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EXISTENCE RESULTS FOR IMPULSIVE NEUTRAL FUNCTIONAL DIFFERENTIAL INCLUSIONS

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Abstract. In this paper we prove existence results for first and second order impulsive neutral functional differential inclusions under the mixed Lipschitz and Carathéodory conditions.

1. Introduction

The theory of impulsive differential equations is emerging as an important area of investigation since it is much richer that the corresponding theory of differential equations; see the monograph of Lakshmikantham *et al* [2]. In this paper, we study the existence of solutions for initial value problems for first and second order impulsive neutral functional differential inclusions. More precisely in Section 3 we consider first order impulsive neutral functional differential inclusions of the form

$$\frac{d}{dt}[x(t) - f(t, x_t)] \in G(t, x_t), \qquad \text{a.e. } t \in I := [0, T], \quad t \neq t_k,\ k = 1, \dots, m, \tag{1.1}$$

$$x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, \dots, m, \tag{1.2}$$

$$x_0 = \phi, \tag{1.3}$$

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where $f: I \times \mathcal{D} \to \mathbb{R}^n$ and $G: I \times \mathcal{D} \to \mathcal{P}_f(\mathbb{R}^n)$, $\mathcal{D} = \{\psi: [-r, 0] \to \mathbb{R}^n \mid \psi$ is continuous everywhere except for a finite number of points s at which the left limit $\psi(s^-)$ and the right limit $\psi(s^+)$ exist and $\psi(s^-) = \psi(s)\}$, $\phi \in \mathcal{D}$, $(0 < r < \infty)$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$, $I_k: \mathbb{R}^n \to \mathbb{R}^n$ $(k = 1, \ldots, m)$, $x(t_k^+)$ and $x(t_k^-)$ are respectively the right and the left limit of x at $t = t_k$, and $\mathcal{P}_f(\mathbb{R}^n)$ denotes the class of all nonempty subsets of $\mathbb{R}^n$.

For any continuous function x defined on the interval $[-r, T] \setminus \{t_1, \ldots, t_m\}$ and any $t \in I$, we denote by x_t the element of $\mathcal{D}$ defined by

$$x_t(\theta) = x(t + \theta), \quad \theta \in [-r, 0].$$

For $\psi \in \mathcal{D}$ the norm of ψ is defined by

$$\|\psi\|_{\mathcal{D}} = \sup\{|\psi(\theta)|, \theta \in [-r, 0]\}.$$

Later, in Section 4, we study the existence of solutions of second order impulsive neutral functional differential inclusions of the form

$$\frac{d}{dt}[x'(t) - f(t, x_t)] \in G(t, x_t), \qquad t \in I := [0, T],\ t \neq t_k,\ k = 1, \ldots, m, \tag{1.4}$$

$$x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \qquad k = 1, \ldots, m, \tag{1.5}$$

$$x'(t_k^+) - x'(t_k^-) = \overline{I}_k(x'(t_k^-)), \qquad k = 1, \ldots, m, \tag{1.6}$$

$$y(t) = \phi(t), \qquad t \in [-r, 0], \quad x'(0) = \eta, \tag{1.7}$$

where f, G, I_k, ϕ are as in problem (1.1)–(1.3), $\overline{I}_k: \mathbb{R}^n \to \mathbb{R}^n$, and $\eta \in \mathbb{R}^n$.

The main tools used in the study are the fixed point theorems of Dhage [1]. In the following section we give some auxiliary results needed in the subsequent part of the paper.

2. Auxiliary results

Throughout this paper X will be a Banach space and let $\mathcal{P}(X)$ denote the class of all subsets of X. Let $\mathcal{P}_f(X)$, $\mathcal{P}_{\mathrm{bd,cl}}(X)$ and $\mathcal{P}_{\mathrm{cp,cv}}(X)$ denote respectively the classes of all nonempty, bounded-closed and compact-convex subsets of X. For $x \in X$ and $Y, Z \in \mathcal{P}_{\mathrm{bd,cl}}(X)$ we denote by $D(x, Y) = \inf\{\|x - y\| \mid y \in Y\}$ and $\rho(Y, Z) = \sup_{a \in Y} D(a, Z)$.

Define a function $H: \mathcal{P}_{\mathrm{bd,cl}}(X) \times \mathcal{P}_{\mathrm{bd,cl}}(X) \to \mathbb{R}^+$ by

$$H(A, B) = \max\{\rho(A, B), \rho(B, A)\}.$$

The function H is called a Hausdorff metric on X. Note that $\|Y\| = H(Y, \{0\})$.

A correspondence $T: X \to \mathcal{P}_f(X)$ is called a multi-valued mapping on X. A point $x_0 \in X$ is called a *fixed point of the multi-valued operator* $T: X \to \mathcal{P}_f(X)$ *if* $x_0 \in T(x_0)$. The fixed points set of T will be denoted by $\mathrm{Fix}(T)$.

Definition 2.1. Let $T: X \to \mathcal{P}_{\mathrm{bd,cl}}(X)$ be a multi-valued operator. Then T is called a multi-valued contraction if there exists a constant $k \in (0,1)$ such that for each $x, y \in X$ we have

$$H(T(x), T(y)) \le k\|x - y\|.$$

The constant k is called a contraction constant of T.

A multi-valued mapping $T: X \to \mathcal{P}_f(X)$ is called *lower semi-continuous* (shortly l.s.c.) (resp. *upper semi-continuous* (shortly u.s.c.)) if B is any open subset of X then $\{x \in X \mid Gx \cap B \neq \emptyset\}$ (resp. $\{x \in X \mid Gx \subset B\}$) is an open subset of X. The multi-valued operator T is called *compact* if $\overline{T(X)}$ is a compact subset of X. Again T is called *totally bounded* if for any bounded subset S of X, $T(S)$ is a totally bounded subset of X. A multi-valued operator $T: X \to \mathcal{P}_f(X)$ is called *completely continuous* if it is upper semi-continuous and totally bounded on X, for each bounded $A \in \mathcal{P}_f(X)$. Every compact multi-valued operator is totally bounded but the converse may not be true. However the two notions are equivalent on a bounded subset of X.

We apply the following form of the fixed point theorem of Dhage [1] in the sequel.

Theorem 2.2. *Let X be a Banach space, $A: X \to \mathcal{P}_{\mathrm{cl,cv,bd}}(X)$ and $B: X \to \mathcal{P}_{\mathrm{cp,cv}}(X)$ two multi-valued operators satisfying:*

(a) *A is contraction with a contraction constant k, and*
(b) *B is completely continuous.*

Then either

(i) *the operator inclusion $\lambda x \in Ax + Bx$ has a solution for $\lambda = 1$, or*
(ii) *the set $\mathcal{E} = \{u \in X \mid \lambda u \in Au + Bu,\ \lambda > 1\}$ is unbounded.*

3. First order impulsive neutral functional differential inclusions

Let us start by defining what we mean by a solution of problem (1.1)–(1.3). In order to define the solutions of the above problems, we shall consider the spaces

$$\begin{aligned} PC([-r,T],\mathbb{R}^n) = \{x: [-r,T] \to \mathbb{R}^n : x(t) &\text{ is continuous almost everywhere} \\ &\text{except for some } t_k \text{ at which } x(t_k^-) \text{ and } x(t_k^+), \\ &k = 1, \ldots, m \text{ exist and } x(t_k^-) = x(t_k)\} \end{aligned}$$

and

$$\begin{aligned} PC^1([0,T],\mathbb{R}^n) = \{x: [0,T] \to \mathbb{R}^n : x(t) &\text{ is continuously differentiable everywhere} \\ &\text{except for some } t_k \text{ at which } x'(t_k^-) \text{ and } x'(t_k^+), \\ &k = 1, \ldots, m \text{ exist and } x'(t_k^-) = x'(t_k)\}. \end{aligned}$$

Let $Z = PC([-r,T],\mathbb{R}^n) \cap PC^1([0,T],\mathbb{R}^n)$. Obviously, for any $t \in [0,T]$ and $x \in Z$, we have $x_t \in \mathcal{D}$ and $PC([-r,T],\mathbb{R}^n)$ and Z are Banach spaces with the norms

$$\|x\| = \sup\{|x(t)| : t \in [-r,T]\} \quad \text{and} \quad \|x\|_Z = \|x\| + \|x'\|,$$

where $\|x'\| = \sup\{|x'(t)| : t \in [0,T]\}$.

In the following we set for convenience $\Omega = PC([-r,T],\mathbb{R}^n)$. Also we denote by $AC(J,\mathbb{R}^n)$ the space of all absolutely continuous functions $x\colon J \to \mathbb{R}^n$.

Definition 3.1. A function $x \in \Omega \cap AC((t_k,t_{k+1}),\mathbb{R}^n)$, $k = 1,\dots,m$ is said to be a solution of (1.1)–(1.3) if $x(t) - f(t,x_t)$ is absolutely continuous on $J \setminus \{t_1,\dots,t_m\}$ and (1.1)–(1.3) are satisfied.

We need the following definitions in the sequel.

Definition 3.2. A multi-valued map map $G\colon J \to \mathcal{P}_{cp,\mathrm{cv}}(\mathbb{R}^n)$ is said to be *measurable* if the function $t \to d(y,G(t)) = \inf\{\|y-x\| : x \in G(t)\}$ is measurable for every $y \in \mathbb{R}^n$.

Definition 3.3. A multi-valued map $G\colon I \times \mathcal{D} \to \mathcal{P}_{\mathrm{cl}}(\mathbb{R}^n)$ is said to be L^1-*Carathéodory* if

(a) $t \mapsto G(t,x)$ is measurable for each $x \in \mathcal{D}$,
(b) $x \mapsto G(t,x)$ is upper semi-continuous for almost all $t \in I$, and
(c) for each real number $\rho > 0$, there exists a function $h_\rho \in L^1(I,\mathbb{R}^+)$ such that

$$\|G(t,u)\| := \sup\{|v| : v \in G(t,u)\} \le h_\rho(t), \quad \text{a.e. } t \in I$$

for all $u \in \mathcal{D}$ with $\|u\|_{\mathcal{D}} \le \rho$.

Then we have the following lemmas due to Lasota and Opial [3].

Lemma 3.4. *If* $\dim(X) < \infty$ *and* $F\colon J \times X \to \mathcal{P}_f(X)$ *is* L^1-*Carathéodory, then* $S^1_G(x) \neq \emptyset$ *for each* $x \in X$.

Lemma 3.5. *Let* X *be a Banach space,* G *an* L^1-*Carathéodory multi-valued map with* $S^1_G \neq \emptyset$ *where*

$$S^1_G(x) := \{v \in L^1(I,\mathbb{R}^n) : v(t) \in G(t,x_t) \text{ a.e. } t \in I\},$$

and $\mathcal{K}\colon L^1(J,X) \to C(J,X)$ *be a linear continuous mapping. Then the operator*

$$\mathcal{K} \circ S^1_G\colon C(J,X) \to \mathcal{P}_{\mathrm{cp,cv}}(C(J,X))$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

We consider the following set of assumptions in the sequel.

(H_1) There exists a function $k \in B(I, \mathbb{R}^+)$ such that, for all $x, y \in \mathcal{D}$ and $\|k\| < 1$,

$$|f(t,x) - f(t,y)| \leq k(t)\|x - y\|_{\mathcal{D}} \quad \text{a.e. } t \in I.$$

(H_2) The multi $G(t,x)$ has compact and convex values for each $(t,x) \in I \times \mathcal{D}$.

(H_3) G is L^1-Carathéodory.

(H_4) There exists a function $q \in L^1(I, \mathbb{R})$ with $q(t) > 0$ for a.e. $t \in I$ and a nondecreasing function $\psi\colon \mathbb{R}^+ \to (0, \infty)$ such that

$$\|G(t,x)\| := \sup\{|v| : v \in G(t,x)\} \leq q(t)\psi(\|x\|_{\mathcal{D}}) \qquad \text{a.e. } t \in I,$$

for all $x \in \mathcal{D}$.

(H_5) The impulsive functions $|I_k|$ are continuous and there exist constants c_k such that $|I_k(x)| \leq c_k$, $k = 1, \ldots, m$ for each $x \in \mathbb{R}^n$.

Theorem 3.6. *Assume that* (H_1)–(H_5) *hold. Suppose that*

$$\int_{c_1}^{\infty} \frac{ds}{\psi(s)} > c_2\|\gamma\|_{L^1} \tag{3.1}$$

where

$$c_1 = \frac{F + \sum_{k=1}^m c_k}{1 - \|k\|}, \quad c_2 = \frac{1}{1 - \|k\|}$$

and

$$F = \|\phi\|_{\mathcal{D}} + |\phi(0) - f(0,\phi)| + \sup_{t \in I} |f(t,0)|.$$

Then the initial value problem (1.1)–(1.3) *has at least one solution on* $[-r, T]$.

Proof. Transform the problem (1.1)–(1.3) into a fixed point problem. Consider the operator $N\colon \Omega \to \mathcal{P}(\Omega)$ defined by:

$$Nx(t) = \left\{ h \in \Omega : h(t) = \begin{cases} \phi(t), & t \in I_0, \\ \phi(0) - f(0, \phi(0)) + f(t, x_t) & \\ \quad + \displaystyle\int_0^t v(s)\,ds + \sum_{0<t_k<t} I_k(x(t_k^-)), & t \in I, \end{cases} \right\}$$

where $v \in S_G^1(x)$.

Define two operators $A\colon \Omega \to \Omega$ by

$$Ax(t) = \begin{cases} 0 & \text{if } t \in I, \\ \{-f(0,\phi) + f(t, x_t)\} & \text{if } t \in I_0, \end{cases} \tag{3.2}$$

and the multi-valued operator $B\colon \Omega \to \mathcal{P}_f(\Omega)$ by

$$Bx(t) = \left\{ h \in \Omega : h(t) = \begin{cases} \phi(t) & \text{if } t \in I_0, \\ \phi(0) + \int_0^t v(s)\,ds & \\ \quad + \sum_{0<t_k<t} I_k(x(t_k^-)) & \text{if } t \in I. \end{cases} \right\} \tag{3.3}$$

Then $N = A + B$. We shall show that the operators A and B satisfy all the conditions of Theorem 2.2 on J.

Step 1. Since Ax is singleton for each $x \in \Omega$, A has closed, convex values on Ω. Also A has bounded values for bounded sets in X. To show this, let S be a bounded subset of Ω. Then, for any $x \in S$ one has

$$\|Ax\| \le \|Ax - A0\| + \|A0\| \le \|k\|\|x\| + \|A0\| \le \|k\|\rho + \|A0\|.$$

Hence A is bounded on bounded subsets of Ω.

Step 2. Next we prove that Bx is a convex subset of Ω for each $x \in \Omega$. Let $u_1, u_2 \in Bx$. Then there exists v_1 and v_2 in $S^1_G(x)$ such that

$$u_j(t) = \phi(0) + \sum_{0<t_k<t} I_k(x(t_k^-)) + \int_0^t v_j(s)\,ds, \quad j = 1, 2.$$

Since $G(t,x)$ has convex values, one has for $0 \le \mu \le 1$,

$$[\mu v_1 + (1-\mu)v_2](t) \in S^1_G(x)(t) \quad \text{for all } t \in J.$$

As a result we have

$$[\mu u_1 + (1-\mu)u_2](t) = \phi(0) + \sum_{0<t_k<t} I_k(x(t_k^-)) + \int_0^t [\mu v_1(s) + (1-\mu)v_2(s)]\,ds.$$

Therefore $[\mu u_1 + (1-\mu)u_2] \in Bx$ and consequently Bx has convex values in Ω. Thus we have $B\colon \Omega \to \mathcal{P}_{\mathrm{cv}}(\Omega)$.

Step 3. We show that A is a contraction on Ω. Let $x, y \in X$. By hypothesis (H_1)

$$|Ax(t) - Ay(t)| \le |f(t,x_t) - f(t,y_t)| \le k(t)\|x_t - y_t\|_{\mathcal{D}} \le \|k\|\|x - y\|.$$

Taking supremum over t, we have $\|Ax - Ay\| \le \|k\|\|x - y\|$. This shows that A is a multi-valued contraction, since $\|k\| < 1$.

Step 4. Now we show that the multi-valued operator B is completely continuous on Ω. First we show that B maps bounded sets into bounded sets in Ω. To see this, let S be a bounded set in Ω. Then there exists a real number $\rho > 0$ such that $\|x\| \le \rho$, for all $x \in S$.

Now for each $u \in Bx$, there exists a $v \in S^1_G(x)$ such that

$$u(t) = \phi(0) + \sum_{0<t_k<t} I_k(u(t_k^-)) + \int_0^t v(s)\,ds.$$

Then for each $t \in I$,

$$\begin{aligned} |u(t)| &\leq |\phi(0)| + \sum_{k=1}^{m} c_k + \int_0^t |v(s)|\, ds \\ &\leq \|\phi\|_{\mathcal{D}} + \sum_{k=1}^{m} c_k + \int_0^t h_\rho(s)\, ds \leq \|\phi\|_{\mathcal{D}} + \sum_{k=1}^{m} c_k + \|h_\rho\|_{L^1}. \end{aligned}$$

This further implies that

$$\|u\| \leq \|\phi\|_{\mathcal{D}} + \sum_{k=1}^{m} c_k + \|h_\rho\|_{L^1}$$

for all $u \in Bx \subset \bigcup B(S)$. Hence $\bigcup B(S)$ is bounded.

Next we show that B maps bounded sets into equi-continuous sets. Let S be, as above, a bounded set and $u \in Bx$ for some $x \in S$. Then there exists $v \in S_G^1(x)$ such that

$$u(t) = \phi(0) + \sum_{k=1}^{m} I_k(u(t_k^-)) + \int_0^t v(s)\, ds.$$

Then for any $\tau_1, \tau_2 \in I$ with $\tau_1 \leq \tau_2$ we have

$$\begin{aligned} |u(\tau_1) - u(\tau_2)| &\leq \left| \int_0^{\tau_1} v(s)\, ds - \int_0^{\tau_2} v(s)\, ds \right| + \sum_{0<t_k<\tau_2-\tau_1} |I_k(x(t_k^-))| \\ &\leq \int_{\tau_1}^{\tau_2} |v(s)|\, ds + \sum_{0<t_k<\tau_2-\tau_1} |I_k(x(t_k^-))| \\ &\leq \int_{\tau_1}^{\tau_2} h_\rho(s)\, ds + \sum_{0<t_k<\tau_2-\tau_1} |I_k(x(t_k^-))| \\ &\leq |p(\tau_1) - p(\tau_2)| + \sum_{0<t_k<\tau_2-\tau_1} |I_k(x(t_k^-))| \end{aligned}$$

where $p(t) = \int_0^t h_\rho(s)\, ds$.

If $\tau_1, \tau_2 \in I_0$ then $|u(\tau_1) - u(\tau_2)| = |\phi(\tau_1) - \phi(\tau_2)|$. For the case where $\tau_1 \leq 0 \leq \tau_2$ we have that

$$\begin{aligned} |u(\tau_1) - u(\tau_2)| &\leq \left| \phi(\tau_1) - \phi(0) - \sum_{0<t_k<\tau_2} I_k(x(t_k^-)) - \int_0^{\tau_2} v(s)\, ds \right| \\ &\leq |\phi(\tau_1) - \phi(0)| + \sum_{0<t_k<\tau_2} |I_k(x(t_k^-))| + \int_0^{\tau_2} |v(s)|\, ds \\ &\leq |\phi(\tau_1) - \phi(0)| + \sum_{0<t_k<\tau_2} |I_k(x(t_k^-))| + \int_0^{\tau_2} h_r(s)\, ds \\ &\leq |\phi(\tau_1) - \phi(0)| + \sum_{0<t_k<\tau_2} |I_k(x(t_k^-))| + |p(\tau_2) - p(0)|. \end{aligned}$$

Hence, in all cases, we have

$$|u(\tau_1) - u(\tau_2)| \to 0 \quad \text{as } \tau_1 \to \tau_2.$$

As a result $\bigcup B(S)$ is an equicontinuous set in Ω. Now an application of Arzelá-Ascoli theorem yields that the multi B is totally bounded on Ω.

Step 5. Next we prove that B *has a closed graph.* Let $\{x_n\} \subset \Omega$ be a sequence such that $x_n \to x_*$ and let $\{y_n\}$ be a sequence defined by $y_n \in Bx_n$ for each $n \in \mathbb{N}$ such that $y_n \to y_*$. We will show that $y_* \in Bx_*$. Since $y_n \in Bx_n$, there exists a $v_n \in S^1_G(x_n)$ such that

$$y_n(t) = \phi(0) + \sum_{0<t_k<t} I_k(y_n(t_k^-)) + \int_0^t v_n(s)\,ds.$$

Consider the linear and continuous operator $\mathcal{K}\colon L^1(J, \mathbb{R}^n) \to C(J, \mathbb{R}^n)$ defined by

$$\mathcal{K}v(t) = \int_0^t v_n(s)\,ds.$$

Now

$$\bigg\| y_n(t) - \phi(0) - \sum_{0<t_k<t} I_k(y_n(t_k^-)) - \bigg(y_*(t) - \phi(0) - \sum_{0<t_k<t} I_k(y_*(t_k^-)) \bigg) \bigg\| \to 0,$$

as $n \to \infty$. From Lemma 3.5 it follows that $(\mathcal{K} \circ S^1_G)$ is a closed graph operator and from the definition of $\mathcal{K}$ one has

$$y_n(t) - \phi(0) - \sum_{0<t_k<t} I_k(y_n(t_k^-)) \in (\mathcal{K} \circ S^1_F(y_n)).$$

As $x_n \to x_*$ and $y_n \to y_*$, there is a $v \in S^1_G(x_*)$ such that

$$y_*(t) = \phi(0) + \sum_{0<t_k<t} I_k(y_*(t_k^-)) + \int_0^t v_*(s)\,ds.$$

Hence the multi B is an upper semi-continuous operator on Ω.

Step 6. Finally we show that the set

$$\mathcal{E} = \{u \in \Omega : \lambda u \in Au + Bu \text{ for some } \lambda > 1\}$$

is bounded.

Let $u \in \mathcal{E}$ be any element. Then there exists $v \in S^1_G(u)$ such that

$$u(t) = \lambda^{-1}[\phi(0) - f(0, \phi)] + \lambda^{-1} f(t, u_t) + \lambda^{-1} \sum_{0<t_k<t} I_k(u(t_k^-)) + \lambda^{-1} \int_0^t v(s)\,ds.$$

Then

$$
\begin{aligned}
|u(t)| &\leq \|\phi\|_{\mathcal{D}} + |\phi(0) - f(0,\phi)| + |f(t,u_t)| + \sum_{k=1}^{m} c_k + \int_0^t |v(s)|\,ds \\
&\leq \|\phi\|_{\mathcal{D}} + |\phi(0) - f(0,\phi)| + |f(t,u_t) - f(t,0)| + |f(t,0)| \\
&\quad + \sum_{k=1}^{m} c_k + \int_0^t q(s)\psi(\|u_s\|_{\mathcal{D}})\,ds \\
&\leq \|\phi\|_{\mathcal{D}} + |\phi(0) - f(0,\phi)| + |f(t,0)| + k(t)\|u_t\|_{\mathcal{D}} \\
&\quad + \sum_{k=1}^{m} c_k + \int_0^t q(s)\psi(\|u_s\|_{\mathcal{D}})\,ds \\
&\leq \|\phi\|_{\mathcal{D}} + |\phi(0) - f(0,\phi)| + \sup_{t\in I} |f(t,0)| + \|k\|\|u_t\|_{\mathcal{D}} \\
&\quad + \sum_{k=1}^{m} c_k + \int_0^t q(s)\psi(\|u_s\|_{\mathcal{D}})\,ds \\
&\leq F + \|k\|\|u_t\|_{\mathcal{D}} + \sum_{k=1}^{m} c_k + \int_0^t q(s)\psi(\|u_s\|_{\mathcal{D}})\,ds,
\end{aligned}
$$

where $F = \|\phi\|_{\mathcal{D}} + |\phi(0) - f(0,\phi)| + \sup_{t\in I} |f(t,0)|$.

Put $w(t) = \max\{|u(s)| : -r \leq s \leq t\}$, $t \in I$. Then $\|u_t\|_{\mathcal{D}} \leq w(t)$ for all $t \in I$ and there is a point $t^* \in [-r,t]$ such that $w(t) = u(t^*)$. Hence we have

$$
\begin{aligned}
w(t) = |u(t^*)| &\leq F + \|k\|\|u_t\|_{\mathcal{D}} + \sum_{k=1}^{m} c_k + \int_0^t q(s)\psi(\|u_s\|_{\mathcal{D}})\,ds \\
&\leq F + \|k\| w(t) + \sum_{k=1}^{m} c_k + \int_0^t q(s)\psi(w(s))\,ds,
\end{aligned}
$$

or

$$
(1 - \|k\|)w(t) \leq F + \sum_{k=1}^{m} c_k + \int_0^t q(s)\psi(w(s))\,ds
$$

and

$$
w(t) \leq c_1 + c_2 \int_0^t q(s)\psi(w(s))\,ds, \quad t \in I,
$$

where

$$
c_1 = \frac{F + \sum\limits_{k=1}^{m} c_k}{1 - \|k\|} \quad \text{and} \quad c_2 = \frac{1}{1 - \|k\|}.
$$

Let

$$
m(t) = c_1 + c_2 \int_0^t q(s)\psi(w(s))\,ds, \quad t \in I.
$$

Then we have $w(t) \leq m(t)$ for all $t \in I$. Differentiating w.r.t. to t, we obtain

$$
m'(t) = c_2 q(t)\psi(w(t)), \quad \text{a.e. } t \in I,\ m(0) = c_1.
$$

This further implies that

$$m'(t) \leq c_2 q(t)\psi(m(t)), \quad \text{a.e. } t \in I,\ m(0) = c_1,$$

that is,

$$\frac{m'(t)}{\psi(m(t))} \leq c_2 q(t) \quad \text{a.e. } t \in J,\ m(0) = c_1.$$

Integrating from 0 to t we get

$$\int_0^t \frac{m'(s)}{\psi(m(t))}\, ds \leq \int_0^t c_2 q(s)\, ds.$$

By the change of variable,

$$\int_{c_1}^{m(t)} \frac{ds}{\psi(s)} \leq c_2 \|q\|_{L^1} < \int_{c_1}^{\infty} \frac{ds}{\psi(s)}.$$

Hence there exists a constant M such that

$$w(t) \leq m(t) \leq M \quad \text{for all } t \in I.$$

Now from the definition of w it follows that

$$\|u\| = \sup_{t \in [-r,a]} |u(t)| = w(a) \leq m(a) \leq M,$$

for all $u \in \mathcal{E}$. This shows that the set $\mathcal{E}$ is bounded in Ω. As a result the conclusion (ii) of Theorem 2.2 does not hold. Hence the conclusion (i) holds and consequently the initial value problem (1.1)–(1.3) has a solution x on J. This completes the proof. □

4. Second order impulsive neutral functional differential inclusions

Definition 4.1. A function $x \in Z \cap AC^1((t_k, t_{k+1}), \mathbb{R}^n)$, $k = 1, \ldots, m$, is said to be a solution of (1.4)–(1.7) if $x'(t) - f(t, x_t)$ is absolutely continuous on $J \setminus \{t_1, \ldots, t_m\}$ and (1.4)–(1.7) are satisfied.

Theorem 4.2. *Assume that* (H_2), (H_3) *and* (H_5) *hold. Moreover, we suppose that:*

(B_1) *There exists a function* $k \in L^1(I, \mathbb{R}^+)$ *such that*

$$|f(t,x) - f(t,y)| \leq k(t)\|x - y\|_{\mathcal{D}} \quad \text{a.e. } t \in I,$$

for all $x, y \in \mathcal{D}$ *and* $\|k\|_{L^1} < 1$.

(B_2) *There exists a function* $p \in L^1(I, \mathbb{R})$ *with* $p(t) > 0$ *for a.e.* $t \in I$ *and a nondecreasing function* $\psi\colon \mathbb{R}^+ \to (0, \infty)$ *such that*

$$\|F(t,u)\| := \sup\{|v| : v \in F(t,u)\} \leq p(t)\psi(\|u\|_{\mathcal{D}})$$

for almost all $t \in J$ *and all* $u \in \mathcal{D}$, *with*

$$\int_0^T M(s)\,ds < \int_{\overline{c}}^{\infty} \frac{ds}{1+s+\psi(s)},$$

where

$$\overline{c} = \|\phi\|_{\mathcal{D}} + T|\eta - f(0,\phi)| + \sum_{k=1}^{m}[c_k + (T-t_k)d_k],$$

and

$$M(t) = \max\{k(t), Tp(t), \sup_{t\in[0,T]} |f(t,0)|\}.$$

(B_3) *The impulsive functions* $|\overline{I}_k|$ *are continuous and there exist constants* d_k *such that* $|\overline{I}_k(x)| \le d_k$, $k = 1,\dots,m$ *for each* $x \in \mathbb{R}^n$.

Then the initial value problem (1.4)–(1.7) *has at least one solution on* $[-r,T]$.

PROOF. Transform the problem (1.4)–(1.7) into a fixed point problem. Consider the operator $\overline{N}\colon \Omega \to \mathcal{P}(\Omega)$ defined by:

$$\overline{N}x(t) = \left\{ h \in \Omega : h(t) = \begin{cases} \phi(t), & t \in I_0, \\ \phi(0) + [\eta - f(0,\phi(0))]t + \displaystyle\int_0^t f(s,x_s)\,ds & \\ + \displaystyle\int_0^t (t-s)v(s)\,ds & \\ + \displaystyle\sum_{0<t_k<t} [I_k(x(t_k^-)) + (t-t_k)\overline{I}_k(x(t_k^-))], & t \in I, \end{cases} \right\}$$

where $v \in S^1_G(x)$. Define two operators $\overline{A}\colon \Omega \to \Omega$ by

$$\overline{A}x(t) = \begin{cases} 0 & \text{if } t \in I_0, \\ \left\{ [\eta - f(0,\phi]t + \displaystyle\int_0^t f(s,x_s)\,ds \right\} & \text{if } t \in I, \end{cases} \tag{4.1}$$

and the multi-valued operator $\overline{B}\colon \Omega \to \mathcal{P}_f(\Omega)$ by

(4.2) $\overline{B}x(t)$

$$= \left\{ h \in \Omega : h(t) = \begin{cases} \phi(t) & \text{if } t \in I_0, \\ \phi(0) + \displaystyle\int_0^t (t-s)v(s)\,ds & \\ + \displaystyle\sum_{0<t_k<t} [I_k(x(t_k^-)) + (t-t_k)\overline{I}_k(x(t_k^-))], & \text{if } t \in I. \end{cases} \right\}$$

Then $\overline{N} = \overline{A} + \overline{B}$. We can prove, as in Theorem 3.6, that the operators $\overline{A}$ and $\overline{B}$ satisfy all the conditions of Theorem 2.2 on J. We omit the details, and we prove only that the set

$$\mathcal{E}(\overline{N}) := \{x \in \Omega : \lambda x \in \overline{A}x + \overline{B}x, \text{ for some } \lambda > 1\}$$

is bounded.

Let $x \in \mathcal{E}(\overline{N})$. Then there exists $v \in S^1_G(x)$ such that

$$\begin{aligned} x(t) = {} & \lambda^{-1}\phi(0) + \lambda^{-1}[\eta - f(0,\phi(0))]t \\ & + \lambda^{-1}\int_0^t f(s,x_s)ds + \lambda^{-1}\int_0^t (t-s)v(s)\,ds \\ & + \lambda^{-1}\sum_{0<t_k<t}[I_k(x(t_k^-)) + (t-t_k)\overline{I}_k(x'(t_k^-))]. \end{aligned}$$

This implies that, for each $t \in [0,T]$, we have

$$\begin{aligned} |x(t)| \leq {} & \|\phi\|_{\mathcal{D}} + T|\eta - f(0,\phi)| + \int_0^t k(s)\|x_t\|_{\mathcal{D}}\,ds + \int_0^t f(s,0)\,ds \\ & + \int_0^t (T-s)p(s)\psi(\|x_s\|_{\mathcal{D}})\,ds + \sum_{k=1}^m [c_k + (T-t_k)d_k]. \end{aligned} \tag{4.3}$$

We consider the function μ defined by

$$\mu(t) := \sup\{|x(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq T.$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |x(t^*)|$. If $t^* \in J$, by the inequality (4.3) we have for $t \in [0,T]$

$$\begin{aligned} \mu(t) \leq {} & \|\phi\|_{\mathcal{D}} + T|\eta - f(0,\phi)| + \int_0^t k(s)\mu(s)\,ds + \int_0^t f(s,0)\,ds \\ & + T\int_0^t p(s)\psi(\mu(s))\,ds + \sum_{k=1}^m [c_k + (T-t_k)d_k]. \end{aligned} \tag{4.4}$$

If $t^* \in [-r,0]$ then $\mu(t) = \|\phi\|_{\mathcal{D}}$ and the inequality (4.4) holds. Let us take the right-hand side of inequality (4.4) as $v(t)$. Then we have

$$\begin{gathered} \mu(t) \leq v(t), \quad t \in [0,T], \\ v(0) := \overline{c} = \|\phi\|_{\mathcal{D}} + T|\eta - f(0,\phi)| + \sum_{k=1}^m [c_k + (T-s)d_k], \end{gathered}$$

and

$$v'(t) = k(t)\mu(t) + f(t,0) + Tp(t)\psi(\mu(t)), \quad t \in [0,T].$$

Using the nondecreasing character of ψ we get

$$v'(t) \leq M(t)[1 + v(t) + \psi(v(t))], \quad t \in [0,T].$$

This inequality implies for each $t \in [0,T]$ that

$$\int_{v(0)}^{v(t)} \frac{d\tau}{1+\tau+\psi(\tau)} \leq \int_0^T M(s)\,ds < \int_{v(0)}^{\infty} \frac{d\tau}{1+\tau+\psi(\tau)}.$$

This inequality implies that there exists a constant b such that $v(t) \leq b$, $t \in [0,T]$, and hence $\mu(t) \leq b$, $t \in [0,T]$. Since for every $t \in [0,T]$, $\|y_t\|_{\mathcal{D}} \leq \mu(t)$, we have

$$\|x\| \leq \max\{\|\phi\|_{\mathcal{D}}, b\},$$

where b depends only on T and on the functions p and ψ. This shows that $\mathcal{E}(\overline{N})$ is bounded. $\square$

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