$$

with (equivalent) norm

$$\|D^m f(x)\| = \max_{|\alpha|=m} \left\| \left(\frac{\partial}{\partial x}\right)^\alpha f(x) \right\|.$$

If Ω is an open subset of $\mathbb{R}^n$ and E is a normed vector space, $\mathcal{C}^m(\Omega, E)$ is the space of m-times continuously and bounded differentiable functions on Ω whose derivatives extend continuously to the closure $\overline{\Omega}$, with the usual norm

$$\|f\|_{\mathcal{C}^m(\Omega,E)} = \max_{0 \le j \le m} \sup_{x \in \Omega} |D^m f(x)|.$$

If $E = \mathbb{R}$, we write simply $\mathcal{C}^m(\Omega)$.

$\mathcal{C}^m_{\text{inif}}(\Omega, E)$ is the closed subspace of $\mathcal{C}^m(\Omega, E)$ of functions whose m-th derivative is uniformly continuous. If Ω is bounded, this is $\mathcal{C}^m(\Omega, E)$.

We say that an open set $\Omega \subset \mathbb{R}^n$ is $\mathcal{C}^m$*-regular* if there exists $\phi \in \mathcal{C}^m(\mathbb{R}^n, \mathbb{R})$, which is at least in $\mathcal{C}^1_{\text{inif}}(\mathbb{R}^n, \mathbb{R})$, such that

$$\Omega = \{x \in \mathbb{R}^n \mid \phi(x) > 0\}$$

and $\phi(x) = 0$ implies $|\nabla\phi| \ge 1$.

Let m be a non negative integer and $p \ge 1$ a real number. We define the *Sobolev spaces* $W^{m,p}(\Omega)$ and $W^{m,p}_0(\Omega)$, as the completion of $\mathcal{C}^m(\Omega)$ and $\mathcal{C}^m_0(\Omega)$, respectively, under the norm

$$\|u\| = \left(\int_\Omega \sum_{|\alpha| \le m} |D^\alpha u|^p \, dx \right)^{1/p}$$

where $\mathcal{C}^m_0(\Omega)$ is the subspace of functions on $\mathcal{C}^m(\Omega)$ with compact support (when $p = 2$ we usually write $H^m(\Omega) = W^{m,2}(\Omega)$ and $H^m_0(\Omega) = W^{m,2}_0(\Omega)$).

We sometimes need to use differential operators (gradient, divergence and Laplacian) in a hypersurface $S \subset \mathbb{R}^n$. The following definitions are all equivalent to the corresponding formulas in Riemannian geometry, in the metric induced in S by the surrounding ambient space. These formulas are intrinsic to S but our interest is precisely in their relation to a neighbourhood of S (see Theorem 2.1).

Let S be a $\mathcal{C}^1$ hypersurface in $\mathbb{R}^n$ and let $\phi \colon S \to \mathbb{R}$ be $\mathcal{C}^1$ (so it can be extended to be $\mathcal{C}^1$ on a neighbourhood of S), then $\nabla_S \phi$ is the tangent vector field in S such that, for each $\mathcal{C}^1$ curve $t \to x(t) \subset S$, we have

$$\frac{d}{dt}\phi(x(t)) = \nabla_S \phi(x(t)) \cdot \dot{x}(t).$$

Let S be a $\mathcal{C}^2$ hypersurface in $\mathbb{R}^n$ and $\overrightarrow{a}: S \to \mathbb{R}^n$ a $\mathcal{C}^1$ vector field tangent to S. Then $\operatorname{div}_S \overrightarrow{a}: S \to \mathbb{R}^n$ is the continuous function such that, for every $\mathcal{C}^1$, $\phi: S \to \mathbb{R}$ with compact support in S,

$$\int_S (\operatorname{div}_S \overrightarrow{a})\phi = -\int_S \overrightarrow{a} \cdot \nabla_S \phi.$$

Finally, if $u: S \to \mathbb{R}$ is $\mathcal{C}^2$, then $\Delta_S u = \operatorname{div}_S(\nabla_S u)$ or, equivalently, for all $\mathcal{C}^1$, $\phi: S \to \mathbb{R}$ with compact support

$$\int_S \phi \Delta_S u = -\int_S \nabla_S \phi \cdot \nabla_S u.$$

Theorem 2.1.

(1) *If S is a $\mathcal{C}^1$ hypersurface and $\phi: \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{C}^1$ in a neighbourhood of S, then, on S, $\nabla_S \phi(x)$ is the component of $\nabla \phi(x)$ tangent S at x, that is*

$$\nabla_S \phi(x) = \nabla \phi(x) - \frac{\partial \phi}{\partial N}(x) N(x)$$

where N is an unit normal field on S.

(2) *If S is a $\mathcal{C}^2$ hypersurface in $\mathbb{R}^n$, $\overrightarrow{a}: S \to \mathbb{R}^n$ is $\mathcal{C}^1$ in a neighbourhood of S, $N: \mathbb{R}^n \to \mathbb{R}^n$ is a $\mathcal{C}^1$ unit normal field in a neighbourhood of S and $H = \operatorname{div} N$ is the mean curvature of S, then*

$$\operatorname{div}_S \overrightarrow{a} = \operatorname{div} \overrightarrow{a} - H \overrightarrow{a} \cdot N - \frac{\partial}{\partial N}(a \cdot N)$$

on S.

(3) *If S is a $\mathcal{C}^2$ hypersurface, $u: \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{C}^2$ in a neighbourhood of S and N is a normal vector field for S, then*

$$\Delta_S u = \Delta u - H \frac{\partial u}{\partial N} - \frac{\partial^2 u}{\partial N^2} + \nabla_S u \cdot \frac{\partial N}{\partial N}$$

on S. We may choose N so that $\partial N / \partial N = 0$ on S and then the final term vanishes.

We often need the *Cauchy's uniqueness theorem for second order elliptic equations.* We state here a fairly general version whose proof can be found in [5, Theorem 8.9.1].

Theorem 2.2. *Suppose $Q \subset \mathbb{R}^n$ is an open connected set, B is a ball which intersects ∂Q in a $\mathcal{C}^2$ hypersurface $B \cap \partial Q$, $a_{ij} = a_{ji}: Q \to \mathbb{R}$ is a $\mathcal{C}^1$ function for $1 \leq i, j \leq n$, with $\sum_{i,j=1}^n a_{ij}(x)\xi_i \xi_j \geq c_0 |\xi|^2$ for all $x \in Q$ and $\xi \in \mathbb{R}^n$ for some constant $c_0 > 0$. Assume $u \in H^2(Q)$ and, for some constant K,*

$$\left| \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i x_j} \right| \leq K(|\nabla u(x)| + |u(x)|)$$

for a.e. $x \in Q$ and $u = 0$, $\partial u / \partial N = 0$ on $B \cap \partial Q$. Then $u = 0$ a.e. in Q.

2.2. Differential calculus of boundary perturbation. Given an open bounded, $\mathcal{C}^m$ region $\Omega_0 \subset \mathbb{R}^n$, consider the following open subset of $\mathcal{C}^m(\Omega, \mathbb{R}^n)$

$$\operatorname{Diff}^m(\Omega) = \{h \in \mathcal{C}^m(\Omega, \mathbb{R}^n) \mid h \text{ is injective and } 1/|\det h'(x)| \text{ is bounded in } \Omega\}$$

and the collection of all regions $\{h(\Omega_0) \mid h \in \operatorname{Diff}^m(\Omega_0)\}$.

We introduce a topology in this set by defining a (sub-basis of) the neighbourhoods of a given Ω by

$$\{h(\Omega) \mid \|h - i_\Omega\|_{\mathcal{C}^m(\Omega,\mathbb{R}^n)} < \varepsilon,\ \varepsilon > 0 \text{ suficiently small}\}.$$

When $\|h - i_\Omega\|_{\mathcal{C}^m(\Omega,\mathbb{R}^n)}$ is small, h is a $\mathcal{C}^m$ imbedding of Ω in $\mathbb{R}^n$, a $\mathcal{C}^m$ diffeomorphism to its image $h(\Omega)$. Michelleti in [7] shows this topology is metrizable, and the set of regions $\mathcal{C}^m$-diffeomorphic to Ω may be considered a separable metric space which we denote by $\mathcal{M}_m(\Omega)$, or simply $\mathcal{M}_m$. We say that a function F defined in the space $\mathcal{M}_m$ with values in a Banach space is $\mathcal{C}^m$ or analytic if $h \mapsto F(h(\Omega))$ is $\mathcal{C}^m$ or analytic as a map of Banach spaces (h near i_Ω in $\mathcal{C}^m(\Omega, \mathbb{R}^n)$). In this sense, we may express problems of perturbation of the boundary of a boundary value problem as problems of differential calculus in Banach spaces.

More specifically, consider a formal non-linear differential operator $u \mapsto v$

$$v(y) = f\left(y, u(y), \frac{\partial u}{\partial y_1}(y), \dots, \frac{\partial u}{\partial y_n}(y), \frac{\partial^2 u}{\partial y_1^2}(x), \frac{\partial^2 u}{\partial y_1 \partial y_2}(y), \dots\right), \quad y \in \mathbb{R}^n.$$

To simplify the notation, we define a constant matrix coefficient differential operator L

$$Lu(y) = \left(u(y), \frac{\partial u}{\partial y_1}(y), \dots, \frac{\partial u}{\partial y_n}(y), \frac{\partial^2 u}{\partial y_1^2}(y), \frac{\partial^2 u}{\partial y_1 \partial y_2}(y), \dots\right), \quad y \in \mathbb{R}^n$$

with as many terms as needed, so our nonlinear operator becomes

$$u \mapsto v(\,\cdot\,) = f(\,\cdot\,, Lu(\,\cdot\,)).$$

More precisely, suppose $Lu(\,\cdot\,)$ has values in $\mathbb{R}^p$ and $f(y, \lambda)$ is defined for (y, λ) in some open set $O \subset \mathbb{R}^n \times \mathbb{R}^p$. For subsets $\Omega \subset \mathbb{R}^n$ define F_Ω by

$$F_\Omega(u)(y) = f(y, Lu(y)), \quad y \in \Omega$$

for sufficiently smooth functions u in Ω such that $(y, Lu(y)) \in O$ for any $y \in \overline{\Omega}$. For example, if f is continuous, Ω is bounded and L involves derivatives of order $\leq m$, the domain of F_Ω is an open subset (perhaps empty) of $\mathcal{C}^m(\Omega)$, and the values of F_Ω are in $\mathcal{C}^0(\Omega)$. (Other function spaces could be used with obvious modifications).

If $h: \Omega \mapsto \mathbb{R}^n$ is a $\mathcal{C}^k$ imbedding, we can also consider $F_{h(\Omega)}: \mathcal{C}^m(h(\Omega)) \mapsto \mathcal{C}^0(h(\Omega))$. But then the problem will be posed in different spaces. To bring it back to the original spaces we consider the "pull-back" of h

$$h^\star: \mathcal{C}^k(h(\Omega)) \mapsto \mathcal{C}^k(\Omega) \quad (0 \le k \le m)$$

defined by $h^\star(\varphi) = \varphi \circ h$ (which is a diffeomorphism) and then $h^\star F_{h(\Omega)} h^{\star -1}$ is again a map from $\mathcal{C}^m(\Omega)$ into $\mathcal{C}_0(\Omega)$. This is more convenient if we wish to use tools like the Implicit Function or Transversality theorems. On the other hand, a new variable h is introduced. We then need to study the differentiability properties of the function $(h, u) \mapsto h^\star F_{h(\Omega)} h^{\star -1} u$. This has been done in [4] where it is shown that, if $(y, \lambda) \mapsto f(y, \lambda)$ is $\mathcal{C}^k$ or analytic then so is the map above, considered as a map from $\mathrm{Diff}^m(\Omega) \times \mathcal{C}^m(\Omega)$ to $\mathcal{C}^0(\Omega)$ (other function spaces can be used instead of $\mathcal{C}^m$). To compute the derivative we then need only compute the Gateaux derivative that is, the t-derivative along a smooth curve $t \mapsto (h(t, \cdot), u(t, \cdot)) \in \mathrm{Diff}^m(\Omega) \times \mathcal{C}^m(\Omega)$.

Suppose we wish to compute

$$\frac{\partial}{\partial t} F_{\Omega(t)}(v)(y) = \frac{\partial}{\partial t} f(y, Lv(y))$$

with $y = h(t, x)$ fixed in $\Omega(t) = h(t, \Omega)$. To keep y fixed we must take $x = x(t)$, $y = h(t, x(t))$ with

$$0 = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} x'(t) \Rightarrow x'(t) = -\left(\frac{\partial h}{\partial x}\right)^{-1} \frac{\partial h}{\partial t},$$

that is, $x(t)$ is the solution of the differential equation $dx/dt = -U(x, t)$ where $U(x, t) = (\partial h/\partial x)^{-1}(\partial h/\partial t)$. The differential operator

$$D_t = \frac{\partial}{\partial t} - U(x, t)\frac{\partial}{\partial x}, \quad U(x, t) = \left(\frac{\partial h}{\partial x}\right)^{-1} \frac{\partial h}{\partial t}$$

is called the *anti-convective derivative*. The results (Theorems 2.3, 2.6) below are the main tools we use to compute derivatives.

THEOREM 2.3. *Suppose $f(t, y, \lambda)$ is $\mathcal{C}^1$ in an open set in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p$, L is a constant-coefficient differential operator of order $\le m$ with $Lv(y) \in \mathbb{R}^p$ (where defined). For open sets $Q \subset \mathbb{R}^n$ and $\mathcal{C}^m$ functions v on Q, let $F_Q(t)v$ be the function*

$$y \to f(t, y, Lv(y)), \quad y \in Q,$$

where defined. Suppose $t \to h(t, \cdot)$ is a curve of imbeddings of an open set $\Omega \subset \mathbb{R}^n$, $\Omega(t) = h(t, \Omega)$ and for $|j| \le m$, $|k| \le m+1$, $(t, x) \to \partial_t \partial_x^j h(t, x)$, $\partial_x^k h(t, x)$, $\partial_x^k u(t, x)$ are continuous on $\mathbb{R} \times \Omega$ near $t = 0$, and $h(t, \cdot)^{*-1} u(t, \cdot)$ is in the domain of $F_{\Omega(t)}$. Then, at points of Ω

$$D_t(h^* F_{\Omega(t)}(t) h^{*-1})(u) = (h^* \dot{F}_{\Omega(t)}(t) h^{*-1})(u) + (h^* F'_{\Omega(t)}(t) h^{*-1})(u) \cdot D_t u$$

where D_t *is the anti-convective derivative defined above,*

$$\dot{F}_Q(t)v(y) = \frac{\partial f}{\partial t}(t, y, Lv(y))$$

and

$$F'_Q(t)v \cdot w(y) = \frac{\partial f}{\partial \lambda}(t, y, Lv(y)) \cdot Lw(y), \quad y \in Q$$

is the linearization of $v \to F_Q(t)v$.

Remark 2.4. Suppose we deal with a linear operator

$$A = \sum_{|\alpha| \leq m} a_\alpha(y) \left(\frac{\partial}{\partial y} \right)^\alpha$$

not explicitly dependent on t, and $h(t, x) = x + tV(x) + o(t)$ as $t \to 0$ and $x \in \Omega$. Then at $t = 0$

$$\begin{aligned} \frac{\partial}{\partial t}(h^* A h^{*-1} u)|_{t=0} &= D_t(h^* A h^{*-1} u)|_{t=0} + h_x^{-1} h_t \nabla(h^* A h^{*-1} u)|_{t=0} \\ &= A\left(\frac{\partial u}{\partial t} - V \cdot \nabla u \right) + V \cdot \nabla(Au) = A\frac{\partial u}{\partial t} + [V \cdot \nabla, A]u \end{aligned}$$

since $(\partial A/\partial t) = 0$. Note that the commutator $[V \cdot \nabla, A](\,\cdot\,)$ is still an operator of order m.

We also need to be able to differentiate boundary conditions, and a quite general form is

$$b(t, y, Lv(y), MN_{\Omega(t)}(y)) = 0 \quad \text{for } y \in \partial\Omega(t),$$

where L, M are constant-coefficient differential operators and $N_{\Omega(t)}(y)$ is the outward unit normal for $y \in \partial\Omega(t)$, extended smoothly as a unit vector field on a neighbourhood of $\partial\Omega(t)$. We choose some extension of N_Ω in the reference region and then define $N_{\Omega(t)} = N_{h(t,\Omega)}$ by

$$h^* N_{h(t,\Omega)}(x) = N_{h(t,\Omega)}(h(x)) = \frac{(h_x^{-1})^T N_\Omega(x)}{\|(h_x^{-1})^T N_\Omega(x)\|} \tag{2.2}$$

for x near $\partial\Omega$, where $(h_x^{-1})^T$ is the inverse-transpose of the Jacobian matrix h_x and $\|\cdot\|$ is the Euclidean norm. This is the extension understood in the above boundary condition: $b(t, y, Lv(y), MN_{\Omega(t)}(y))$ is defined for $y \in \Omega$ near $\partial\Omega$ and has limit zero (in some sense, depending on the functional space employed) as $y \to \partial\Omega$.

Lemma 2.5. *Let* Ω *be a* $\mathcal{C}^2$*-regular region,* $N_{\Omega(\,\cdot\,)}$ *a* $\mathcal{C}^1$ *unit-vector field defined on a neighbourhood of* $\partial\Omega$ *which is the outward normal on* $\partial\Omega$*, and for a* $\mathcal{C}^2$

function $h\colon \Omega \to \mathbb{R}^n$ *define* $N_{h(\Omega)}$ *on a neighbourhood of* $h(\partial\Omega) = \partial h(\Omega)$ *by* (2.2) *above. Suppose* $h(t,\,\cdot\,)$ *is an imbedding for each* t, *defined by*

$$\frac{\partial}{\partial t}h(t,x) = V(t,h(t,x)) \quad \text{for } x \in \Omega,\ h(0,x) = x,$$

$(t,y) \to V(t,y)$ *is* $\mathcal{C}^2$ *and* $\Omega(t) = h(t,\Omega)$, $N_{\Omega(t)} = N_{h(t,\Omega)}$. *Then for* x *near* $\partial\Omega$, $y = h(t,x)$ *near* $\partial\Omega(t)$, *we may compute the derivative* $(\partial/\partial t)_{y=\text{constant}}$ *and, if* $y \in \partial\Omega$,

$$\frac{\partial}{\partial t}N_{\Omega(t)}(y) = D_t(h^* N_{h(t,\Omega)})(x) = -\bigg(\nabla_{\partial\Omega(t)}\sigma + \sigma\frac{\partial N_{\Omega(t)}}{\partial N_{\Omega(t)}}(y)\bigg)$$

where $\sigma = V \cdot N_{\Omega(t)}$ *is the normal velocity and* $\nabla_{\partial\Omega(t)}\sigma$ *is the component of the gradient tangent to* $\partial\Omega$.

Theorem 2.6. *Let* $b(t,y,Lv(y),MN_{\Omega(t)}(y))$ *be a* $\mathcal{C}^1$ *function on an open set of* $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^p\times\mathbb{R}^q$ *and let* L, M *be constant-coefficient differential operators with order* $\le m$ *of appropriate dimensions so* $b(t,y,Lv(y),MN_{\Omega(t)}(y))$ *makes sense. Assume that* Ω *is a* $\mathcal{C}^{m+1}$ *region,* $N_\Omega(x)$ *is a* $\mathcal{C}^m$ *unit-vector field near* $\partial\Omega$ *which is the outward normal on* $\partial\Omega$, *and define* $N_{h(t,\Omega)}$ *by* (2.2) *when* $h\colon\Omega\to\mathbb{R}^n$ *is a* $\mathcal{C}^{m+1}$ *smooth imbedding. Also define* $\mathcal{B}_{h(\Omega)}(t)$ *by*

$$\mathcal{B}_{h(\Omega)}v(y) = b(t,y,Lv(y),MN_{h(\Omega)}(y))$$

for $y \in h(\Omega)$ *near* $\partial h(\Omega)$. *If* $t \to h(t,\,\cdot\,)$ *is a curve of* $\mathcal{C}^{m+1}$ *imbeddings of* Ω *and for* $|j| \le m$, $|k| \le m+1$, $(t,x) \to (\partial_t\partial_x^j h, \partial_x^k, \partial_t\partial_x^j u, \partial_x^k u)(t,x)$ *are continuous on* $\mathbb{R}\times\Omega$ *near* $t=0$, *then at points of* Ω *near* $\partial\Omega$

$$\begin{aligned} D_t(h^*\mathcal{B}_{h(\Omega)}{h^*}^{-1})(u) &= (h^*\dot{\mathcal{B}}_{h(\Omega)}{h^*}^{-1})(u) + (h^*\mathcal{B}'_{h(\Omega)}{h^*}^{-1})(u)\cdot D_t u \\ &\quad + \bigg(h^*\frac{\partial\mathcal{B}_{h(\Omega)}}{\partial N}{h^*}^{-1}\bigg)(u)\cdot D_t(h^*N_{\Omega(t)}) \end{aligned}$$

where $h = h(t,\,\cdot\,)$, $\dot{\mathcal{B}}_{h(\Omega)}$ *and* $\mathcal{B}'_{h(\Omega)}$ *are defined as in Theorem* 2.3,

$$\frac{\partial\mathcal{B}_{h(\Omega)}}{\partial N}(v)\cdot n(y) = \frac{\partial b}{\partial\mu}(t,y,Lv(y),MN_{h(\Omega)}(y))\cdot Mn(y)$$

and $D_t(h^*N_{\Omega(t)})|_{\partial\Omega}$ *is computed in Lemma* 2.5.

2.3. The Transversality Theorem. A basic tool for our results will be the Transversality Theorem in the form below, due to D. Henry (see [4]). We first recall some definitions.

A map $T \in \mathcal{L}(X,Y)$ where X and Y are Banach spaces is a *semi-Fredholm* map if the range of T is closed and at least one (or both, for Fredholm) of $\dim\ker(T)$, $\operatorname{codim}\operatorname{Im}(T)$ is finite; the *index* of T is then

$$\operatorname{index}(T) = \operatorname{ind}(T) = \dim\ker(T) - \operatorname{codim}\operatorname{Im}(T).$$

We say that a subset F of a topological space X is *rare* if its closure has empty interior and *meager* if it is contained in a countable union of rare subsets of X. We say that F is *residual* if its complement in X is meager. We also say that X is a *Baire space* if any residual subset of X is dense.

Let f be a $\mathcal{C}^k$ map between Banach spaces. We say that x is a *regular point* of f if the derivative $f'(x)$ is surjective and its kernel is finite-dimensional. Otherwise, x is called a *critical* point of f. A point is *critical* if it is the image of some critical point of f.

Let now X be a Baire space and $I = [0,1]$. For any closed or σ-closed $F \subset X$ and any nonnegative integer m we say that the codimension of F is greater or equal to m ($\operatorname{codim} F \geq m$) if the subset $\{\phi \in \mathcal{C}(I^m, X) \mid \phi(I^m) \cap F \text{ is non-empty }\}$ is meager in $\mathcal{C}(I^m, X)$. We say $\operatorname{codim} F = k$ if k is the largest integer satisfying $\operatorname{codim} F \geq m$.

Theorem 2.7. *Suppose given positive numbers k and m, Banach manifolds X, Y, Z of class $\mathcal{C}^k$, an open set $A \subset X \times Y$, a $\mathcal{C}^k$ map $f\colon A \mapsto Z$ and a point $\xi \in Z$. Assume for each (x,y) $\in f^{-1}(\xi)$ that:*

(1) $(\partial f/\partial x)(x,y)\colon T_x X \mapsto T_\xi Z$ *is semi-Fredholm with* index $< k$.

(2) *Either*

(α) $Df(x,y) = (\partial f/\partial x, \partial f/\partial y)\colon T_x X \times T_y Y \mapsto T_\xi Z$ *is surjective*

or

(β) $\dim\{\operatorname{Im}(Df(x,y))/\operatorname{Im}(\partial f(x,y)/\partial x)\} \geq m + \dim\ker(\partial f(x,y)/\partial x)$.

Further assume:

(3) $(x,y) \mapsto y\colon f^{-1}(\xi) \mapsto Y$ *is σ-proper,* $f^{-1}(\xi) = \bigcup_{j=1}^{\infty} \mathcal{M}_j$ *is a countable union of sets $\mathcal{M}_j$ such that $(x,y) \mapsto y\colon \mathcal{M}_j \mapsto Y$, is a proper map for each j. (Given $(x_\nu, y_\nu) \in \mathcal{M}_j$ such that y_ν converges in Y, there exists a subsequence (or subnet) with limit in $\mathcal{M}_j$).*

We note that (3) *holds if $f^{-1}(\xi)$ is Lindelöf (every open cover has a countable subcover) or, more specifically, if $f^{-1}(\xi)$ is a separable metric space, or if X, Y are separable metric spaces.*

Let $A_y = \{x \mid (x,y) \in A\}$ and

$$Y_{\text{crit}} = \{y \mid \xi \textit{ is a critical value of } f(\,\cdot\,,y)\colon A_y \mapsto Z\}.$$

Then Y_{crit} is a meager set in Y and, if $(x,y) \mapsto y$ such that $f^{-1}(\xi) \mapsto Y$ is proper, Y_{crit} is also closed. If ind $\partial f/\partial x \leq -m < 0$ on $f^{-1}(\xi)$, then (2)(α) *implies* (2)(β) *and*

$$Y_{\text{crit}} = \{y \mid \xi \in f(A_y, y)\}$$

has codimension $\geq m$ in Y. (Note Y_{crit} is meager if and only if $\operatorname{codim} Y_{\text{crit}} \geq 1$).

Remark 2.8. The usual hypothesis is that ξ is a regular value of f, so $(2)(\alpha)$ holds. If $(2)(\beta)$ holds at some point then $\operatorname{ind}(\partial f/\partial x) \le -m$ at this point, since

$$\operatorname{codim}\operatorname{Im}\left(\frac{\partial f}{\partial x}\right) \ge \dim\left\{\frac{\operatorname{Im}(Df)}{\operatorname{Im}(\partial f/\partial x)}\right\}.$$

If $\operatorname{ind}(\partial f/\partial x) \le -m$ and $(2)(\alpha)$ holds, then $(2)(\beta)$ also holds. Thus $(2)(\beta)$ is more general for the case of negative index.

3. A generic property for the eigenfunctions of the Dirichlet Problem

We will show that, generically in the set of open, connected, bounded $\mathcal{C}^2$ regions $\Omega \subset \mathbb{R}^n$ with $n \ge 2$, the normalized eigenfunctions u of

$$\Delta u + \lambda u = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,\ u \ne 0 \tag{3.1}$$

satisfy $\int_\Omega u^3 \ne 0$. We need first some preliminary results

Lemma 3.1. *Given $h_0 \in \operatorname{Diff}^2(\Omega)$ there exists a neighbourhood V_0 of h_0 in $\operatorname{Diff}^2(\Omega)$ such that, for all $h \in V_0$ and $u \in H^2 \cap H_0^1(\Omega)$*

$$\|(h^*\Delta h^{*-1} - h_0^*\Delta h_0^{*-1})u\|_{L^2(\Omega)} \le \varepsilon(h)\|u\|_{H^2\cap H_0^1(\Omega)}$$

with $\varepsilon(h) \to 0$ as $h \to h_0$ in $\mathcal{C}^2(\Omega, \mathbb{R}^n)$.

Proof. It is sufficient to consider the case $h_0 = i_\Omega$. We have

$$\begin{aligned} h^*\frac{\partial}{\partial y_i}h^{*-1}u(x) &= \frac{\partial}{\partial y_i}(u\circ h^{-1})(h(x)) \\ &= \sum_{j=1}^n \frac{\partial u}{\partial x_j}(x)(h_x^{-1})_{ji}(x) = \sum_{j=1}^n b_{ij}(x)\frac{\partial u}{\partial x_j}(x) \end{aligned}$$

where $b_{ij}(x) = (h_x^{-1})_{ji}(x)$, that is, $b_{ij}(x)$ is the i,j-th entry in the transposed inverse of the Jacobian matrix of $h_x = (\partial h_i/\partial x_j)_{i,j=1}^n$. Therefore

$$\begin{aligned} h^*\frac{\partial^2}{\partial y_i^2}h^{*-1}u(x) &= \sum_{k=1}^n b_{ik}(x)\frac{\partial}{\partial x_k}(x)\left(\sum_{j=1}^n b_{ij}\frac{\partial u}{\partial x_j}\right)(x) \\ &= \sum_{k=1}^n b_{ik}(x)\sum_{j=i}^n\left[\left(\frac{\partial}{\partial x_k}b_{ij}\right)(x)\frac{\partial u}{\partial x_j}(x) + b_{ij}(x)\frac{\partial^2 u}{\partial x_k\partial x_j}(x)\right] \\ &= \sum_{j,k=1}^n b_{ik}(x)b_{ij}(x)\left(\frac{\partial^2 u}{\partial x_k\partial x_j}\right)(x) \\ &\quad + \sum_{j,k=1}^n b_{ik}(x)\left(\frac{\partial}{\partial x_k}b_{ij}(x)\right)(x)\frac{\partial u}{\partial x_j}(x) \\ &= \left(\frac{\partial^2}{\partial x_i^2}(u)\right)(x) + L_i(u)(x) \end{aligned}$$

where

$$\begin{aligned} L_i(u)(x) = {} & (b_{ii}^2(x)-1)\bigg(\frac{\partial^2}{\partial x_i^2}(u)\bigg)(x) \\ & + \sum_{j,k=1}^{n}(1-\delta_{i,j,k})b_{i,k}(x)b_{i,j}(x)\bigg(\frac{\partial^2 u}{\partial x_k \partial x_j}\bigg)(x) \\ & + \sum_{j,k=1}^{n} b_{i,k}(x)\bigg(\frac{\partial}{\partial x_k}b_{ij})(x)\bigg)(x)\frac{\partial u}{\partial x_j}(x). \end{aligned}$$

Thus $(h^*\Delta h^{*-1}(u)) = \Delta u + Lu$ with $Lu = \sum_{i=1}^n L_i u$.

Since $b_{j,k} \to \delta_{j,k}$ in $\mathcal{C}^2(\Omega,\mathbb{R}^n)$ when $h \to i_\Omega$ in $\mathcal{C}^2(\Omega,\mathbb{R}^n)$ the coefficients of L go to 0 uniformly in x as $h \to i_\Omega$ in $\mathcal{C}^2(\Omega,\mathbb{R}^n)$. It follows that

$$\|Lu\|_{L^2(\Omega)} \le \varepsilon(h)\|u\|_{H^2\cap H_0^1(\Omega)}$$

where $\varepsilon(h)$ goes to zero as $h \to i_\Omega$ in $\mathcal{C}^2(\Omega,\mathbb{R}^n)$. □

Let $\Omega \subset \mathbb{R}^n$ be a $\mathcal{C}^k$ $(k \ge 2)$, open, bounded, connected region and consider the set

$$\begin{aligned} D_M = \{h \in \operatorname{Diff}^k(\Omega) \mid {} & M \text{ is not an eigenvalue of (3.1) in } h(\Omega) \\ & \text{and all the eigenvalues } \lambda \in (0,M) \text{ in } h(\Omega) \text{ are simple}\}. \end{aligned}$$

Lemma 3.2. *D_M is an open and dense subset of* $\operatorname{Diff}^k(\Omega)$.

Proof. Define

$$D = \{h \in \operatorname{Diff}^k(\Omega) \mid \text{ all the eigenvalues of (3.1) in } h(\Omega) \text{ are simple}\}$$

and

$$\widetilde{D_M} = \{h \in \operatorname{Diff}^k(\Omega) \mid \text{ all the eigenvalues } \lambda \in (0,M) \text{ in } h(\Omega) \text{ are simple}\}.$$

We first show that D_M is open. Let $h_0 \in D_M$ and let $\lambda_1,\ldots,\lambda_k$ be the (simple) eigenvalues of Δ in $h_0(\Omega)$ smaller M. Let also γ be the circle of radius M with center in the origin.

From the previous lemma and Theorems 2.14, 3.16 of [6] it follows that there exists a neighbourhood V_0 of h_0 such that the dimension of the eigenspace associated to the eigenvalues smaller than M of $h^*\Delta h^{*-1}$ is constant and there are no eigenvalues in γ for $h \in V_0$. From the implicit function theorem (see [4] for details) the simple eigenvalues of ${h_0}^*\Delta {h_0}^{*-1}$ depend continuously of h in a neighbourhood of h_0 in $\mathcal{C}^k$. Therefore, for each $1 \le i \le k$ there exists a neighbourhood $V_i \subset \operatorname{Diff}^k(\Omega)$ of h_0 and continuous functions $\Lambda_i\colon V_i \to (0,M)$ such that $\Lambda_i(h)$ is a simple eigenvalue of $h^*\Delta h^{*-1}$ for any $h \in V_i$ with $\Lambda_i(h_0) = \lambda_i$ and the sets $\Lambda_i(V_i)$ are pairwise disjoint. Define then $V = \bigcap_{i=0}^k V_i$, neighbourhood of h_0 in $\operatorname{Diff}^k(\Omega)$. Observe that for all $h \in V$, $h^*\Delta h^{*-1}$ has k eigenvalues smaller than M, which are all simple. Therefore, D_M is open.

To prove density we observe that D is dense in $\mathrm{Diff}^k(\Omega)$ (see [4] or [7]) and therefore $\widetilde{D_M}$ is also dense. To conclude the proof we just need to show that, if M is an eigenvalue of (3.1) in Ω, there exists h near i_Ω such that this does not hold anymore in $h(\Omega)$. To this end, it is enough to take $h(x) = (1+\varepsilon)x$. A simple computation shows that each eigenvalue λ of Δ in Ω changes to $\lambda/(1+\varepsilon)^2$ in $h(\Omega)$. $\square$

Before proceeding, we try to outline the main steps of our argument. Let $\Omega \subset \mathbb{R}^n$ be an open, connected, bounded $\mathcal{C}^2$-regular region and consider the mapping

$$\begin{aligned} &F \colon H^2 \cap H_0^1(\Omega) - \{0\} \times (0, M) \times D_M \to L^2(\Omega) \times \mathbb{R} \times \mathbb{R}, \\ &(u, \lambda, h) \to \bigg(h^*(\Delta + \lambda) h^{*-1} u, \int_\Omega u^2 \det h', \int_\Omega u^3 \det h' \bigg). \end{aligned}$$

We would like to show that, for each $M \in \mathbb{N}$, the set

$$B_M = \{ h \in D_M \mid (0,1,0) \in F(H^2 \cap H_0^1(\Omega) - \{0\} \times (0, M), h) \}$$

is meager in D_M. Since the operator $\partial F(u, \lambda, h)/\partial(u, \lambda)$ from $H^2 \cap H_0^1(\Omega) \times \mathbb{R}$ into $L^2(\Omega) \times \mathbb{R} \times \mathbb{R}$ is Fredholm with $\mathrm{ind}\,(\partial F(u,\lambda,h)/\partial(u,\lambda)) \le -1$ for all $(u, \lambda, h) \in F^{-1}(0,1,0)$ (see Theorem 3.7 below), this would follow from the Transversality Theorem 2.7 if we could prove that $(0,1,0)$ is a regular value of F. We try to do that and fail. However, we do show that a critical point must have very special properties, which enables us to show that they can only occur in a "exceptional" set of regions. Repeating the argument in the complement of this set we can, finally, prove our result.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^n$ be an open connected, bounded, $\mathcal{C}^5$-regular region. If $(u, \lambda, h) \in H^2 \cap H_0^1(\Omega) - \{0\} \times (0, M) \times D_M$ is a critical point of F, with $F(u, \lambda, h) = (0, 1, 0)$ then there exists $\psi \in H_0^2(h(\Omega))$ satisfying $(\Delta + \lambda)\psi = -u^2$.*

Proof. By "transferring the origin", we can suppose $h = i_\Omega$. We prove below (see proof of Theorem 3.7) that the "partial derivative" $\partial F/\partial(u, \lambda)$ is Fredlholm and thus, its range has finite codimension. It follows that $\mathrm{Im}\, DF(u, \lambda, i_\Omega)$ also has finite codimension and, therefore, is closed. Suppose $(u, \lambda, i_\Omega) \in H^2 \cap H_0^1(\Omega) - \{0\} \times (0, M) \times D_M$ is a critical point of F with $F(u, \lambda, i_\Omega) = (0, 1, 0)$. We prove below (see proof of Theorem 3.7), that then, there exists $(\psi, \alpha, \theta) \in L^2(\Omega) \times \mathbb{R} \times \mathbb{R}$ orthogonal to $\mathrm{Im}\, DF(u, \lambda, i_\Omega)$, that is,

$$\begin{aligned} 0 = \int_\Omega \{ &\psi[(\Delta + \lambda)(\dot{u} - \dot{h} \cdot \nabla u) + \dot{\lambda} u] \\ &+ \alpha[2u\dot{u} + u^2 \mathrm{div}(\dot{h})] + \theta[3u^2 \dot{u} + u^3 \mathrm{div}(\dot{h})] \} \end{aligned} \tag{3.2}$$

for all $(\dot{u}, \dot{\lambda}, \dot{h}) \in H^2 \cap H_0^1(\Omega) - \{0\} \times \mathbb{R} \times \mathcal{C}^5(\Omega, \mathbb{R}^n)$.

Taking $\dot{u} = \dot{h} = 0$ in (3.2), we obtain $\int_\Omega \psi u = 0$. Taking $\dot{h} = \dot{\lambda} = 0$, we have

$$\int_\Omega \{\psi(\Delta+\lambda)\dot{u} + 2\alpha u\dot{u} + 3\theta u^2\dot{u}\} = 0 \quad \text{for all } \dot{u} \in H^2 \cap H_0^1(\Omega). \tag{3.3}$$

If $\dot{u} = u$ in (3.3) it follows that $\alpha = 0$ and so, by regularity of solutions of elliptic problems we conclude that $\psi \in H^2 \cap H_0^1(\Omega) \cap \mathcal{C}_\alpha^2(\Omega)$ for all $0 < \alpha < 1$ and $(\Delta+\lambda)\psi = -3\theta u^2$. Taking now, $\dot{u} = \dot{\lambda} = 0$ in (3.2)

$$\int_\Omega \psi(\Delta+\lambda)(\dot{h}\cdot\nabla u) = \int_\Omega \theta u^3 \operatorname{div}(\dot{h}) \quad \text{for all } \dot{h} \in \mathcal{C}^5(\Omega,\mathbb{R}^n). \tag{3.4}$$

Let N a unit vector field normal to $\partial\Omega$. Since

$$\begin{aligned}\int_\Omega \psi(\Delta+\lambda)(\dot{h}\cdot\nabla u) &= \int_\Omega (\dot{h}\cdot\nabla u)(\Delta+\lambda)\psi - \int_{\partial\Omega} \frac{\partial u}{\partial N}\frac{\partial\psi}{\partial N}\dot{h}\cdot N \\ &= -\int_\Omega 3\theta u^2(\dot{h}\cdot\nabla u) - \int_{\partial\Omega} \frac{\partial u}{\partial N}\frac{\partial\psi}{\partial N}\dot{h}\cdot N,\end{aligned}$$

we obtain, substituting in (3.4)

$$\int_\Omega \theta u^3 \operatorname{div}(\dot{h}) = -\int_\Omega 3\theta u^2(\dot{h}\cdot\nabla u) - \int_{\partial\Omega} \frac{\partial u}{\partial N}\frac{\partial\psi}{\partial N}\dot{h}\cdot N \tag{3.5}$$

for all $\dot{h} \in \mathcal{C}^5(\Omega,\mathbb{R}^n)$. Observe now that $\operatorname{div}(u^3\dot{h}) = 3u^2\nabla u\cdot\dot{h} + u^3\operatorname{div}(\dot{h})$ and so

$$\int_\Omega \theta u^3 \operatorname{div}(\dot{h}) = -\int_\Omega 3\theta u^2(\dot{h}\cdot\nabla u). \tag{3.6}$$

Therefore, substituting (3.6) in (3.5), we have

$$\int_{\partial\Omega} \frac{\partial u}{\partial N}\frac{\partial\psi}{\partial N}\dot{h}\cdot N = 0 \quad \text{for all } \dot{h} \in \mathcal{C}^5(\Omega,\mathbb{R}^n) \tag{3.7}$$

from which, $(\partial u/\partial N)(\partial\psi/\partial N) = 0$ on $\partial\Omega$. Since u is not identically zero it follows from Theorem 2.2 that $\partial\psi/\partial N = 0$ on $\partial\Omega$ and (multiplying ψ by a constant if needed) our result follows. □

Remark 3.4. Observe that,by regularity in the elliptic problem, $\psi \in H^4 \cap H_0^2(\Omega) \cap \mathcal{C}^{4,\alpha}(\Omega)$ since $u^2 \in H^2 \cap H_0^1(\Omega) \cap \mathcal{C}^{2,\alpha}(\Omega)$ for all $0 < \alpha < 1$.

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^n$ be an open connected, bounded $\mathcal{C}^5$-regular region. If $\psi \in H_0^2(\Omega)$ satisfies $(\Delta+\lambda)\psi = u^2$ for some $u \in H^2 \cap H_0^1(\Omega) \cap \mathcal{C}_\alpha^2(\Omega)$, then*

(1) *$\partial\psi/\partial x_i = 0$ in $\partial\Omega$ for all $1 \le i \le n$,*
(2) *$\partial^2\psi/(\partial x_i\partial x_j) = 0$ in $\partial\Omega$ for all $1 \le i,j \le n$,*
(3) *$\partial^3\psi/(\partial x_i\partial x_j\partial x_k) = 0$ in $\partial\Omega$ for all $1 \le i,j,k \le n$.*

Proof. From $\psi = 0$ and $\partial\psi/\partial N = 0$ in $\partial\Omega$ it follows that $\nabla\psi = (\partial\psi/\partial N)\cdot N = 0$ in $\partial\Omega$ and thus $\partial\psi/\partial x_i = 0$ on $\partial\Omega$ for all $1 \le i \le n$.

From (2.1) we obtain

$$0 = u^2 = (\Delta+\lambda)\psi = \frac{\partial^2\psi}{\partial N^2} + H\frac{\partial\psi}{\partial N}$$

in $\partial\Omega$ where $H = \mathrm{div}(N)$ which implies $\partial^2\psi/\partial N^2 = 0$ in $\partial\Omega$.

Now, since $\partial\psi/\partial N = 0$ and $\partial^2\psi/\partial N^2 = 0$ in $\partial\Omega$ we have

$$\nabla\left(\frac{\partial\psi}{\partial N}\right) = \frac{\partial}{\partial N}\frac{\partial\psi}{\partial N}\cdot N = 0$$

and then $\nabla(\partial\psi/\partial N) = 0$ in $\partial\Omega$. Therefore, for all $0 \le i \le n$ we have

$$\frac{\partial}{\partial x_i}\frac{\partial\psi}{\partial N} = 0$$

in $\partial\Omega$ from which it follows that

$$\frac{\partial}{\partial x_i}\sum_{k=1}^{n} N_k \frac{\partial\psi}{\partial x_k} = 0$$

in $\partial\Omega$, that is,

$$\sum_{k=1}^{n} N_k \frac{\partial^2\psi}{\partial x_k\partial x_i} = \frac{\partial}{\partial N}\frac{\partial\psi}{\partial x_i} = 0$$

in $\partial\Omega$, for all $0 \le i \le n$. Therefore we have

$$\frac{\partial\psi}{\partial x_i} = \frac{\partial}{\partial N}\frac{\partial\psi}{\partial x_i} = 0$$

on $\partial\Omega$ which implies $\nabla(\partial\psi/\partial x_i) = 0$ in $\partial\Omega$, that is, $\partial^2\psi/(\partial x_i\partial x_j) = 0$ in $\partial\Omega$ for all $1 \le i, j \le n$.

To obtain the last equality, observe that

$$\frac{\partial}{\partial x_i}(u^2) = \frac{\partial}{\partial x_i}(\Delta + \lambda)\psi = (\Delta + \lambda)\frac{\partial\psi}{\partial x_i}$$

in Ω, and so

$$\begin{aligned} 0 &= 2u\frac{\partial u}{\partial x_i} = (\Delta + \lambda)\frac{\partial\psi}{\partial x_i} \\ &= \Delta_{\partial\Omega}\frac{\partial\psi}{\partial x_i} + H\frac{\partial}{\partial N}\frac{\partial\psi}{\partial x_i} + \frac{\partial^2}{\partial N^2}\frac{\partial\psi}{\partial x_i} + \lambda\frac{\partial\psi}{\partial x_i} = \frac{\partial^2}{\partial N^2}\frac{\partial\psi}{\partial x_i} \end{aligned}$$

on $\partial\Omega$, since $\partial\psi/\partial x_i = \partial^2\psi/(\partial x_i\partial x_j) = 0$ on $\partial\Omega$, $1 \le i, j \le n$. Now, since

$$\frac{\partial}{\partial N}\frac{\partial\psi}{\partial x_i} = \frac{\partial^2}{\partial N^2}\frac{\partial\psi}{\partial x_i} = 0$$

on $\partial\Omega$ we have

$$\nabla\frac{\partial}{\partial N}\frac{\partial\psi}{\partial x_i} = 0$$

on $\partial\Omega$, and so,

$$0 = \frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial N}\frac{\partial\psi}{\partial x_i}\right) = \sum_{j=1}^{n}\left(\frac{\partial N_j}{\partial x_k}\frac{\partial^2\psi}{\partial x_i\partial x_j} + N_j\frac{\partial^3\psi}{\partial x_i\partial x_j\partial x_k}\right) = \frac{\partial}{\partial N}\frac{\partial^2\psi}{\partial x_i\partial x_k}$$

on $\partial\Omega$. Therefore

$$\frac{\partial^2\psi}{\partial x_i\partial x_j} = \frac{\partial}{\partial N}\frac{\partial^2\psi}{\partial x_i\partial x_j} = 0$$

on $\partial\Omega$ which implies $\nabla(\partial^2\psi/\partial x_i\partial x_j) = 0$ on $\partial\Omega$, that is, $\partial^3\psi/(\partial x_i\partial x_j\partial x_k) = 0$ on $\partial\Omega$ for all $1 \le i, j, k \le n$. $\square$

LEMMA 3.6. *Let $\Omega \subset \mathbb{R}^n$ be an open, connected, bounded, $\mathcal{C}^5$-regular region. Consider the mapping*

$$G\colon H^2 \cap H_0^1(\Omega) \times [0, M] \times H^4 \cap H_0^2(\Omega) \times D_M \to L^2(\Omega) \times L^2(\Omega) \times H^{-1/2}(\partial\Omega)$$

defined by

$$G(u, \lambda, \psi, h) = \bigg(h^*(\Delta + \lambda)h^{*-1}u, h^*(\Delta + \lambda)h^{*-1}\psi + u^2, h^*\frac{\partial^3}{\partial N^3}h^{*-1}\psi|_{\partial h(\Omega)}\bigg).$$

Then, the set

$$C_M = \{h \in D_M \mid (0, 0, 0) \in G(H^2 \cap H_0^1(\Omega) \times [0, M] \times H^4 \cap H_0^2(\Omega), h)\}$$

is meager and closed in D_M.

PROOF. We will apply the Transversality Theorem. We note that, as mentioned previously, the mapping G is analytic in h. It is clearly also analytic in the other variables.

Let $(u, \lambda, \psi, h) \in G^{-1}(0, 0, 0)$. As before, we may assume that $h = i_\Omega$. The partial derivative $(\partial G/\partial(u, \lambda, \psi))(u, \lambda, \psi, i_\Omega)$ defined from $H^2 \cap H_0^1(\Omega) \times \mathbb{R} \times H^4 \cap H_0^2(\Omega)$ into $L^2(\Omega) \times L^2(\Omega) \times H^{-1/2}(\partial\Omega)$ is given by

$$\begin{aligned}&\frac{\partial G}{\partial(u, \lambda, \psi)}(u, \lambda, \psi, i_\Omega)(\,\cdot\,)\\ &= \bigg(\frac{\partial G_1}{\partial(u, \lambda, \psi)}(u, \lambda, \psi, i_\Omega), \frac{\partial G_2}{\partial(u, \lambda, \psi)}(u, \lambda, \psi, i_\Omega), \frac{\partial G_3}{\partial(u, \lambda, \psi)}(u, \lambda, \psi, i_\Omega)\bigg)(\,\cdot\,)\end{aligned}$$

where

$$\begin{aligned}\frac{\partial G_1}{\partial(u, \lambda, \psi)}(u, \lambda, \psi, i_\Omega)(\dot u, \dot\lambda, \dot\psi, \dot h) &= (\Delta + \lambda)\dot u + \dot\lambda u,\\ \frac{\partial G_2}{\partial(u, \lambda, \psi)}(u, \lambda, \psi, i_\Omega)(\dot u, \dot\lambda, \dot\psi, \dot h) &= (\Delta + \lambda)\dot\psi + \dot\lambda\psi + 2u\dot u,\\ \frac{\partial G_3}{\partial(u, \lambda, \psi)}(u, \lambda, \psi, i_\Omega)(\dot u, \dot\lambda, \dot\psi, \dot h) &= \frac{\partial^3}{\partial N^3}\dot\psi.\end{aligned}$$

Now $DG(u, \lambda, \psi, i_\Omega)$ defined from $H^2 \cap H_0^1(\Omega) \times \mathbb{R} \times H^4 \cap H_0^2(\Omega) \times \mathcal{C}^5(\Omega, \mathbb{R}^n)$ into $L^2(\Omega) \times L^2(\Omega) \times H^{-1/2}(\partial\Omega)$ is given by

$$DG(u, \lambda, \psi, i_\Omega)(\,\cdot\,) = (DG_1(u, \lambda, \psi, i_\Omega), DG_3(u, \lambda, \psi, i_\Omega), DG_3(u, \lambda, \psi, i_\Omega))(\,\cdot\,)$$

where

$$\begin{aligned}DG_1(u, \lambda, \psi, i_\Omega)(\dot u, \dot\lambda, \dot\psi, \dot h) &= (\Delta + \lambda)(\dot u - \dot h \cdot \nabla u) + \dot\lambda u,\\ DG_2(u, \lambda, \psi, i_\Omega)(\dot u, \dot\lambda, \dot\psi, \dot h) &= (\Delta + \lambda)(\dot\psi - \dot h \cdot \nabla\psi) + \dot\lambda\psi + 2u\dot u,\end{aligned}$$

$$DG_3(u,\lambda,\psi,i_\Omega)(\dot u,\dot\lambda,\dot\psi,\dot h) = \frac{\partial^3}{\partial N^3}(\dot\psi - \dot h\cdot\nabla\psi) + (\dot h\cdot N)\frac{\partial^4\psi}{\partial N^4}.$$

We observe here that, since $\psi\in H^4$ then $\partial^4\psi/\partial N^4$ is in L^2 so its restriction to the boundary is actually in $H^{-1/2}(\partial\Omega)$.

The first two components are easy to compute. To compute the third component we first observe that

$$\begin{aligned}
\frac{\partial^3}{\partial N^3}\psi &= \nabla[\nabla(\nabla\psi\cdot N)\cdot N]\cdot N \\
&= \sum_{k=1}^n N_k\frac{\partial}{\partial x_k}\bigg[\sum_{j=1}^n N_j\frac{\partial}{\partial x_j}\bigg(\sum_{i=1}^n N_i\frac{\partial\psi}{\partial x_i}\bigg)\bigg] \\
&= \sum_{i,j,k=1}^n\bigg[N_k\frac{\partial N_j}{\partial x_k}\frac{\partial N_i}{\partial x_j}\frac{\partial\psi}{\partial x_i} + N_kN_j\frac{\partial^2 N_i}{\partial x_j\partial x_k}\frac{\partial\psi}{\partial x_i} \\
&\quad + N_kN_j\frac{\partial N_i}{\partial x_j}\frac{\partial^2\psi}{\partial x_i\partial x_k} + N_kN_j\frac{\partial N_i}{\partial x_k}\frac{\partial^2\psi}{\partial x_i\partial x_j} + N_kN_jN_i\frac{\partial^3\psi}{\partial x_i\partial x_j\partial x_k}\bigg].
\end{aligned}$$

Using Theorem 2.6, we obtain

$$h^*\frac{\partial^3}{\partial N^3}h^{*-1}\psi = h^*\mathcal{B}_{h(\Omega)}h^{*-1}\psi = b(Lv(y), MN_{h(\Omega)}(y))$$

where $v=h^{*-1}\psi$, $y=h(x)$,

$$\begin{aligned}
MN_{h(\Omega)} &= \bigg(((N_{h(\Omega)})_i,\ 1\le i\le n),\ \bigg(\frac{\partial(N_{h(\Omega)})_i}{\partial y_j},\ 1\le i,j\le n,\bigg), \\
&\qquad\qquad \bigg(\frac{\partial^2(N_{h(\Omega)})_i}{\partial y_j\partial y_k},\ 1\le i,j,k\le n\bigg)\bigg), \\
Lv &= \bigg(\bigg(\frac{\partial v}{\partial y_i},\ 1\le i\le n\bigg),\ \bigg(\frac{\partial^2 v}{\partial y_i\partial y_j},\ 1\le i,j\le n\bigg), \\
&\qquad\qquad \bigg(\frac{\partial^3 v}{\partial y_i\partial y_j\partial y_k},\ 1\le i,j,k\le n\bigg)\bigg),
\end{aligned}$$

and $b\colon\mathbb{R}^{n+n^2+n^3}\times\mathbb{R}^{n+n^2+n^3}\to\mathbb{R}$ is given by

$$\begin{aligned}
b(\lambda,\mu) = \sum_{i,j,k=1}^n &\{\mu_k\mu_{ij}\mu_{jk}\lambda_i + \mu_k\mu_j\mu_{ijk}\lambda_i + \mu_k\mu_j\mu_{ij}\lambda_{ki} \\
&+ \mu_k\mu_j\mu_{ik}\lambda_{ij} + \mu_k\mu_i\mu_{jk}\lambda_{ij} + \mu_k\mu_i\mu_j\lambda_{ijk}\}
\end{aligned}$$

if

$$\begin{aligned}
\lambda &= ((\lambda_i, 1\le i\le n), (\lambda_{i,j}, 1\le i,j\le n), (\lambda_{i,j,k}, 1\le i,j,k\le n)), \\
\mu &= ((\mu_i, 1\le i\le n), (\mu_{i,j}, 1\le i,j\le n), (\mu_{i,j,k}, 1\le i,j,k\le n)),
\end{aligned}$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}(h^*\mathcal{B}_{h(\Omega)}h^{*-1})(\psi)|_{t=0}\\
&= D_t(h^*\mathcal{B}_{h(\Omega)}h^{*-1})(\psi)|_{t=0} + h_x^{-1}h_t\nabla[(h^*\mathcal{B}_{h(\Omega)}h^{*-1})(\psi)]|_{t=0}\\
&= (h^*\dot{\mathcal{B}}_{h(\Omega)}h^{*-1})(\psi)|_{t=0} + (h^*\mathcal{B}'_{h(\Omega)}h^{*-1})(\psi)\cdot D_t\psi|_{t=0}\\
&\quad + \left(h^*\frac{\partial\mathcal{B}_{h(\Omega)}}{\partial N}h^{*-1}\right)(\psi)\cdot D_t(h^*N_{h(\Omega)})|_{t=0}\\
&\quad + h_x^{-1}h_t\nabla[(h^*\mathcal{B}_{h(\Omega)}h^{*-1})(\psi)]|_{t=0}.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\dot{\mathcal{B}}_{h(\Omega)} &\equiv 0,\\
\mathcal{B}'_{h(\Omega)}(v(y))\cdot w(y) &= \frac{\partial^3}{\partial N^3}w(y)\\
\frac{\partial\mathcal{B}_{h(\Omega)}}{\partial N}(v)\cdot n(y) &= \frac{\partial b}{\partial\mu}(Lv(y), MN_{h(\Omega)})\cdot n(y).
\end{aligned}
$$

Now

$$
\begin{aligned}
&\frac{\partial b}{\partial\mu}(Lv, MN_{h(\Omega)})|_{t=0}\\
&= \sum_{i,j,k=1}^{n}\Bigg\{\frac{\partial N_i}{\partial x_j}\frac{\partial N_i}{\partial x_k}\frac{\partial\psi}{\partial x_i}n_k + N_k\frac{\partial N_j}{\partial x_k}\frac{\partial\psi}{\partial x_i}\frac{\partial n_i}{\partial x_j} + N_k\frac{\partial N_i}{\partial x_j}\frac{\partial\psi}{\partial x_i}\frac{\partial n_i}{\partial x_j}\\
&\quad + N_j\frac{\partial^2 N_i}{\partial x_j\partial x_k}\frac{\partial\psi}{\partial x_i}n_k + N_k\frac{\partial^2 N_i}{\partial x_j\partial x_k}\frac{\partial\psi}{\partial x_i}n_j + N_kN_j\frac{\partial\psi}{\partial x_i}\frac{\partial^2 n_i}{\partial x_j\partial x_k}\\
&\quad + N_j\frac{\partial N_i}{\partial x_j}\frac{\partial^2\psi}{\partial x_i\partial x_k}n_k + N_k\frac{\partial N_i}{\partial x_j}\frac{\partial^2\psi}{\partial x_i\partial x_k}n_j + N_kN_j\frac{\partial^2\psi}{\partial x_i\partial x_k}\frac{\partial n_i}{\partial x_j}\\
&\quad + N_k\frac{\partial N_i}{\partial x_k}\frac{\partial^2\psi}{\partial x_i\partial x_j}n_j + N_j\frac{\partial N_i}{\partial x_k}\frac{\partial^2\psi}{\partial x_i\partial x_j}n_k + N_kN_j\frac{\partial^2\psi}{\partial x_i\partial x_j}\frac{\partial n_i}{\partial x_k}\\
&\quad + N_i\frac{\partial N_j}{\partial x_k}\frac{\partial^2\psi}{\partial x_i\partial x_j}n_k + N_k\frac{\partial N_j}{\partial x_k}\frac{\partial^2\psi}{\partial x_i\partial x_j}n_i + N_kN_i\frac{\partial^2\psi}{\partial x_i\partial x_j}\frac{\partial n_i}{\partial x_k}\\
&\quad + N_iN_j\frac{\partial^3\psi}{\partial x_i\partial x_j\partial x_k}n_k + N_kN_j\frac{\partial^3\psi}{\partial x_i\partial x_j\partial x_k}n_i + N_iN_k\frac{\partial^3\psi}{\partial x_i\partial x_j\partial x_k}n_j\Bigg\} = 0.
\end{aligned}
$$

In fact, by Lemma 3.5, $\partial\psi/\partial x_i = 0$ for all $1 \le i \le n$, $\partial^2\psi/(\partial x_i\partial x_j) = 0$ for all $1 \le i,j \le n$ and $\partial^3\psi/(\partial x_i\partial x_j\partial x_k) = 0$ for all $1 \le i,j,k \le n$ on $\partial\Omega$.

Now, we can easily see that the hypothesis (1) of the Transversality Theorem is satisfied, in fact $\ker((\partial G/\partial(u,\lambda,\psi))(u,\lambda,\psi,i_\Omega))$ is one dimensional and generated by $(u,0,2\psi)$ since λ is a simple eigenvalue of Δ and $(\Delta+\lambda)$ is injective in $H^4\cap H_0^2(\Omega)$ by Theorem 2.2. Therefore, $\operatorname{ind}(\partial G(u,\lambda,\psi,i_\Omega)/\partial(u,\lambda,\psi)) \le 1$.

We now prove that (2β) also holds, that is, we show that

$$
\dim\left\{\frac{\operatorname{Im}(DG(u,\lambda,\psi,i_\Omega))}{\operatorname{Im}(\partial G(u,\lambda,\psi,i_\Omega)/\partial(u,\lambda,\psi))}\right\} = \infty.
$$

Suppose this is not true and so, there exist $\theta_1, \ldots, \theta_m \in L^2(\Omega) \times H^{-1/2}(\Omega) \times L^2(\partial\Omega)$ such that, for all $\dot{h} \in \mathcal{C}^5(\Omega, \mathbb{R}^n)$ there exist $\dot{u}$, $\dot{\psi}$, $\dot{\lambda}$ and $c_1, \ldots, c_m$ with

$$DG(u, \lambda, \psi, i_\Omega)(\dot{u}, \dot{\lambda}, \dot{\psi}, \dot{h}) = \sum_{j=1}^{n} c_j \theta_j, \tag{3.8}$$

that is

$$\bigg((\Delta + \lambda)(\dot{u} - \dot{h} \cdot \nabla u) + \dot{\lambda} u, (\Delta + \lambda)(\dot{\psi} - \dot{h} \cdot \nabla \psi) + \dot{\lambda}\psi + 2u\dot{u},$$
$$\frac{\partial^3}{\partial N^3}(\dot{\psi} - \dot{h} \cdot \nabla\psi) + (\dot{h} \cdot N)\frac{\partial^4 \psi}{\partial N^4}\bigg) = \sum_{j=1}^{n} c_j \theta_j$$

with $(\dot{u}, \dot{\lambda}, \dot{\psi}, \dot{h}) \in H^2 \cap H^1_0(\Omega) \times \mathbb{R} \times H^4 \cap H^2_0(\Omega) \times \mathcal{C}^5(\Omega, \mathbb{R}^n)$, where $\theta_j = (\theta_j^1, \theta_j^2, \theta_j^3)$.

Define the operators

$$\mathcal{A}_{\Delta+\lambda}: L^2(\Omega) \to H^2 \cap H^1_0(\Omega), \quad \mathcal{S}_{\Delta+\lambda}: H^2(\Omega) \to H^4 \cap H^2_0(\Omega)$$

by

$$\begin{aligned} v &= \mathcal{A}_{\Delta+\lambda} f \quad \text{where } (\Delta+\lambda)v - f \in \ker(\Delta+\lambda), \quad v \perp \ker(\Delta+\lambda) \\ \varphi &= \mathcal{S}_{\Delta+\lambda} g \quad \text{where } (\Delta+\lambda)\varphi - g \in \ker(\Delta+\lambda) \text{ in } H^4 \cap H^2_0(\Omega), \\ &\qquad\qquad\quad \varphi \perp \ker(\Delta+\lambda). \end{aligned}$$

From the first component in (3.8), we obtain

$$\dot{u} - \dot{h} \cdot \nabla u = \xi u + \sum_{j=1}^{m} c_j \mathcal{A}_{\Delta+\lambda} \theta_j^1$$

and similarly for $\dot{\psi} - \dot{h} \cdot \nabla\psi$. Substituting in the third component of (3.8), we conclude that

$$(\dot{h} \cdot N)\frac{\partial^4 \psi}{\partial N^4}$$

belongs to a finite dimensional subspace of $H^{-1/2}(\partial\Omega)$ for each $\dot{h} \in \mathcal{C}^5(\Omega, \mathbb{R}^n)$. But this can only occur (in dimension ≥ 2) if $\partial^4\psi/\partial N^4 \equiv 0$ in $\partial\Omega$.

Now, since $(\Delta + \lambda)\psi = u^2$ in Ω we have

$$\frac{\partial^2}{\partial N^2}(\Delta + \lambda)\psi = \frac{\partial^2}{\partial N^2} u^2$$

on $\partial\Omega$, and so

$$\frac{\partial^2}{\partial N^2}\Delta\psi = \frac{\partial^2}{\partial N^2}u^2 - \lambda\frac{\partial^2 \psi}{\partial N^2} = 2\bigg(\frac{\partial^2 u}{\partial N^2}\bigg)^2 \quad \text{on } \partial\Omega.$$

Observe that

$$
\begin{aligned}
\frac{\partial^2}{\partial N^2}\Delta\psi = \Delta\frac{\partial^2\psi}{\partial N^2} - \sum_{i,j,k=1}^{n}\Bigg[&\frac{\partial^2 N_j}{\partial {x_k}^2}\left(\frac{\partial N_i}{\partial x_j}\frac{\partial\psi}{\partial x_i} + N_i\frac{\partial^2\psi}{\partial x_i\partial x_j}\right)\\
&+ 2\frac{\partial N_i}{\partial x_k}\left(\frac{\partial^2 N_i}{\partial x_k\partial x_j}\frac{\partial\psi}{\partial x_i} + \frac{\partial N_i}{\partial x_j}\frac{\partial^2\psi}{\partial x_i\partial x_k} + \frac{\partial N_i}{\partial x_k}\frac{\partial^2\psi}{\partial x_i\partial x_j}\right.\\
&\left. + N_i\frac{\partial^3\psi}{\partial x_i\partial x_j\partial x_k}\right)\\
&+ N_j\left(\frac{\partial^3 N_i}{\partial {x_k}^2\partial x_j}\frac{\partial\psi}{\partial x_i} + 2\frac{\partial^2 N_i}{\partial x_k\partial x_j}\frac{\partial^2\psi}{\partial x_i\partial x_k} + \frac{\partial^2 N_i}{\partial {x_k}^2}\frac{\partial^2\psi}{\partial x_i\partial x_j}\right.\\
&\left. + 2\frac{\partial N_i}{\partial x_k}\frac{\partial^3\psi}{\partial x_i\partial x_j\partial x_k}\right)\Bigg] = \Delta\frac{\partial^2\psi}{\partial N^2}
\end{aligned}
$$

on $\partial\Omega$, by Lemma 3.5 and, therefore

$$
2\left(\frac{\partial u}{\partial N}\right)^2 = \frac{\partial^2}{\partial N^2}\Delta\psi = \Delta\frac{\partial^2\psi}{\partial N^2} = \Delta_{\partial\Omega}\frac{\partial^2\psi}{\partial N^2} + H\frac{\partial^3\psi}{\partial N^3} + \frac{\partial^4\psi}{\partial N^4} = \frac{\partial^4\psi}{\partial N^4} = 0
$$

on $\partial\Omega$, that is , $\partial u/\partial N = 0$ on $\partial\Omega$. By uniqueness in the Cauchy Problem (Theorem 2.2) $u \equiv 0$, which is a contradiction.

Since the spaces are separable, the hypothesis (3) is automatically satisfied. The result is, therefore, proved. □

Theorem 3.7. *For a generic set of open, connected, bounded $\mathcal{C}^2$-regular regions $\Omega \subset \mathbb{R}^n$, $(n \geq 2)$ the eigenfunctions u of (3.1) satisfy $\int_\Omega u^3 \neq 0$.*

Proof. We prove first that the property holds for any eigenfunction associated to eigenvalues smaller than a fixed natural number M, in a open dense set of $\mathrm{Diff}^3(\Omega)$. The result then follows easily, taking intersection. The openness property is easy to obtain using the continuity of the (simple) eigenfunctions. To prove density, we may first approximate (in the $\mathcal{C}^2$ topology) by a more regular region and then use stronger norms.

Consider the map

$$
\begin{gathered}
F\colon H^2 \cap H_0^1(\Omega) - \{0\} \times (0, M) \times D_M - C_M \to L^2(\Omega) \times \mathbb{R} \times \mathbb{R},\\
(u, \lambda, h) \to \left(h^*(\Delta + \lambda)h^{*-1}u, \int_\Omega u^2 det\ h', \int_\Omega u^3 det\ h'\right).
\end{gathered}
$$

Observe that, by Lemmas 3.2 and 3.6, $D_M - C_M$ is an open dense subset of $\mathrm{Diff}^5(\Omega)$. We wish to apply the Transversality Theorem to conclude that the set

$$
B_M = \{h \in D_M - C_M \mid (0, 1, 0) \in F(H^2 \cap H_0^1(\Omega) - \{0\} \times (0, M), h)\}
$$

is a meager set in $D_M - C_M$ and, therefore, its complement is dense in $\mathrm{Diff}^5(\Omega)$.

We claim first that the operator $\partial F(u,\lambda,h)/\partial(u,\lambda)$ from $H^2 \cap H_0^1(\Omega) \times \mathbb{R} \to L^2(\Omega) \times \mathbb{R} \times \mathbb{R}$ is Fredholm with $\operatorname{ind}(\partial F(u,\lambda,h)/\partial(u,\lambda)) \leq -1$, for all $(u,\lambda,h) \in F^{-1}(0,1,0)$.

Let $(u,\lambda,h) \in F^{-1}(0,1,0)$. Again, we assume without loss of generality that $h = i_\Omega$. Computing the derivatives (using (2.3)), we have

$$\begin{gathered}DF(u,\lambda,i_\Omega)\colon H^2 \cap H_0^1(\Omega) - \{0\} \times \mathbb{R} \times \mathcal{C}^5(\Omega,\mathbb{R}^n) \to L^2(\Omega) \times \mathbb{R} \times \mathbb{R},\\ (\dot u,\dot\lambda,\dot h) \to (DF_1(u,\lambda,i_\Omega), DF_2(u,\lambda,i_\Omega), DF_3(u,\lambda,i_\Omega))(\dot u,\dot\lambda,\dot h),\end{gathered}$$

where

$$\begin{aligned}DF_1(u,\lambda,i_\Omega)(\dot u,\dot\lambda,\dot h) &= (\Delta+\lambda)\dot u + \dot\lambda u + [\dot h\cdot\nabla,(\Delta+\lambda)]u\\ &= (\Delta+\lambda)(\dot u - \dot h\cdot\nabla u) + \dot\lambda u,\\ DF_2(u,\lambda,i_\Omega)(\dot u,\dot\lambda,\dot h) &= \int_\Omega \{2u\dot u + u^2\operatorname{div}(\dot h)\},\\ DF_3(u,\lambda,i_\Omega)(\dot u,\dot\lambda,\dot h) &= \int_\Omega \{3u^2\dot u + u^3\operatorname{div}(\dot h)\},\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial F}{\partial(u,\lambda)}(u,\lambda,i_\Omega)(\dot u,\dot\lambda)\\ &\qquad= \bigg(\frac{\partial F_1}{\partial(u,\lambda)}(u,\lambda,i_\Omega), \frac{\partial F_2}{\partial(u,\lambda)}(u,\lambda,i_\Omega), \frac{\partial F_3}{\partial(u,\lambda)}(u,\lambda,i_\Omega)\bigg)(\dot u,\dot\lambda)\\ &\qquad= \Big((\Delta+\lambda)\dot u + \dot\lambda u, \int_\Omega 2u\dot u, \int_\Omega 3u^2\dot u\Big).\end{aligned}$$

Clearly $\partial F(u,\lambda,i_\Omega)/\partial(u,\lambda)$ is Fredholm, since $\partial F_1(u,\lambda,i_\Omega)/\partial(u,\lambda)$ is Fredholm and F_2, F_3 have finite dimensional range. Observe now that the mapping

$$\bigg(\frac{\partial F_1}{\partial(u,\lambda)}(u,\lambda,i_\Omega), \frac{\partial F_2}{\partial(u,\lambda)}(u,\lambda,i_\Omega)\bigg)\colon H^2 \cap H_0^1(\Omega) \times \mathbb{R} \to L^2(\Omega) \times \mathbb{R} \tag{3.9}$$

is surjective. In fact, given $(f,x) \in L^2(\Omega) \times \mathbb{R}$, let $(v,\xi) \in H^2 \cap H_0^1(\Omega) \times \mathbb{R}$ be defined by

$$v = v_0 + \frac{xu}{2} \quad \text{and} \quad \xi = \int_\Omega uf$$

where $v_0 \in H^2 \cap H_0^1(\Omega)$ satisfy $(\Delta+\lambda)v_0 = f - \xi u$ and $v_0 \perp u$. Note that such a v_0 exists, since $(f - \xi u)\perp u$. Thus

$$\begin{aligned}\frac{\partial F_1}{\partial(u,\lambda)}(u,\lambda,i_\Omega)(\dot u,\dot\lambda) &= (\Delta+\lambda)v + \xi u = f - \xi u + \frac{x}{2}(\Delta+\lambda)u + \xi u = f,\\ \frac{\partial F_2}{\partial(u,\lambda)}(u,\lambda,i_\Omega)(\dot u,\dot\lambda) &= \int_\Omega 2u\Big(v_0 + \frac{x}{2}u\Big) = x\int_\Omega u^2 = x.\end{aligned}$$

Observe also that $(\partial F_1(u,\lambda,i_\Omega)/\partial(u,\lambda), (\partial F_2(u,\lambda,i_\Omega)/\partial(u,\lambda)))$ is injective, since

$$\left(\frac{\partial F_1}{\partial(u,\lambda)}(u,\lambda,i_\Omega), \frac{\partial F_2}{\partial(u,\lambda)}(u,\lambda,i_\Omega)\right)(v,\xi) = (0,0)$$
$$\Leftrightarrow (\Delta+\lambda)v + \xi u = 0 \text{ and } \int_\Omega 2uv = 0.$$

Now $(\Delta+\lambda)v + \xi u = 0 \Rightarrow u(\Delta+\lambda)v + \xi u^2 = 0$ from which $-\int_\Omega u(\Delta+\lambda)v = \xi$ if and only if $\xi = 0$. Therefore, $(\Delta+\lambda)v = 0$ with $\int_\Omega 2uv = 0$, that is, $u \perp v$ and $(\Delta+\lambda)v = 0$. Since λ is a simple eigenvalue associated to u, it follows that $v \equiv 0$. Now, since (3.9) is a continuous surjective operator with domain $H^2 \cap H_0^1(\Omega)$ it follows, from the Closed Graph Theorem, that its inverse is continuous in $L^2(\Omega)$ and thus, (3.9) is an isomorphism so $\partial F(u,\lambda,i_\Omega)/\partial(u,\lambda)$ is not surjective. Furthermore, since its kernel is trivial, we have $\operatorname{ind}(\partial F(u,\lambda,i_\Omega)/\partial(u,\lambda)) \le -1$. Therefore, for all $(u,\lambda,h) \in F^{-1}(0,1,0)$, $\operatorname{ind}(\partial F(u,\lambda,h)/\partial(u,\lambda)) \le -1$, as we wish to show.

Now, by Lemma 3.3 and the definition of C_M, (see also Remark 3.4) it follows that $(0,1,0)$ is a regular value of F. Therefore, by the Transversality Theorem, we conclude that B_M is meager as claimed. The result is, therefore, proved. $\square$

4. A generic property for the eigenfunctions of the Neumann Problem

We now consider the same property of the previous section in the case of Neumann boundary conditions. We show that, generically in the set of open, connected, bounded $\mathcal{C}^3$ regions $\Omega \subset \mathbb{R}^n$ with $n \ge 2$, the normalized eigenfunctions u of

$$\begin{aligned} \Delta u + \lambda u &= 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial N} &= 0 \quad \text{on } \partial\Omega, \\ u &\ne 0, \end{aligned} \tag{4.1}$$

satisfy $\int_\Omega u^3 \ne 0$.

Remark 4.1. We could prove the result for $\mathcal{C}^2$ regions as in the previous section. However, we have chosen to work here in the setting of $\mathcal{C}^3$ regions, which slightly simplify the arguments.

We first observe that the result is trivial if u is a constant eigenfunction and, therefore, we do not need to consider the eigenvalue 0.

Let us define as before the set

$$D_M = \{h \in \operatorname{Diff}^3(\Omega) \mid M \text{ is not an eigenvalue of (4.1) in } h(\Omega) \text{ and all the eigenvalues } \lambda \in (0,M) \text{ in } h(\Omega) \text{ are simple}\}.$$

This is again an open and dense subset of $\mathrm{Diff}^3(\Omega)$. The proof is very similar to the Dirichlet case. However, in the present case we need to consider the following subset of D_M

$$E_M = \{h \in D_M \mid \nabla u \not\equiv 0 \text{ on } \partial\Omega,$$
$$\text{for any eigenfunction associated to an eigenvalue in } (0, M)\}.$$

LEMMA 4.2. *E_M is an open dense subset of* $\mathrm{Diff}^3(\Omega)$.

PROOF. Openness is easy to obtain, by continuity of the eigenfunctions. To prove density, we apply the Transversality Theorem to the map

$$G\colon H^2_N(\Omega) \setminus \{0\} \times (0, M) \times D_M \to L^2(\Omega) \times (L^2(\partial\Omega))^n$$

defined by

$$G(u, \lambda, h) = \bigg(h^*(\Delta + \lambda)h^{*-1}u, h^* \frac{\partial}{\partial x_i} h^{*-1}u|_{\partial h(\Omega)}, 1 \le i \le n\bigg)$$

where

$$H^2_N(\Omega) = \bigg\{u \in H^2(\Omega) \;\bigg|\; \frac{\partial u}{\partial N} = 0 \text{ on } \partial\Omega\bigg\}.$$

Let (u, λ, h) be such that $G(u, \lambda, h) = (0, \ldots, 0)$. As before, we may assume $h = i_\Omega$. Now, the kernel of $\partial G(u, \lambda, h)/\partial(u, \lambda)$ is finite-dimensional. Therefore, to use the Transversality Theorem, we need to prove that

$$\dim \bigg\{\frac{\mathrm{Im}\,(DG(u, \lambda, h))}{\mathrm{Im}\,(\partial G(u, \lambda, h)/\partial(u, \lambda))}\bigg\} = \infty. \tag{4.2}$$

The partial derivative $(\partial G/\partial(u, \lambda))(u, \lambda, i_\Omega)\colon H^2_N(\Omega) \times \mathbb{R} \to L^2(\Omega) \times (L^2(\partial\Omega))^n$ is given by

$$\frac{\partial G}{\partial(u, \lambda)}(u, \lambda, i_\Omega)(\,\cdot\,) = \bigg(\frac{\partial G_1}{\partial(u, \lambda)}(u, \lambda, i_\Omega), \frac{\partial G_{i+1}}{\partial(u, \lambda)}(u, \lambda, i_\Omega), 1 \le i \le n\bigg)(\,\cdot\,)$$

where

$$\begin{aligned}
\frac{\partial G_1}{\partial(u, \lambda)}(u, \lambda, i_\Omega)(\dot u, \dot\lambda, \dot h) &= (\Delta + \lambda)\dot u + \dot\lambda u,\\
\frac{\partial G_{i+1}}{\partial(u, \lambda)}(u, \lambda, i_\Omega)(\dot u, \dot\lambda, \dot h) &= \frac{\partial \dot u}{\partial x_i}\bigg|_{\partial\Omega}, \quad 1 \le i \le n.
\end{aligned}$$

On the other hand, $DG(u, \lambda, i_\Omega)$ defined from $H^2_N(\Omega) \times \mathbb{R} \times \mathcal{C}^3(\Omega, \mathbb{R}^n)$ into $L^2(\Omega) \times (L^2(\partial\Omega))^n$ is given by

$$DG(u, \lambda, i_\Omega)(\,\cdot\,) = (DG_1(u, \lambda, i_\Omega), DG_{i+1}(u, \lambda, i_\Omega),\ 1 \le i \le n)(\,\cdot\,)$$

where

$$\begin{aligned}
DG_1(u, \lambda, i_\Omega)(\dot u, \dot\lambda, \dot h) &= (\Delta + \lambda)(\dot u - \dot h \cdot \nabla u) + \dot\lambda u,\\
DG_{i+1}(u, \lambda, i_\Omega)(\dot u, \dot\lambda, \dot h) &= \bigg\{\frac{\partial}{\partial x_i}(\dot u - \dot h \cdot \nabla u) + \dot h \cdot \nabla\bigg(\frac{\partial u}{\partial x_i}\bigg)\bigg\}\bigg|_{\partial\Omega},
\end{aligned}$$

for $1 \leq i \leq n$. Suppose (4.2) is false, that is, there exist $\theta_1, \dots, \theta_m \in L^2(\Omega) \times (L^2(\partial\Omega))^n$ such that, for any $\dot{h} \in \mathcal{C}^3(\Omega, \mathbb{R}^n)$ there exist $\dot{u}$, $\dot{\lambda}$ and $c_1, \dots, c_m$ with

$$DG(u, \lambda, i_\Omega)(\dot{u}, \dot{\lambda}, \dot{h}) = \sum_{j=1}^{n} c_j \theta_j, \tag{4.3}$$

where $\theta_j = (\theta_j^1, \dots, \theta_j^{n+1})$. Define the operator

$$\mathcal{L}_{\Delta+\lambda}: L^2(\Omega) \to H_N^2(\Omega) \tag{4.4}$$

by

$$v = \mathcal{L}_{\Delta+\lambda} f \quad \text{where } (\Delta + \lambda)v - f \in \ker(\Delta + \lambda) \text{ in } H_N^2(\Omega),\ v \perp \ker(\Delta + \lambda).$$

We obtain, from the first equation in (3.8),

$$\dot{u} - \dot{h} \cdot \nabla u = \xi u + \sum_{j=1}^{m} c_j \mathcal{L}_{\Delta+\lambda} \theta_j^1.$$

Substituting in the $(i+1)$-th component of (4.3), we conclude that

$$\dot{h} \cdot \nabla \left(\frac{\partial u}{\partial x_i} \right) \Big|_{\partial\Omega}$$

belongs to a finite dimensional subspace of $L^2(\partial\Omega)$ when $\dot{h}$ varies in $\mathcal{C}^3(\Omega, \mathbb{R}^n)$. But this can only happen (in $\dim \Omega \geq 2$) if $\nabla(\partial u/\partial x_i) \equiv 0$ in $\partial\Omega$, for $1 \leq i \leq n$, that is, $\partial^2 u/\partial x_i \partial x_j \equiv 0$ in $\partial\Omega$ for $1 \leq i, j \leq n$. Therefore, for each $1 \leq i \leq n$, $\partial u/\partial x_i$ satisfies (4.1) in Ω and $\partial u/\partial x_i = 0$ on $\partial\Omega$. By uniqueness in the Cauchy problem, we have $\partial u/\partial x_i = 0$ in Ω and so u is constant in Ω contradicting the hypothesis. Since our spaces are separable, the hypothesis (3) of the Transversality Theorem is verified, and the result claimed follows. □

THEOREM 4.3. *For a generic set of open, connected, bounded $\mathcal{C}^3$-regular regions $\Omega \subset \mathbb{R}^n$, $(n \geq 2)$ the eigenfunctions u of* (4.1) *satisfy $\int_\Omega u^3 \neq 0$.*

PROOF. We prove first that the property holds for any eigenfunction associated to eigenvalues smaller than a fixed natural number M, in a open dense set of $\mathrm{Diff}^3(\Omega)$. The result then follows easily, taking intersection. The openness property is, as in lemma (4.2), easy to obtain. To prove density, we again use the Transversality Theorem.

Consider the mapping

$$F: H_N^2(\Omega) \times (0, M) \times E_M \to L^2(\Omega) \times \mathbb{R} \times \mathbb{R},$$
$$(u, \lambda, h) \to \left(h^*(\Delta + \lambda) h^{*-1} u, \int_\Omega u^2 \det h', \int_\Omega u^3 \det h' \right).$$

We wish to prove that the set $\{h \in E_M \mid (0,1,0) \in F(H_N^2(\Omega) - \{0\} \times (0, M), h)\}$ is a meager set in E_M and, therefore, in $\mathrm{Diff}^3(\Omega)$.

We claim first that $\partial F(u,\lambda,h)/\partial(u,\lambda)$ is Fredholm, with $\operatorname{ind}(\partial F(u,\lambda,h)/\partial(u,\lambda)) \le -1$ for all $(u,\lambda,h) \in F^{-1}(0,1,0)$. The proof is almost the same as the one in Theorem 3.7. We need to prove that hypotheses (2)(α) of the Transversality Theorem is satisfied. Suppose it is not, and $(u,\lambda,i_\Omega) \in H^2_N(\Omega) \times (0,M) \times E_M$ is a critical point, with $F(u,\lambda,i_\Omega) = (0,1,0)$. Then, there exists $(\psi,\alpha,\beta) \in L^2(\Omega) \times \mathbb{R} \times \mathbb{R}$ orthogonal to $\operatorname{Im} DF(u,\lambda,i_\Omega)$, that is,

$$0 = \int_\Omega \{\psi[(\Delta+\lambda)(\dot{u}-\dot{h}\cdot\nabla u)+\dot{\lambda}u]+\alpha[2u\dot{u}+u^2\operatorname{div}(\dot{h})]+\beta[3u^2\dot{u}+u^3\operatorname{div}(\dot{h})]\} \tag{4.5}$$

for all $(\dot{u},\dot{\lambda},\dot{h}) \in H^2_N(\Omega) \times \mathbb{R} \times \mathcal{C}^3(\Omega,\mathbb{R}^n)$.

If $\dot{u} = \dot{h} = 0$ in (4.5) then $\int_\Omega \psi u = 0$. If $\dot{h} = \dot{\lambda} = 0$, then

$$\int_\Omega \{\psi(\Delta+\lambda)\dot{u} + 2\alpha u\dot{u} + 3\beta u^2\dot{u}\} = 0 \quad \text{for all } \dot{u} \in H^2_N. \tag{4.6}$$

If we take $\dot{u} = u$ in (4.6), then $\alpha = 0$ and by regularity of solutions in the Cauchy problem we conclude that $\psi \in H^2_N(\Omega) \cap \mathcal{C}^2_\alpha(\Omega)$ for all $0 < \alpha < 1$ and satisfies

$$(\Delta+\lambda)\psi = -3\beta u^2 \quad \text{in } \Omega.$$

If now we take $\dot{u} = \dot{\lambda} = 0$ in (4.5) then, since $\alpha = 0$

$$0 = -\int_\Omega \psi(\Delta+\lambda)(\dot{h}\cdot\nabla u) + \int_\Omega \beta u^3 \operatorname{div}(\dot{h}) \tag{4.7}$$

for all $\dot{h} \in \mathcal{C}^3(\Omega,\mathbb{R}^n)$. Now, we have

$$\begin{aligned}
&\int_\Omega \psi(\Delta+\lambda)(\dot{h}\cdot\nabla u)\\
&\quad= \int_\Omega (\dot{h}\cdot\nabla u)(\Delta+\lambda)\psi + \int_{\partial\Omega} \psi\frac{\partial}{\partial N}(\dot{h}\cdot\nabla u) - (\dot{h}\cdot\nabla u)\frac{\partial\psi}{\partial N}\\
&\quad= -\int_\Omega 3\beta u^2(\dot{h}\cdot\nabla u) + \int_{\partial\Omega} \psi\frac{\partial}{\partial N}(\dot{h}\cdot\nabla u)\\
&\quad= \int_\Omega \beta\{u^3\operatorname{div}(\dot{h}) - \operatorname{div}(u^3\dot{h})\} + \int_{\partial\Omega} \psi\frac{\partial}{\partial N}(\dot{h}\cdot\nabla u)\\
&\quad= \int_\Omega \beta u^3\operatorname{div}(\dot{h}) + \int_{\partial\Omega} \{\psi\frac{\partial}{\partial N}(\dot{h}\cdot\nabla u) - \beta u^3(\dot{h}\cdot N)\}.
\end{aligned}$$

Substituting in (4.7), we obtain

$$\int_{\partial\Omega} \{\beta u^3(\dot{h}\cdot N) - \psi\frac{\partial}{\partial N}(\dot{h}\cdot\nabla u)\} = 0 \tag{4.8}$$

for all $\dot{h} \in \mathcal{C}^3(\Omega,\mathbb{R}^n)$.

If τ is any vector field in $\mathcal{C}^3(\Omega,\mathbb{R}^n)$ with $\tau \perp N = 0 \in \partial\Omega$, and $\dot h = g\tau$, for some $g \in \mathcal{C}^3(\Omega,\mathbb{R})$, $g \equiv 0$ in $\partial\Omega$ then

$$\begin{aligned} 0 &= \int_{\partial\Omega} \left\{ \beta u^3 (\dot h \cdot N) - \psi \frac{\partial}{\partial N} (\dot h \cdot \nabla u) \right\} = - \int_{\partial\Omega} \psi \frac{\partial}{\partial N} (\dot h \cdot \nabla u) \\ &= - \int_{\partial\Omega} \psi \left\{ \frac{\partial g}{\partial N} \frac{\partial u}{\partial \tau} + g \frac{\partial}{\partial N} (\tau \cdot \nabla u) = - \int_{\partial\Omega} \psi \frac{\partial g}{\partial N} \frac{\partial u}{\partial \tau}. \right. \end{aligned}$$

Since $\partial g/\partial N$ can be arbitrily chosen in $\partial\Omega$ and $\nabla u \not\equiv 0$ we must have

$$\psi \equiv 0 \tag{4.9}$$

in a neighbourhood of $\partial\Omega$.

On the other hand, if $\dot h = gN$, we have

$$\begin{aligned} 0 &= \int_{\partial\Omega} \left\{ \beta u^3 (\dot h \cdot N) - \psi \frac{\partial}{\partial N} (\dot h \cdot \nabla u) \right\} \\ &= \int_{\partial\Omega} \beta u^3 g - \psi \frac{\partial g}{\partial N} \frac{\partial u}{\partial N} - \psi g \frac{\partial^2 u}{\partial N^2} = \int_{\partial\Omega} \left(\beta u^3 - \psi \frac{\partial^2 u}{\partial N^2} \right) g \end{aligned}$$

for any $g \in \mathcal{C}^3(\Omega,\mathbb{R})$. Therefore, we must have

$$\beta u^3 - \psi \frac{\partial^2 u}{\partial N^2} = 0 \quad \text{on } \partial\Omega. \tag{4.10}$$

But then, it follows from (4.9) and (4.10) that $u \equiv 0$ in a neighbourhood of $\partial\Omega$ and, by uniqueness in the Cauchy problem $u \equiv 0$, a contradiction. The result is, therefore, proved. □

5. Appendix. A Proof of the Transversality Theorem

For the sake of completeness we give here a proof of Theorem 2.7. Apart from a change of order and some other minor modification the proof is the same as in [4].

Lemma 5.1. *Suppose $f(x_0,y_0) = \xi$, $(\partial f/\partial x)(x_0,y_0)$ is left-Fredholm and f is continuously differentiable on a neighbourhood W_0 of (x_0,y_0). Then there is a neighbourhood W of (x_0,y_0) such that $\overline{W} \subset W_0$ and $(x,y) \to y\colon f^{-1}(\xi) \cap \overline{W} \to Y$ is proper.*

Proof. The result is local, so we may assume X, Y, Z are Banach spaces. Now $L := (\partial f/\partial x)(x_0,y_0)$ is left-Fredholm so $X_1 = \ker L$ is finite dimensional and splits $X = X_1 \oplus X_2$, and the restriction of L is an isomorphism from X_2 onto $\operatorname{Im} L$, with a continuous inverse since $\operatorname{Im} L$ is closed. There exists $C_0 > 0$ so $|Lx_2| \geq C_0 |x_2|$ for all $x_2 \in X_2$. Also, if $K = 1 + \|\partial f(x_0,y_0)/\partial y\|$, there is a bounded neighbourhood W of (x_0, y_0), so small that $\overline{W} \subset W_0$ and

$$\begin{aligned}|f(x,y) - f(u,v) - L(x-u)| \\ &= \Big|f(x,y) - f(u,v) - L(x-u) \pm \frac{\partial f}{\partial y}(x_0,y_0)(y-v)\Big|, \\ &\le \frac{C_0}{2}|x-u| + K|y-v|\end{aligned}$$

(x,y), $(u,v) \in W$. Now suppose $\{(x^n, y^n)\}_{n\ge 1}$ is a sequence in $f^{-1}(\xi) \cap \overline{W}$ such that $\{y^n\}$ converges. Then x_1^n, the component of x^n in X_1, is bounded in a finite-dimensional space and has a convergent subsequence, in fact, we suppose that $\{x_1^n\}$ converges.

$$\begin{aligned}C_0|x_2^n - x_2^m| &\le |L(x_2^n - x_2^m)| \\ &\le |L(x^n - x^m)| = |f(x^n,y^n) - f(x^m,y^m) - L(x^n - x^m)| \\ &\le \frac{C_0}{2}|x^n - x^m| + K|y^n - y^m| \\ &\le \frac{C_0}{2}(|x_1^n - x_1^m| + |x_2^n - x_2^m|) + K|y^n - y^m|\end{aligned}$$

so

$$|x_2^n - x_2^m| \le |x_1^n - x_1^m| + \frac{2K}{C_0}|y^n - y^m|$$

that converges for 0 as $n, m \to \infty$. Thus $\{x^n\}$ converges, which proves the lemma. $\square$

Remark 5.2. Next we show $f^{-1}(\xi)$ Lindelöf implies (3). Indeed, by Lemma 5.1, each point of $f^{-1}(\xi)$ has an open neighbourhood $W \subset \overline{W} \subset A$ such that $(x,y) \to y$: $f^{-1}(\xi) \cup \overline{W}$ is proper. By hypothesis, there is a countable subcover $\{f^{-1}(\xi) \cup \overline{W}\}_{j=1}^{\infty}$ so (3) holds with $M_j = \{f^{-1}(\xi)\}$.

Lemma 5.3. *Let k, $m = 1$, X, Y, Z, A, f, ξ be given as in the Transversality Theorem, $(x_0, y_0) \in f^{-1}(\xi)$, and assume hypothesis* (1) *and* (2), (α) *or* (β), *hold at* (x_0, y_0). *Then there exists open neighbourhoods U of x_0, V of y_0, and an open dense subset $V^0 \subset V$, such that $\overline{U} \times \overline{V} \subset A$ and ξ is a regular value of $f(\,\cdot\,, y)|_U$ whenever $y \in V^0$.*

Proof. Since the result is local, near $(x_0, y_0) \in X \times Y$ and $\xi \in Z$, we may assume X, Y, Z are Banach spaces, $x_0 = 0$, $y_0 = 0$, $\xi = 0$, f is $\mathcal{C}^k$ on a neighbourhood of $(0,0) \in X \times Y$, $f(x,y) = Lx + My + o(|x| + |y|)$, L is semi-Fredholm with index less then k, and either

(α) $\operatorname{Im}(L, M) = \{Lx + My \mid \text{for all } x, y\} = Z$ or

(β) $\dim\{\operatorname{Im}(L,M)/\operatorname{Im} L\} > \dim\ker L$.

Since $(\alpha) \Rightarrow (\beta)$ for negative index, it is enough to prove the result in case (α) when $\operatorname{ind} L \ge 0$ and in case (β) when $\operatorname{ind} L < 0$.

Case (α). $\operatorname{ind} L \geq 0$, $\operatorname{Im}(L, M) = Z$. L is Fredholm so $X = X_1 \oplus X_2$, $Z = Z_1 \oplus Z_2$, $X_1 = \ker L$, $Z_2 = \operatorname{Im} L$, $L_2 := L|_{X_2}\colon X_2 \to Z_2$ is a isomorphism, and $\operatorname{ind} L = \dim X_1 - \dim Z_1$. The complement Z_1 to $\operatorname{Im} L$ is not unique and we may choose $Z_1 \subset \operatorname{Im} M$. Then there is a subspace $Y_1 \subset Y$ so $M_1 := M|_{Y_1}\colon Y_1 \to Z_1$ is an isomorphism; defining $Y_2 = M^{-1} Z_2$, it follows that $Y = Y_1 \oplus Y_2$. Writing f in terms of its components in these spaces,

$$f(x_1, x_2, y_1, y_2) = (M_1 y_1 + g(x, y), L_2 x_2 + h(x, y))$$

where g, h are $\mathcal{C}^k$ and g, g_x, g_y, h, h_x all vanish at $(0,0)$, but perhaps $h_y \neq 0$. By the implicit function theorem, we may solve $f(x,y) = (0,0)$ for $y_1 = \phi(x_1, y_2)$ and $x_2 = \psi(x_1, y_2)$ with ϕ, ψ of class $\mathcal{C}^k$ near $(0,0)$ since

$$\begin{aligned} &Df_{x_2,y_1}(0,0)\colon X_2 \oplus Y_1 \to Z_1 \oplus Z_2, \\ &(\dot{x_2}, \dot{y_1}) \to (M_1 \dot{y_1}, L_2 \dot{x_2} + h_{y_1}(0,0)\dot{y_1}) \end{aligned}$$

is an isomorphism. In matrix form, $(\partial f/\partial x)(x,y)$ is

$$\begin{pmatrix} g_{x_1} & g_{x_2} \\ h_{x_1} & L_2 + h_{x_2} \end{pmatrix} = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta & 0 \\ 0 & L_2 + h_{x_2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}$$

where $p\colon Z_2 \to Z_1$, $q\colon X_1 \to X_2$ and $\Delta\colon X_1 \to Z_1$ are defined by $p = g_{x_2}(L_2 + h_{x_2})^{-1}$, $q = g_{x_2}(L_2 + h_{x_2})^{-1}$ and $\Delta = g_{x_1} - g_{x_2}(L_2 + h_{x_2})^{-1} h_{x_1}$. Then $\partial f/\partial x$ is surjective if and only is Δ is surjective. Now, by the definition of ϕ and ψ, we have near $(0,0)$

$$\begin{aligned} M_1 \phi(x_1, y_2) + g(x_1, \psi(x_1, y_2), \phi(x_1, y_2), y_2) &= 0, \\ L_2 \phi(x_1, y_2) + h(x_1, \psi(x_1, y_2), \phi(x_1, y_2), y_2) &= 0, \end{aligned}$$

which implies

$$\begin{aligned} M_1 \phi_{x_1} + g_{x_1} + g_{x_2} \psi_{x_1} + g_{y_1} \phi_{x_1} &= 0, \\ L_2 \phi_{x_1} + h_{x_1} + h_{x_2} \psi_{x_1} + h_{y_1} \phi_{x_1} &= 0. \end{aligned}$$

Now, by (5.2), $\psi_{x_1} = -(L_2 + h_{x_2})^{-1}[h_{x_1} + h_{y_1}\phi_{x_1}]$. Substitution in (5.1) gives

$$\{M_1 + g_{y_1} - g_{x_2}(L_2 + h_{x_2})^{-1} h_{y_1}\}\phi_{x_1} + \Delta = 0$$

and the coefficient of ϕ_{x_1} in equation above is an isomorphism when we are close to $(0,0)$.

Thus in a neighbourhood $\max\{|y_1|, |y_2|\} < \delta$ of $y = 0$ in Y, 0 is regular value of $f(\,\cdot\,, y)|_{\max\{|x_1|,|x_2|\}<\varepsilon}$, with $y = y_1 + y_2$, if and only if y_1 is a regular value of $\phi(\,\cdot\,, y_2)|_{|x_2|<\varepsilon}$. Since $\phi(\,\cdot\,, y_2)\colon X_1 \to Y_1$ is $\mathcal{C}^k$ near 0 and $k > \dim X_1 - \dim Y_1$, Sard‘s theorem says, for every small y_2, there is a dense set of y_1 such that 0 is a regular value of $f(\,\cdot\,, y_1 + y_2)$ on $\{|x_1| < \varepsilon, |x_2| < \varepsilon\}$. This proves that V_0 is dense.

Now, suppose that $V_0 \subset Y$ is not open. Then there is $(x_n, y_n) \in X \times Y$, $y_n \to y_0 \in V_0 \subset Y$, with $f(x_n, y_n) = 0$ and (x_n, y_n) is a critical point of f. By Lemma 5.1, we can suppose that $(x_n, y_n) \to (x_0, y_0)$ with $x_0 \in X$ and $f(x_0, y_0) = 0$. Since (x_n, y_n) is a critical point for all $n \in \mathbb{N}$, we have (x_0, y_0) is a critical point, a contradiction. Thus V_0 is open.

Case (β). $\operatorname{ind} L < 0$, $k = 1$, $\dim\{\operatorname{Im}(L, M)/\operatorname{Im} L\} > \dim\ker L$. Let $n = \dim\ker L$. There exist $\{f_1, \ldots, f_{n+1}\}$ in $\operatorname{Im}(L, M)$ independent relative to $\operatorname{Im} L$, i.e. $\sum_{i=1}^{n+1} c_i f_i \in \operatorname{Im} L$ implies all $c_i = 0$. Then $f_i = Lx_i + My_i$ where $\{y_1, \ldots, y_{n+1}\}$ are linearly independent in Y, a basis for subspace $Y_1 \subset Y$ such that $MY_1 \cap \operatorname{Im} L = \{0\}$ and M is injective on Y_1. Let $Z_1 = MY_1$ and choose $Z_2 \supset \operatorname{Im} L$ so that $Z = Z_1 \oplus Z_2$. Let $Y_2 = M^{-1}Z_2$; then $Y = Y_1 \oplus Y_2$. Also let $X_1 = \ker L$, $X = X_1 \oplus X_2$, so $n = \dim X_1 < \dim Z_1 = \dim Y_1$.

Now f has the form

$$f(x_1, x_2, y_1, y_2) = (M_1 y_1 + g(x, y), L_2 x_2 + h(x, y))$$

where $M_1 = M|_{Y_1}\colon Y_1 \to Z_1$ is an isomorphism, and $L_2 = L|_{X_2}\colon X_2 \to Z_2$ is injective with closed image. Further, g, h are $\mathcal{C}^1$ and at $(0,0)$, g, g_x, g_y, h, h_x all vanish. By the implicit function theorem, we may solve $M_1 y_1 + g(x, y) = 0$ for $y_1 = \psi(x, y_2)$ a $\mathcal{C}^1$ function with $\psi = 0$ and $\psi_x = 0$ at the origin.

Choose small $\delta > 0$. Fix $y_2 \in Y_2$, $|y_2| < \delta$, and let

$$S_{y_2} = \{x \in X \mid |x_1| \le \delta,\ |x_2| \le \delta,\ f(x_1, x_2, \psi(x, y_2), y_2) = 0\}.$$

Also let $P_1\colon X \to X_1$ be the projection on X_1 and $\pi_{y_2} = P_1|_{S_{y_2}}\colon S_{y_2} \to X_1$. If δ is small, π_{y_2} is injective with Lipschitz inverse. Assuming this

$$\psi(S_{y_2}, y_2) = \psi(\pi_{y_2}^{-1} \circ \pi_{y_2}(S_{y_2}), y_2) \subset Y_1$$

is the Lipschitz image of a set in X_1, and $\dim X_1 < \dim Y_1$. So $\psi(S_{y_2}, y_2)$ has measure zero in Y_1. Thus given any $|y_1| < \delta$, $|y_2| < \delta$, there exist $\widetilde{y_1}$ arbitrarily close to y_1 but outside $\psi(S_{y_2}, y_2)$, hence $f(x_1, x_2, \widetilde{y_1}, y_2) \ne 0$ for all $|x_1| < \delta$, $|x_2| < \delta$. Openess follows from Lemma 5.1 as above, so it only remains to show π_{y_2} has Lipschitz inverse.

Now $|Lx_2| \ge c_0|x_2|$ for all $x_2 \in X_2$ and some constant $c_0 > 0$. Since $\psi_x(0,0) = 0$, for sufficiently small $\delta > 0$ and $|x| \le \delta$, $|\widetilde{x}| \le \delta$, $|y_2| \le \delta$

$$|f(x, \psi(x, y_2), y_2) - f(\widetilde{x}, \psi(\widetilde{x}, y_2), y_2) - L(x - \widetilde{x})| \le \frac{c_0}{2}|x - \widetilde{x}|.$$

If also $x, \widetilde{x} \in S_{y_2}$ then

$$c_0|x_2 - \widetilde{x_2}| \le |L(x - \widetilde{x})| \le \frac{c_0}{2}|x - \widetilde{x}| \le \frac{c_0}{2}(|x_1 - \widetilde{x_1}| + |x_2 - \widetilde{x_2}|)$$

so $|x_2 - \widetilde{x_2}| \le |x_1 - \widetilde{x_1}| = |\pi_{y_2} x - \pi_{y_2}\widetilde{x_2}|$ implies $|x - \widetilde{x}| \le 2|\pi_{y_2} x - \pi_{y_2}\widetilde{x}|$, which completes the proof. $\square$

5.1. Proof of the Transversality Theorem. Assuming the Lemma 5.3 and that $(x,y) \to y\colon f^{-1}(\xi) \to Y$ is proper, we prove

$$Y_{\mathrm{crit}} = \{y \in Y \mid \xi \text{ is a critical value of } f(\,\cdot\,,y)\colon A_y \to Z\}$$

is closed and without interior.

Let $\{y_n\}_{n\geq 1}$ be a sequence in this set which converges in Y; for each n, there exist $x_n \in X$ so $(x_n,y_n) \in f^{-1}(\xi)$ and x_n is a critical point of $f(\,\cdot\,,y_n)$. By hypothesis, we may suppose $(x_n,y_n) \to (x,y)$, so $(x,y) \in f^{-1}(\xi)$. If $\partial f(x,y)/\partial x$ were onto, then so would be $(\partial f/\partial x)(x_n,y_n)$ for n large; hence x is critical point of $f(\,\cdot\,,y)$ and closeness is proved. It remains to show that for each $y \in Y$, there exists $\widetilde{y}$ arbitrarily close to y such that ξ is a regular value of $f(\,\cdot\,,\widetilde{y})$.

Let $K_y = \{x \in X \mid f(x,y) = \xi\}$, by the properness assumption, this is a compact set. By Lemma 5.3, for each $x \in K_y$ there are open sets U_x, V_x neighbourhood of x and y respective and V_x^0, an open dense subset of V_x, such that $\overline{U_x} \times \overline{V_x} \subset A$ and $f(\,\cdot\,,\widetilde{y})|_{U_x}$ has ξ as a regular value for all $\widetilde{y} \in V_x^0$. Choose a finite subcover $U_{x_1},\dots,U_{x_N}$ for K_y and let $\widetilde{U} = \bigcup_{i=1}^N U_{x_i}$, $\widetilde{V} = \bigcap_{i=1}^N V_{x_i}$ and $\widetilde{V^0} = \widetilde{V^0_{x_i}}$. $\widetilde{V^0}$ is open and dense in $\widetilde{V}$, $\widetilde{V}$ and $\widetilde{U}$ are open, $y \in \widetilde{V}$, $K_y \subset \widetilde{U}$, and ξ is a regular value of $f(\,\cdot\,,\widetilde{y})$ for all $\widetilde{y} \in \widetilde{V^0}$ sufficiently close to y. Otherwise there would exist $y_n \to y$, $y_n \in \widetilde{V^0}$, and critical points x_n of $f(\,\cdot\,,y_n)$ with $(x_n,y_n) \in f^{-1}(\xi)$, such that $\lim_{n\to\infty} x_n = x$ exists, then $(x,y) \in f^{-1}(\xi)$, $x \in K_y$, and $x_n \in \widetilde{U}$, $y_n \in \widetilde{V^0}$ for n large, so x_n is not a critical point of $f(\,\cdot\,,y_n)$, a contradiction.

If Lemma 5.3 and (3) hold, the same argument shows

$$\{y \in Y \mid \text{ there is a critical point } x \text{ of } f(\,\cdot\,,y) \text{ with } (x,y) \in M_j \subset f^{-1}(\xi)\}$$

is closed and nowhere dense for each $j \in \mathbb{N}$. Hence the union of these,

$$\{y \in Y \mid \text{ there is a critical point } x \text{ of } f(\,\cdot\,,y) \text{ with } (x,y) \in f^{-1}(\xi)\}$$

is meager. This completes the first step of the demonstration of the theorem.

Now, we show the case $m > 1$ of Transversality Theorem may be reduced to the case $m = 1$, by change of variables. Suppose therefore $m > 1$, $k = 1$ and $\widetilde{(2)(\beta)}$ holds and let $\widetilde{X} = X \times S^{m-1}$, $\widetilde{Y} = \mathcal{C}^1(S^{m-1},Y)$, $\widetilde{A} = \{(x,t,\widetilde{y}) \in \widetilde{X} \times \widetilde{Y} \mid (x,\widetilde{y(t)}) \in A\}$ and $\widetilde{f}\colon \widetilde{A} \to Z\colon (x,t,\widetilde{y}) \to f(x,\widetilde{y(t)})$. Then $\widetilde{f}$ is $\mathcal{C}^1$ and the new problem satisfies the same hypothesis as the original problem, except that m is replaced by 1. If (3) holds for the original problem, it also holds for the new problem. If $f(x,y) = \xi$ and $y = \widetilde{y(t)}$, so $\widetilde{f}(x,t,\widetilde{y}) = \xi$, we choose a maximal subset $\{\dot{t}_1,\dots,\dot{t}_q\} \subset T_t(S^{m-1})$ so $\{(\partial f\widetilde{y}'/\partial y)\cdot \dot{t}_i\}_{i=1}^q$ are independent relative to $\operatorname{Im}(\partial f/\partial x)$. Then

$$
\begin{aligned}
\dim\ker\frac{\partial\widetilde{f}}{\partial(x,t)} &= \dim\ker\frac{\partial f}{\partial x} + \dim\ker\frac{\partial f}{\partial y}\widetilde{y}' - \dim\left(\operatorname{Im}\frac{\partial f}{\partial x}\cap\operatorname{Im}\frac{\partial f}{\partial y}\widetilde{y}'\right)\\
&= \dim\ker\frac{\partial f}{\partial x} + m - 1 - \widetilde{q} + \widetilde{q} - q = \dim\ker\frac{\partial f}{\partial x} + m - 1 - q
\end{aligned}
$$

where $\widetilde{q}$ is the rank of $\partial f\widetilde{y}'/\partial y < \infty$. Then we choose $\{\dot{y}_1,\dots,\dot{y}_p\}\subset T_yY$ so $\{\partial f\dot{y}/\partial y\}$ are independent relative to

$$
\operatorname{Im}\frac{\partial\widetilde{f}}{\partial(x,t)} = \operatorname{Im}\frac{\partial f}{\partial x} + \operatorname{Im}\frac{\partial f}{\partial y}\widetilde{y}'(t).
$$

By (2)(β), we may assume $p+q\geq m+\dim\ker(\partial f/\partial x)$, that is

$$
\dim\left\{\frac{\operatorname{Im}D\widetilde{f}}{\operatorname{Im}(\partial\widetilde{f}/\partial(x,t))}\right\}\geq p\geq m-q+\dim\ker\frac{\partial f}{\partial x}\geq 1+\dim\ker\frac{\partial\widetilde{f}}{\partial(x,t)}.
$$

Thus, assuming the theorem is valid for $m=1$, we see

$$
\{\widetilde{y}\in\mathcal{C}^1(S^{m-1},Y)\mid f(x,\widetilde{y}(t))=\xi\text{ for some }x,\,t\text{ with }(x,\widetilde{y}(t))\in A\}
$$

is a meager set in $\mathcal{C}^1(S^{m-1},Y)$, which means $\mathcal{C}^1$-codimen$\{y\in Y\mid\xi\in f(A_y,y)\}$ $> m-1$, the $\mathcal{C}^1$-codimension is $\geq m$ and so the codimension is $\geq m$, which completes the proof.

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