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ADDENDUM

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“Dependence on parameters for the Dirichlet problem with superlinear nonlinearities” (Topol. Methods Nonlinear Anal. 16 (2000), 145–160)

6. Example

Consider the problem

$$\begin{aligned} &x''(t) + W_x(t,x(t)) = 0, \quad \text{a.e. in } [0,1], \\ &x'(0) = 0 = x'(1) \end{aligned} \tag{6.1}$$

where $W(\,\cdot\,,x)$, $x \in \mathbb{R}^n$, is a measurable function in $[0,1]$, $W(t,\,\cdot\,)$, $t \in [0,1]$, is a convex, continuously Frechet differentiable function, such that its Fenchel conjugate has the derivative $dW_q^*(t,q)/dt$ at $(t,0)$, $t \in [0,1]$ and W satisfies the following growth condition:

- there exist $0 < \beta_1 < \beta_2$, $q_1 > 1$, $q > 2$, $k_1 \geq 0$, $k_2 > 0$ such that for $x \in \mathbb{R}^n$

$$k_1 + \frac{\beta_1}{q_1}\|x\|^{q_1} \leq W(t,x) \leq \frac{\beta_2}{q}\|x\|^q + k_2.$$

In the notation of the paper we have $L(t,x') = |x'|^2/2$ and $V(t,x) = W(t,x)$. It is easily seen that assumptions (H) and (H1) are satisfied. Therefore what we have to do is to construct a nonempty set X defined in Section 1. To this effect let us take any $k > 0$ and let $\overline{X}$ denote the same as in Section 1 with the new L and V. We assume the following hypotheses:

(H1′) $k_3 > \left(\dfrac{\beta_2}{q}\right) k^q + k_2$,

$$k_3 > k\left(\frac{q\beta_2^{1/(q-1)}}{q-1}\right)(k + k_2 - k_1) + 1)^{q-1} + \int_0^1 W(t,0)\,dt,$$

$$\left(\frac{q\beta_2^{1/(q-1)}}{q-1}\right)(k + k_2 - k_1) + 1)^{q-1} \le \frac{1}{3}\pi k,$$

$$\left(\frac{q_1}{q}\right)^{1/q_1}\left(\frac{k}{3}\right)^{q/q_1} + ((k_2 - k_1)q_1)^{1/q_1} \le \frac{k}{3},$$

(H2) $\dfrac{d}{dt}W_q^*(0,0) \neq 0$ or $\dfrac{d}{dt}W_q^*(1,0) \neq 0$.

We shall show that the set $X = \{v \in \widetilde{X} : 0 < \|v\|_{L^\infty} \le k\}$ where

$$\widetilde{X} = \bigg\{x + c_x \in \overline{X} : x \in A_0,\ c_x \in \mathbb{R}^n \text{ is such that } \int_0^1 W_x(t, x(t) + c_x)\,dt = 0, \text{ and } p(t) = x'(t),\ t \in [0,1] \text{ belongs to } A_{0,0}\bigg\}$$

is the set X which we are looking for. That means: we must prove that for each function $v \in X$ the appropriate primitive of the function

$$t \to \int_0^t W_x(\tau, v(\tau))\,d\tau = w'(t), \tag{6.2}$$

belongs to X i.e. $w(t) = c_w + \int_0^t w'(s)\,ds$ with c_w such that $\int_0^1 W_x(\tau, w(\tau))\,d\tau = 0$. It is obvious that $w' \in A_{0,0}$. Therefore we have to show that $\|w\|_{L^\infty} \le k$ — by the first two of assumptions (H1′) we shall get then also the inequality $\int_0^1 W(t, w(t))\,dt \le (1/2)\int_0^1 |w'(t)|^2 dt + k_3$. Moreover, we have also to check that w is not identically equal to 0. If we take $p(t) = w'(t)$ ($w'(t)$ defined by (6.2)) then by estimation theorem for subgradients of convex functions (taking into account the estimations on $W(t,x)$) we observe that

$$\|p'\|_{L^\infty} \le \left(\frac{q\beta_2^{1/(q-1)}}{q-1}\right)(k + k_2 - k_1) + 1)^{q-1}$$

and next applying the estimation for the function by its derivative (for functions with zero at ends) we have

$$\|w'\|_{L^2} \le \frac{1}{\pi}\|p'\|_{L^\infty}.$$

Using the last two assumptions of (H1′) we obtain

$$\|w\|_{L^\infty} \le k.$$

Moreover, $w \neq 0$. Actually, if $w(t) \equiv 0$ for some $v \in X$ then $W_x(t, v(t)) = 0$ for all $t \in [0,1]$. This, by convexity of $W(t, \cdot\,)$ means that $v(t) = W_q^*(t, 0)$ for all $t \in [0,1]$. By (H2), the von Neumann's boundary conditions of v could not be satisfied. Therefore $w \neq 0$ and it belongs to X. It is also clear that the set X is nonempty and, again by (H2) the zero function is not a solution to (6.1). Thus all assumptions of Theorem 4.2 are satisfied, so we come to the following theorem.

Theorem 6.1. *There exists a pair* $(\overline{x}, \overline{p})$ *being a solution to* (6.1) *and such that*

$$J(\overline{x}) = \min_{x \in X} J(x) = \min_{p \in X^d} J_D(p) = J_D(\overline{p}).$$

"Periodic Solutions of Lagrange Equations"
(Topol. Methods Nonlinear Anal. 22 (2003), 167–180)

5. Example

Let us denote by P the positive cone in $\mathbb{R}^n$ i.e. $P = \{x \in \mathbb{R}^n : x_i > 0,\ i = 1, \dots, n\}$ and by $\overline{P} = \{x \in \mathbb{R}^n : x_i \geq 0,\ i = 1, \dots, n\}$. We say that $x \geq y$ for $x, y \in \mathbb{R}^n$ if $x - y \in \overline{P}$.

Consider the problem

$$(k(t)x'(t))' + V_x(t, x(t)) = 0, \quad \text{a.e. in } \mathbb{R}, \tag{5.1}$$

$$x(0) = x(1), \quad x'(0) = x'(1) \tag{5.1'}$$

where $V(\,\cdot\,, x)$ is a 1-periodic, measurable function in $\mathbb{R}$, $V(t, \cdot\,)$ is a continuously Frechet differentiable function. In the notation of the paper we have $L(t, x') = (1/2)k(t)|x'|^2$. If $b, c \in \mathbb{R}^n$ by bc we always mean a vector $[b_i c_i]_{i=1,\dots,n}$. We set the basic hypotheses we need:

- (H1') The function $k(t)$ is absolutely continuous, periodic and positive for $t \in [0,1]$, $k(1) = 1$, $\int_0^1 V_x(t, 0)\,dt \neq 0$, and let $c_0 \in \mathbb{R}^n$ be such that $\int_0^1 V_x(t, c_0)\,dt = 0$,
- (H2') For a given $\theta \in P$, there exists $v \in P$ and $w \in -P$ such that $V_x(t, c_0 + \beta v)$ is positive, $V_x(t, c_0 + \beta w)$ is negative, for $t \in [0,1]$, and

$$\int_0^1 V_x(t, c_0 + \beta(v - w))\,dt \leq \theta v, \quad \int_0^1 V_x(t, c_0 + \beta(v - w))\,dt \leq -\theta w \tag{5.2}$$

and

$$-\int_0^1 V_x(t, c_0 + \beta(w - v))\,dt \leq \int_0^1 V_x(t, c_0 + \beta(v - w))\,dt.$$

where $\theta v = [\theta_i v_i]_{i=1,\dots,n}$, $\beta v = [\beta_i v_i]_{i=1,\dots,n}$, and $\beta = 2\theta \int_0^1 (1/k(r))\,dr$ and the growth condition is satisfied i.e. there exist $0 < \beta_1$, $q_1 > 1$,

$k_1 \in \mathbb{R}$ such that for each $y \in \overline{X} = \{x \in A : x' \in A,\ x(t) \in I,\ t \in [0,1],\ x(0) = x(1) = 0,\ x'(0) = x'(1)\}$, where $I = \{x \in \mathbb{R}^n : \beta w \le x \le \beta v\}$ and for all $c \in \mathbb{R}^n$

$$k_1 + \beta_1 |c|^{q_1} \le \int_0^1 V(t, c_0 + y(t) + c)\, dt. \tag{5.3}$$

(H3') We assume that if c_v is a minimizer of the functional $c \to \int_0^1 V(t, v+c)\, dt$ and c_w is a minimizer of the functional $c \to \int_0^1 V(t, w+c)\, dt$ then $w < v+c_v$, $w+c_w < v$ and $V(t,\cdot)$ is convex in the set $\operatorname{co}(D)$ for $t \in [0,1]$, where $D = \{x \in \mathbb{R}^n : c_0+w-v \le x \le c_0+v-w\}$. Moreover, assume that there exist $l, l_1 \in L^2([0,1],\mathbb{R})$ such that $\sup\{V(t,x) : x \in \operatorname{co}(D)\} \le l(t)$ and $\sup\{V_{x_i}(t,x) : x \in \operatorname{co}(D)\} \le l_1(t)$ for $t \in [0,1]$, $i = 1,\ldots,n$ (here $\operatorname{co}(D)$ denotes the convex hull of D).

We would like to stress that because of (H3') each function $x_j \to V_{x_j}(t,(x_1,\ldots,x_j,\ldots,x_n))$, $j = 1,\ldots,n$, $t \in [0,1]$, is increasing for $(x_1,\ldots,x_j,\ldots,x_n) \in D$ and in consequence for each $x \in \overline{X}$ the following inequalities hold: $V_x(t, c_0 + \beta(w-v)) \le V_x(s, x(s) + c_x) \le V_x(t, c_0 + \beta(v-w))$ (we use the observation that $c_0 - \beta v \le c_x \le c_0 - \beta w$).

It is easily seen that assumptions (H) and (H1) are satisfied. Therefore, what we have to do is to construct a nonempty set X. We prove that $\overline{X}$ is our set X. To this effect let us define in $\overline{X}$ the operator A by the formula

$$\begin{aligned} \mathrm{A}\, x(t) = {} & \int_0^t \frac{1}{k(r)} \bigg(\int_r^1 V_x(s, x(s) + c_x)\, ds \bigg) dr \\ & - \frac{\int_0^t (1/k(r))\, dr}{\int_0^1 (1/k(r))\, dr} \int_0^1 \frac{1}{k(r)} \bigg(\int_r^1 V_x(s, x(s) + c_x)\, ds \bigg) dr. \end{aligned} \tag{5.4}$$

Then by (H2')

$$\begin{aligned} \mathrm{A}\, x(t) \le {} & \int_0^t \frac{1}{k(r)} \int_r^1 V_x(s, c_0 + \beta(v-w))\, ds\, dr \\ & - \frac{\int_0^t (1/k(r))\, dr}{\int_0^1 (1/k(r))\, dr} \int_0^1 \frac{1}{k(r)} \bigg(\int_r^1 V_x(s, c_0 + \beta(w-v))\, ds \bigg) dr \\ \le {} & 2 \int_0^1 \frac{1}{k(r)} \int_0^1 V_x(s, c_0 + \beta(v-w))\, ds\, dr \le 2\theta \int_0^1 \frac{1}{k(r)}\, dr \cdot v = \beta v. \end{aligned}$$

Similarily, again using (H2'), we prove the second needed inequality. Hence $\mathrm{A}\, x \in \overline{X}$. Observe that if we take $\widetilde{p}(t) = k(t)(\mathrm{A}\, x(t))'$ then by (5.4) $-\widetilde{p}'(t) = V_x(t, x(t) + c_x)$. It is clear that $\overline{X}$ contains at least one element w such that $w(0) = w(1) = 0$. What we still have to check is the relation (1.5). By (H3') $V(t,\cdot)$ is convex and by (H1') it is continuously differentiable. However subdifferential is a global notion thus we need to extend convexity of $V(t,\,\cdot\,)$ to the whole space. To this effect let us define

$$\breve{V}(t,x) = \begin{cases} V(t,x) & \text{if } x \in \operatorname{co}(D),\ t \in [0,1], \\ \infty & \text{if } x \notin \operatorname{co}(D),\ t \in [0,1]. \end{cases}$$

As our all investigation reduce to the set D, therefore $\breve{V} = V$ in it. We need this notation only for the purpose of duality in Section 2. Of course (1.5) is satisfied for $\breve{V}$ in $\overline{X}$. Therefore $\overline{X}$ is our set X and problem (5.1) has at least one nonzero (because of (H1')) periodic solution.

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