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THREE SOLUTIONS FOR A NEUMANN PROBLEM

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Dedicated to Professor Andrzej Granas

Abstract. In this paper we consider a Neumann problem of the type

$$(\mathrm{P}_\lambda) \qquad \begin{cases} -\Delta u = \alpha(x)(|u|^{q-2}u - u) + \lambda f(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Applying the theory developed in [13], we establish, under suitable assumptions, the existence of an open interval $\Lambda \subseteq \mathbb{R}$ and of a positive real number ϱ, such that, for each $\lambda \in \Lambda$, problem (P_λ) admits at least three weak solutions in $W^{1,2}(\Omega)$ whose norms are less than ϱ.

Let us recall that a Gâteaux differentiable functional J on a real Banach space X is said to satisfy the Palais–Smale condition if each sequence $\{x_n\}$ in X such that $\sup_{n\in\mathbb{N}} |J(x_n)| < \infty$ and $\lim_{n\to\infty} \|J'(x_n)\|_{X^*} = 0$ admits a strongly converging subsequence.

In [13], we proved the following result:

Theorem A ([13, Theorem 3]). *Let X be a separable and reflexive real Banach space, $I \subseteq \mathbb{R}$ an interval, and $g\colon X \times I \to \mathbb{R}$ a continuous function satisfying the following conditions:*

(i) *for each $x \in X$, the function $g(x, \cdot)$ is concave,*

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(ii) *for each* $\lambda \in I$, *the function* $g(\cdot, \lambda)$ *is sequentially weakly lower semicontinuous and continuously Gâteaux differentiable, satisfies the Palais–Smale condition and*

$$\lim_{\|x\|\to\infty} g(x,\lambda) = \infty,$$

(iii) *there exists a continuous concave function* $h: I \to \mathbb{R}$ *such that*

$$\sup_{\lambda\in I}\inf_{x\in X}(g(x,\lambda)+h(\lambda)) < \inf_{x\in X}\sup_{\lambda\in I}(g(x,\lambda)+h(\lambda)).$$

Then, there exist an open interval $\Lambda \subseteq I$ *and a positive real number* ϱ, *such that, for each* $\lambda \in \Lambda$, *the equation*

$$g'_x(x,\lambda) = 0$$

admits at least three solutions in X *whose norms are less than* ϱ.

There are already several applications of Theorem A to nonlinear boundary value problems (see [2]–[7], [9], [11]–[13]; see also [8], and [10] for the non-smooth case). In the mentioned papers, to satisfy the key assumption (iii), one assumes that the involved nonlinearities have a suitable behaviour in some neighbourhood of 0.

The aim of the present paper is to offer an application of Theorem A to a Neumann problem where no assumption of local character is made. Our result is as follows:

Theorem 1. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open connected set, with boundary of class* C^1, *and let* $f: \Omega \times \mathbb{R} \to \mathbb{R}$ *be a Carathéodory function. Assume that there exist* $b > 0$, $p \geq 0$ *with* $p < (n+2)/(n-2)$ *if* $n \geq 3$, $s \in [0,2[$, $\beta \in L^{2n/(n+2)}(\Omega)$, $\gamma \in L^{2n/(2n-(n-2)s)}(\Omega)$, $\delta \in L^1(\Omega)$ *such that*

(a_1) $|f(x,\xi)| \leq b|\xi|^p + \beta(x)$ *for a.e.* $x \in \Omega$ *and every* $\xi \in \mathbb{R}$,
(a_2) $|\int_0^\xi f(x,t)\,dt| \leq \gamma(x)|\xi|^s + \beta(x)|\xi| + \delta(x)$ *for a.e.* $x \in \Omega$ *and every* $\xi \in \mathbb{R}$,
(a_3) *for a.e.* $x \in \Omega$, *the function* $f(x,\cdot)$ *is even.*

Then, for every $\alpha \in L^\infty(\Omega)$ *with* $\operatorname{ess\,inf}_\Omega \alpha > 0$, *every* $q \in]1,2[$ *and every* $c > 0$ *satisfying*

$$\int_\Omega \left(\int_0^{c^{1/(2-q)}} f(x,\xi)\,d\xi \right) dx \neq 0, \tag{1}$$

there exist an open interval $\Lambda \subseteq \mathbb{R}$ *and a positive real number* ϱ, *such that, for each* $\lambda \in \Lambda$, *the Neumann problem*

$$(\mathrm{P}_\lambda) \qquad \begin{cases} -\Delta u = \alpha(x)(c|u|^{q-2}u - u) + \lambda f(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

(*where* ν *is the outer unit normal to* $\partial\Omega$) *admits at least three weak solutions in* $W^{1,2}(\Omega)$ *whose norms are less than* ϱ.

In general, let us recall that, if $\varphi\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, a weak solution of the problem

$$\begin{cases} -\Delta u = \varphi(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

(where ν is the outer unit normal to $\partial\Omega$) is any $u \in W^{1,2}(\Omega)$ such that

$$\int_\Omega \nabla u(x) \nabla v(x)\,dx - \int_\Omega \varphi(x,u(x))v(x)\,dx = 0 \quad \text{for all } v \in W^{1,2}(\Omega).$$

To prove Theorem 1 we use Theorem 3 below which is, in turn, a corollary of the following consequence of Theorem A:

Theorem 2. *Let* X *be a separable and reflexive real Banach space, and let* $\Phi, \Psi\colon X \to \mathbb{R}$ *be two continuously Gâteaux differentiable functionals. Assume that* Φ *is sequentially weakly lower semicontinuous, that* Ψ *is sequentially weakly continuous and that, for each* $\lambda \in \mathbb{R}$, *the functional* $\Phi + \lambda\Psi$ *satisfies the Palais–Smale condition and*

$$\lim_{\|x\| \to \infty} (\Phi(x) + \lambda\Psi(x)) = \infty.$$

Finally, suppose that there exist $x_1, x_2 \in X$ *and* $r \in \mathbb{R}$ *such that*

$$\inf_{x \in X} \Phi(x) < \inf_{x \in \Psi^{-1}(r)} \Phi(x), \tag{2}$$

$$\Phi(x_1) = \Phi(x_2) = \inf_{x \in X} \Phi(x), \tag{3}$$

$$\Psi(x_1) < r < \Psi(x_2). \tag{4}$$

Then, there exist an open interval $\Lambda \subseteq \mathbb{R}$ *and a positive real number* ϱ, *such that, for each* $\lambda \in \Lambda$, *the equation*

$$\Phi'(x) + \lambda\Psi'(x) = 0$$

admits at least three solutions in X *whose norms are less than* ϱ.

Proof. To get the conclusion, we apply Theorem A taking $I = \mathbb{R}$ and

$$g(x,\lambda) = \Phi(x) + \lambda(\Psi(x) - r)$$

for all $(x,\lambda) \in X \times \mathbb{R}$. Clearly, g is continuous and conditions (i) and (ii) are satisfied. It remains to show that condition (iii) holds too. To see this, assume the contrary. In particular, suppose that

$$\sup_{\lambda \in \mathbb{R}} \inf_{x \in X} (\Phi(x) + \lambda(\Psi(x) - r)) = \inf_{x \in X} \sup_{\lambda \in \mathbb{R}} (\Phi(x) + \lambda(\Psi(x) - r)).$$

Next, observe that

$$\inf_{x\in X}\sup_{\lambda\in\mathbb{R}}(\Phi(x)+\lambda(\Psi(x))-r))=\inf_{x\in\Psi^{-1}(r)}\Phi(x).$$

So, we are assuming that

$$\sup_{\lambda\in\mathbb{R}}\inf_{x\in X}(\Phi(x)+\lambda(\Psi(x)-r))=\inf_{x\in\Psi^{-1}(r)}\Phi(x). \tag{5}$$

Since, by (4),

$$\lim_{\lambda\to\infty}\inf_{x\in X}(\Phi(x)+\lambda(\Psi(x)-r)) \le \lim_{\lambda\to\infty}(\Phi(x_1)+\lambda(\Psi(x_1)-r))=-\infty,$$
$$\lim_{\lambda\to-\infty}\inf_{x\in X}(\Phi(x)+\lambda(\Psi(x)-r)) \le \lim_{\lambda\to-\infty}(\Phi(x_2)+\lambda(\Psi(x_2)-r))=-\infty,$$

it follows that the real-valued function $\lambda\to\inf_{x\in X}(\Phi(x)+\lambda(\Psi(x)-r))$ (which is continuous in $\mathbb{R}$ being concave) attains its supremum. So, let $\lambda^*\in\mathbb{R}$ be such that

$$\inf_{x\in X}(\Phi(x)+\lambda^*(\Psi(x)-r))=\sup_{\lambda\in\mathbb{R}}\inf_{x\in X}(\Phi(x)+\lambda(\Psi(x)-r)).$$

Hence, from (5), we get

$$\inf_{x\in X}(\Phi(x)+\lambda^*(\Psi(x)-r))=\inf_{x\in\Psi^{-1}(r)}\Phi(x). \tag{6}$$

Observe that, by (2), one has $\lambda^*\neq 0$. Finally, putting (3), (4) and (6) together, we get

$$\inf_{x\in\Psi^{-1}(r)}\Phi(x)\le\min\{\Phi(x_1)+\lambda^*(\Psi(x_1)-r),\Phi(x_2)+\lambda^*(\Psi(x_2)-r)\}<\inf_{x\in X}\Phi(x)$$

which is absurd. □

As a corollary of Theorem 2, we get

Theorem 3. *Let X be a separable and reflexive real Banach space, and let $\Phi,\Psi\colon X\to\mathbb{R}$ be two continuously Gâteaux differentiable functionals. Assume that Φ is sequentially weakly lower semicontinuous and even, that Ψ is sequentially weakly continuous and odd, and that, for each $\lambda\in\mathbb{R}$, the functional $\Phi+\lambda\Psi$ satisfies the Palais–Smale condition and*

$$\lim_{\|x\|\to\infty}(\Phi(x)+\lambda\Psi(x))=\infty.$$

Finally, assume that

$$\inf_{x\in X}\Phi(x)<\inf_{x\in\Psi^{-1}(0)}\Phi(x).$$

Then, the conclusion of Theorem 2 holds.

Proof. Let u be a global minimum of Φ. So, by assumption, one has $\Psi(u)\neq 0$. For instance, assume $\Psi(u)<0$. Then, since Φ is even and Ψ is odd, to satisfy conditions (2)–(4) of Theorem 2, we can take $r=0$, $x_1=u$ and $x_2=-u$. □

We now are in a position to prove Theorem 1.

Proof of Theorem 1. Let $\alpha \in L^\infty(\Omega)$ with ess $\inf_\Omega \alpha > 0$, $q \in]1,2[$ and let $c > 0$ satisfy (1). We are going to apply Theorem 3 taking $X = W^{1,2}(\Omega)$ with the norm

$$\|u\| = \left(\int_\Omega (|\nabla u(x)|^2 + \alpha(x)|u(x)|^2)\,dx \right)^{1/2},$$

which is equivalent to the usual one, and setting

$$J(u) = \frac{c}{q}\int_\Omega \alpha(x)|u(x)|^q\,dx, \quad \Phi(u) = \frac{1}{2}\|u\|^2 - J(u)$$

and

$$\Psi(u) = -\int_\Omega \left(\int_0^{u(x)} f(x,\xi)\,d\xi \right) dx$$

for all $u \in X$. By classical results, the functionals Φ and Ψ are (well-defined and) continuously Gâteaux differentiable in X, the critical points of $\Phi + \lambda\Psi$ being precisely the weak solutions of problem (P_λ). Moreover, by the Rellich–Kondrachov theorem, the operators J' and Ψ' are compact, and so, in particular, the functional Φ is sequentially weakly lower semicontinuous and the functional Ψ is sequentially weakly continuous. Moreover, by (a_2), Sobolev embedding theorem and Hölder inequality, for a suitable constant $\eta > 0$, we have

$$\Phi(u) + \lambda\Psi(u) \geq \frac{1}{2}\|u\|^2 - \eta(\|u\|^q + |\lambda|(\|u\|^s + \|u\| + 1))$$

for every $u \in X$ and every $\lambda \in \mathbb{R}$, and so, since $q, s < 2$,

$$\lim_{\|u\|\to\infty} (\Phi(u) + \lambda\Psi(u)) = \infty.$$

This fact, together with the compactness of J' and Ψ', implies that the functional $\Phi + \lambda\Psi$ satisfies the Palais–Smale condition (see, for instance, Example 38.25 of [15]). We also observe that the functional Φ is even and that, by (a_3), the functional Ψ is odd. So, to get the conclusion directly from Theorem 3, it remains to show that

$$\inf_{u\in X} \Phi(u) < \inf_{u\in\Psi^{-1}(0)} \Phi(u). \tag{7}$$

To this end, we first observe that, for a.e. $x \in \Omega$, the points $c^{1/(2-q)}$ and $-c^{1/(2-q)}$ are the only two global minima of the function $\xi \to (\alpha(x)/2)|\xi|^2 - (c\alpha(x)/q)|\xi|^q$. Denote by w the constant function in Ω taking the value $c^{1/(2-q)}$. Then, for every $u \in X$ with $|u| \neq w$, we have

$$\begin{aligned}\Phi(u) &\geq \frac{1}{2}\int_\Omega \alpha(x)|u(x)|^2\,dx - \frac{c}{q}\int_\Omega \alpha(x)|u(x)|^q\,dx \\ &> \left(\frac{1}{2} - \frac{1}{q}\right) c^{2/(2-q)} \int_\Omega \alpha(x)\,dx = \Phi(w).\end{aligned}$$

This means that w and $-w$ are the only two global minima of the functional Φ over X (take into account that Ω is connected and that, by classical regularity results, the minima of Φ are continuous). Since $\Psi^{-1}(0)$ is sequentially weakly closed, the functional $\Phi_{|\Psi^{-1}(0)}$ has a global minimum which, by (1), is different from w and $-w$. From this (7) follows, and the proof is complete. □

REMARK 1. It is an open question to know whether the conclusion of Theorem 1 is still true without assuming condition (1).

REMARK 2. Another open question is to know whether the open interval Λ in the conclusion of Theorem 1 can actually be taken of the form $]-\lambda^*, \lambda^*[$ for some $\lambda^* > 0$.

Note, in particular, the following consequence of Theorem 1.

PROPOSITION 1. *Let* $\alpha, \gamma \in L^\infty(\Omega)$, *with* $\text{ess inf}_\Omega \alpha > 0$, $\beta \in L^{2n/(n+2)}(\Omega)$, $q \in]1, 2[$, *h a positive even integer, k a positive odd integer, with $h < k$, and* $c > 0$. *Assume that*

$$\frac{k}{h+k} c^{h/(k(2-q))} \int_\Omega \gamma(x)\, dx \neq -\int_\Omega \beta(x)\, dx. \tag{8}$$

Then, there exist an open interval $\Lambda \subseteq \mathbb{R}$ and a positive real number ϱ such that, for each $\lambda \in \Lambda$, the Neumann problem

$$\begin{cases} -\Delta u = \alpha(x)(c|u|^{q-2}u - u) + \lambda(\gamma(x)u^{h/k} + \beta(x)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

admits at least three weak solutions in $W^{1,2}(\Omega)$ whose norms are less than ϱ.

Of course, if Theorem 1 was true without condition (1), then Proposition 1 would be true without condition (8).

For other multiplicity results on problem (P_λ) see [1] and the references therein, in particular [14].

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