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DYNAMICS OF NORMALIZED SYSTEMS ON SURFACES

Marco Sabatini

Abstract. We extend to normalized systems several properties of commuting systems proved in [11]. A rough classification of the dynamics induced by normalized vector fields on two-dimensional compact connected oriented manifolds is given.

1. Introduction

Let us consider a couple of differential systems defined on an open, connected subset Ω of the plane:

$$z' = V(z), \tag{1.1}$$

$$z' = W(z), \tag{1.2}$$

where $z \equiv (x, y) \in \Omega$, $V \equiv (v_1, v_2)$, $W \equiv (w_1, w_2)$, $V, W \in C^2(\Omega, \mathbb{R}^2)$. We denote by $\phi_V(t, z)$ ($\phi_W(s, z)$) the local flow defined by (1.1) (resp. (1.2)). Let us set $V \wedge W = v_1 w_2 - v_2 w_1$. Denoting by $[V, W] = \partial_V W - \partial_W V$ the Lie brackets of V and W, we say that W is a *normalizer* of V (W normalizes V) if $[V, W] \wedge V = 0$ (see [13]). If V and W are transversal on Ω, then they normalize each other if and only if $[V, W] = 0$ on Ω. In this case they are said to be *commutators*, or to commute with each other. If V and W commute, and if there exist $S, T \in \mathbb{R}$ such that $\phi_V(t, \phi_W(s, z))$ and $\phi_W(s, \phi_V(t, z))$ are both defined for all $(s, t) \in [0, S] \times$

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$[0, T]$, then for all $(s, t) \in [0, S] \times [0, T]$ one has $\phi_V(t, \phi_W(s, z)) = \phi_W(s, \phi_V(t, z))$. The dynamical properties of couples of commuting systems were studied in [11]. Among other things, centers are isochronous, critical points have positive index, limit cycles do not exist, attraction and central regions are unbounded.

Centers' isochronicity does not require so strong a property as commutativity. In fact, in order to prove that a center of (1.1) is isochronous, it is sufficient to prove that V normalizes a transversal W, as in [8] and [12]. This is a substantial improvement over methods based on commutators, since finding a normalizer is equivalent to find a solution to *one* PDE, $[V, W] \wedge V = 0$, while finding a commutator is equivalent to find a solution to *two* PDE's, $[V, W] = 0$.

Passing from commutators to normalizers means considering much wider a class of systems. In fact, given a couple of commuting vector fields V^* and W^*, every vector field of the form $\alpha(z)V^* + \beta(z)W^*$, with $\alpha(z)$ arbitrary smooth function and $\beta(z)$ first integral of W^* is normalized by W^*. For instance, taking

$$V(x, y) = (y, -x), \quad W(x, y) = (x, y), \quad \alpha(x, y) = 1 + x, \quad \beta(x, y) = 1,$$

one gets a quadratic vector field

$$x' = x + y(1 + x), \quad y' = y - x(1 + x)$$

normalized by

$$x' = x, \quad y' = y,$$

which is not a commutator.

This suggests to re-examine the results obtained in [11], trying to find which ones still hold by only requesting a normalizing property. Roughly speaking, orbital properties described in [11] hold also for normalized systems, while properties depending on the parametrization of the solutions of (1.1) do not hold. For instance, complete normalized systems do not have the properties described in Section 2 of [11] for complete commuting systems. On the other hand, the classification of surfaces admitting commuting systems extends with minor changes to normalized ones. We also extend a characterization of centers appeared in [1].

As far as possible, we have tried to give proofs based on orbital properties, so that several results, as the absence of limit cycles, are extendable to normalized local flows, even if not defined by differentiable vector fields.

2. Results

For the reader's convenience, we give a proof of the main property of normalized vector fields used in this paper. Other proofs may be found in textbooks concerned with Lie symmetries of differential equations (see, e.g. [10], p. 134). Given a vector field V, we denote by $\gamma_V(z)$ the V-orbit passing through z.

THEOREM 2.1. *Let* $V, W \in C^\infty(\Omega, \mathbb{R}^2)$, $[V, W] \wedge V \equiv 0$ *on* Ω. *Let* $z^* \in \Omega$ *be such that* $V(z^*) \neq (0,0)$, $W(z^*) \neq (0,0)$. *Then there exists a neighbourhood of* Ω_{z^*} *of* z^* *such that*

$$\phi_W(s, \gamma_V(z^*) \cap \Omega_{z^*}) \subset \gamma_V(\phi_W(s, z^*)) \cap \Omega_{z^*}.$$

PROOF. Since $V(z^*) \neq (0,0)$, in a neighbourhood of z^* we can consider the function

$$\mu(z) = \frac{[V(z), W(z)] \cdot V(z)}{|V(z)|^2}.$$

One has $[V(z), W(z)] = \mu(z)V(z)$. Let us look for a function α such that the vector fields αV and W commute, that is $[\alpha V, W] = 0$. One has

$$0 = [\alpha V, W] = \alpha \partial_V W - (\partial_W \alpha)V - \alpha \partial_W V = \alpha[V, W] - (\partial_W \alpha)V = (\alpha\mu - \partial_W \alpha)\, V.$$

Let us restrict to a neighbourhood Ω_1 of z^* where, by the local rectification theorem, the flows ϕ_V and ϕ_W are locally parallelizable. Then one can solve the equation $\alpha\mu - \partial_W \alpha = 0$ by integrating along the flow ϕ_W, since such an equation is a linear equation with respect to α,

$$\alpha(\phi_W(s, z)) = \exp\left(\int_0^s \mu(\phi_W(\sigma, z))d\sigma\right). \tag{2.1}$$

The vector field αV so defined commutes with W. Denoting by $\phi_{\alpha V}(r, z)$ the local flow defined by αV, let us take $R, S \in \mathbb{R}$, $R, S > 0$, such that $\phi_{\alpha V}(r, \phi_W(s, z^*))$ and $\phi_W(s, \phi_{\alpha V}(r, z^*))$ are both defined and contained in Ω_1 for all $(r, s) \in [-R, R] \times [-S, S]$. By Theorem 1.34 in [10], one has, for all $(r, s) \in [-R, R] \times [-S, S]$,

$$\phi_W(s, \phi_{\alpha V}(r, z^*)) = \phi_{\alpha V}(r, \phi_W(s, z^*)).$$

Let us restrict to the new neighbourhood Ω_{z^*} of z^* obtained by considering only the points of the form $\phi_W(s, \phi_{\alpha V}(r, z^*))$, for $(r, s) \in [-R, R] \times [-S, S]$. From (2.1) one has in particular that

$$\phi_W(s, \gamma_{\alpha V}(z^*) \cap \Omega_{z^*}) \subset \gamma_{\alpha V}(\phi_W(s, z^*)) \cap \Omega_{z^*},$$

Since the orbits of $\phi_{\alpha V}$ coincide with those ones of ϕ_V, we can conclude that

$$\phi_W(s, \gamma_V(z^*) \cap \Omega_{z^*}) \subset \gamma_V(\phi_W(s, z^*)) \cap \Omega_{z^*}. \qquad \square$$

The property of the above theorem is usually referred to by saying that ϕ_W takes arcs of V-orbits into arcs of V-orbits.

Throughout this paper the following set of hypotheses will be referred to as (NT). In what follows Ω denotes an open subset of $\mathbb{R}^2$. We refer to [3] for definitions related to dynamical systems, and to [9] for what is concerned with plane differential systems.

(i) $V, W \in C^\infty(\Omega, \mathbb{R}^2)$; $V(z) = 0$ if and only if $W(z) = 0$; critical points are isolated;

(ii) $V(z) \wedge W(z) \neq 0$ at non-critical points;

(iii) $[V, W] \wedge V \equiv 0$ on Ω.

Due to (iii), the vectors $[V, W]$ and V are proportional. If z is not a critical point, we set

$$\mu(z) = \frac{[V(z), W(z)] \cdot V(z)}{|V(z)|^2},$$

where $\cdot$ denotes the scalar product, so that

$$[V(z), W(z)] = \mu(z)V(z).$$

A remarkable class of normalized systems is given by hamiltonian systems of the following type (see [7], Lemma 7),

$$x' = -G'(y), \quad y' = F'(x).$$

The vector field associated to the system

$$x' = \frac{F(x)}{F'(x)}, \quad y' = \frac{G(y)}{G'(y)},$$

is a normalizer of the above hamiltonian system on every open set where $F'(x) \neq 0$ and $G'(y) \neq 0$. In this case, one has

$$\mu(x, y) = \left(\frac{F(x)}{F'(x)}\right)' + \left(\frac{G(y)}{G'(y)}\right)' - 1.$$

For $F(x) = 1 - \cos(x)$, $G(y) = \frac{y^2}{2}$, we get a system equivalent to the classical pendulum equation

$$x' = -y, \quad y' = \sin(x),$$

whose normalizer is

$$x' = \frac{1 - \cos(x)}{\sin(x)}, \quad y' = \frac{y}{2},$$

defined on the region $\{(x, y) : -\pi < x < \pi\}$.

Most of the results we prove are consequences of next theorem.

Theorem 2.2. *Let* (NT) *hold. Then*

(a) (1.1) *has no limit cycles,*

(b) *a critical point of* (1.1) *has no hyperbolic sectors.*

Proof. (a) Assume, by absurd, V to have a limit cycle $\gamma(t) = \phi_V(t, z^*)$. Let Γ be the bounded region having γ as boundary. Without loss of generality, we may assume Γ to be positively invariant for (1.2). Then $\phi_W(s, z)$ exists for all $z \in \mathrm{cl}(\Gamma) = \Gamma \cup \gamma$ and for all $s \in [0, \infty)$. By the normalizing property, for every $s > 0$ the set $\phi_W(s, \gamma)$ is V-invariant. Since $\phi_W(s, \cdot)$ is a diffeomorphism, $\phi_W(s, \gamma)$ is a cycle for every $s \in [0, \infty)$. As a conclusion, γ cannot be approached by orbits starting inside Γ.

Since γ is compact and Ω open, there exists $\overline{s} < 0$ such that for every $z \in \gamma$ the solution $\phi_W(s, z)$ exists for all $s \in [0, \overline{s}]$. Working as in the previous step, one proves that every orbit starting at a point of $\phi_W(s, z^*)$ lies on a cycle, so that γ cannot be approached by other orbits from outside Γ.

(b) Let O be a critical point of (1.1). Let us restrict to a neighbourhood U of O where there are no other critical points. Assume by absurd O to have a hyperbolic sector, with adjacent separatrices γ_1, γ_2. Possibly passing to $-V$ or $-W$ or both, we may assume that there exist $z_1 \in \gamma_1$, $z_2 \in \gamma_2$, $s_1 < 0$, such that every V-orbit starting at $\phi_W(s, z_1)$, $s \in [s_1, 0]$ meets $\phi_W(\cdot, z_2)$. In this case O is the positive limit set of γ_1 and the negative limit set of γ_2. Let t^* and $-s_2$ the lowest positive numbers such that $\phi_V(t^*, (\phi_W(s_1, z_1)) = \phi_W(s_2, z_2)$.

By the continuous dependance on initial position, there exists $\varepsilon > 0$ such that every solution $\phi_W(s, z)$ starting at a point $\phi_V(t, z_1)$, $t \in [0, \varepsilon)$ meets γ_1. Let us define $\overline{t}$ as follows

$$\overline{t} = \sup\{t \in [0, t^*] : \phi_W(s, \phi_V(t, z_1)) \text{ meets } \gamma_1\}.$$

Let us set $\overline{z} = \phi_V(\overline{t}, z_1)$. The W-orbit $\phi_W(s, \overline{z})$ does not meet γ_1, since otherwise by continuity every W-orbit in a neighbourhood of $\overline{z}$ should meet γ_1, contradicting he definition of $\overline{t}$. Similarly, $\phi_W(s, \overline{z})$ does not meet γ_2. Hence $\phi_W(s, \overline{z})$ is contained in the compact region bounded by the arcs $\phi_W([s_1, 0], z_1)$, $\phi_V([0, +\infty), z_1)$, $\phi_V((-\infty, 0], z_2)$, $\phi_W([s_2, 0], z_2)$, $\phi_V([0, t^*], \phi_W(s_1, z_1))$, and by the point O. Hence $\phi_W(s, \overline{z})$ exists for all $s > 0$. By construction, $\phi_V(t, z_1) \neq \phi_W(s, \overline{z})$ for all $t > 0$, $s > 0$. But $\phi_W(-s_1, \cdot)$ takes $\phi_W(s_1, z_1)$ into z_1, hence by the uniqueness of solutions all the arc $\phi_V([0, \overline{t}], \phi_W(s_1, z_1))$ is taken into γ_1, contradicting the absence of intersections of $\phi_W(s, \overline{z})$ and γ_1. □

If O is a critical point of a differential system, we denote by $j(O)$ its index ([9]). If O is a center, then N_O is its *central region*, the largest connected punctured neighbourhood of O covered with non-trivial cycles surrounding O. A connected set covered with non-trivial cycles is usually called *period annulus*. Not every period annulus is contained in a central region. If O is asymptotically stable, we denote by A_O its region of attraction ([3]).

Corollary 2.3. *Let* (NT) *hold. Then*

(a) *If a V-orbit γ has non-empty limit set $\omega(\gamma)$, and $\omega(\gamma)$ is not a cycle, then $\omega(\gamma)$ is a singleton.*
(b) *If O is a critical point of* (1.1)*, then $j(O) > 0$.*

(c) *Every period annulus of* (1.1) *is contained in a central region.*

(d) *If* $j(O) = 1$, *then* O *is a center or* (*negatively*) *asymptotically stable.*

(e) *If* O *is a center, then its central region* N_O *is unbounded; if* O *is an asymptotically stable, then its attraction region* A_O *is unbounded.*

Proof. (a) As in [11, Corollary 1.1].

(b) As in [11, Corollary 1.2].

(c) Assume, by absurd, a period annulus P not to be contained in any central region.

Let $\partial_i P$ be the internal part of its boundary, that is the component of ∂P encircled by one of the cycles of P. If P is not contained in a central region, then $\partial_i P$ is not a critical point, so that there exists a non-critical point $z_0 \in \partial_i P$. By point (a), the positive limit set of z_0 is a singleton $\{z_1\}$, contained in $\partial_i P$. But in this case the positive semi-orbit through z_0 is a separatrix bounding a hyperbolic sector of z_1, with the cycles of P crossing such a sector, contradicting Theorem 2.2.

Hence $\omega(\phi_V(t, z_0))$ has to be a cycle, so that, by the absence of limit cycles, $\partial_i P$ is a cycle. Working as in the proof of Theorem 2.2, one shows that $\partial_i P$ has a neighbourhood filled with cycles, contradicting the hypothesis that $\partial_i P$ is part of the boundary of P.

(d) As in [11, Theorem 1.3].

(e) Both properties can be proved as in [11, Corollary 1.4i)]. In fact, the absence of limit cycles and hyperbolic sector implies that both ∂N_O and ∂A_O cannot be bounded. ☐

As in [11], we consider parametrized families of normalized systems.

We say that $\{V(\theta) \in C^2(\Omega, \mathbb{R}^2), \theta \in [0, 2\pi)\}$ is a *complete family of normalized vector fields on* Ω, if $V(\theta)$ is a complete family of rotated vector fields, as in [5], and for all $\theta \in [0, 2\pi)$ there exists a vector field $W(\theta) \in C^2(\Omega, \mathbb{R}^2)$ that normalizes $V(\theta)$ on Ω.

In the following we consider complete families of normalized vector fields in which the normalizer $W(\theta)$ does not actually depend on the angle θ. The simplest situation is that of a family generated by V and its normalizer W by considering linear combinations,

$$V(\theta) = V\cos(\theta) + W\sin(\theta), \quad \theta \in [0, 2\pi). \tag{2.2}$$

Next theorem holds for an arbitrary complete family of normalized vector fields.

Theorem 2.4. *Let* (NT) *hold and* $V(\theta)$ *be a complete family of normalized vector fields on* Ω. *If* $O \in U$ *is a critical point of index* 1 *of* $V(\theta)$, *for any* $\theta \in [0, 2\pi)$, *then there exists* $\theta^* \in [0, \pi)$ *such that*

(a) O *is a center for* $V(\theta^*)$ *and for* $V(\theta^* + \pi)$,

(b) *O is asymptotically stable (negatively asymptotically stable) for all* $\theta \in (\theta^*, \theta^* + \pi)$,

(c) *O is negatively asymptotically stable (asymptotically stable) for all* $\theta \notin [\theta^*, \theta^* + \pi]$.

PROOF. As in [11, Theorem 1.4]. Also this proof is a consequence of the absence of limit cycles. □

Theorem 2.4 shows that the existence of a transversal normalizer does not imply that a critical point of index 1 is a center. As an example, the system

$$x' = x, \quad y' = y,$$

normalizes every system

$$\begin{aligned} x' &= y(x^2 + y^2)\cos(\theta) + x\sin(\theta), \\ y' &= -x(x^2 + y^2)\cos(\theta) + y\sin(\theta), \end{aligned}$$

where the only systems having a center at O are those satisfying $\theta = 0, \pi$. Even a non-degeneracy condition on (1.1) is not sufficient to ensure that O be a center. In fact, the system

$$x' = -y, \quad y' = x$$

normalizes

$$\begin{aligned} x' &= -y + x(x^2 + y^2), \\ y' &= x + y(x^2 + y^2), \end{aligned}$$

which has imaginary eigenvalues at O, but does not have a center at O.

In order to get a sufficient condition for (1.1) to have a center, one has to impose some additional condition. Next corollary extends a similar result in [1], where it was required that

$$W(x, y) = \left(x + o\left(\sqrt{x^2 + y^2}\right), y + o\left(\sqrt{x^2 + y^2}\right)\right).$$

COROLLARY 2.5. *Let* (NT) *hold and O be a critical point of* (1.1). *If* (1.1) *has imaginary eigenvalues at O and* (1.2) *has non-imaginary eigenvalues at O, then O is a center of* (1.1).

PROOF. By possibly performing a change of variables, we may assume that the linearization of (1.1) at O is

$$x' = y, \quad y' = -x.$$

Let

$$x' = ax + by, \quad y' = cx + dy$$

be the linearization of (1.2) after the change of variables. The new vector fields will be called as well V and W. Their eigenvalues at O do not change. In particular, $a + d \neq 0$. The lowest order term of $[V, W] \wedge V$ is

$$(b + c)x^2 + 2(d - a)xy - (b + c)y^2,$$

hence a necessary conditions for W to normalize V are $a = d$, $c = -b$. By hypothesis, $a \neq 0$.

The family (3.1) satisfies the hypotheses of Theorem 2.4, so that there exists $\theta^* \in [0, \pi)$ such that $V(\theta^*)$ has a center at O. Assume, by absurd, that this occurs for $\theta^* \neq 0$. The linearization of $V(\theta^*) = V \cos(\theta^*) + W \sin(\theta^*)$ at O is

$$\begin{aligned} x' &= ax \sin\theta^* + y(\cos\theta^* + b \sin\theta^*), \\ y' &= x(-\cos\theta^* - b \sin\theta^*) + ay \sin\theta^*. \end{aligned}$$

Such a system is non-degenerate, since the determinant is $(a^2 + b^2)\sin^2\theta + 2b \sin\theta \cos\theta + \cos^2\theta = 0$, which does not vanish because its discriminant is $-4a^2 < 0$. Since O is a center, the linearization of $V(\theta^*)$ has zero trace. If the trace $2a \sin\theta^*$ vanishes, then $\theta^* \neq 0, \pi$ implies $a = 0$, contradicting the hypothesis. $\square$

We now examine some properties of the function $A(z) = V(z) \wedge W(z)$. We denote by $d_V(z)$ the divergence of V at z. In [6, Lemma 2(a)], the equality $\partial_V A = A d_V$ was proved. This is usually expressed by saying that A is an *inverse integrating factor*, because in this case the vector field VA^{-1} has zero divergence. As a consequence, one has

$$A(\phi_V(t, z)) = A(z) \exp\left[\int_0^t d_V(\phi_V(\tau, z))\, d\tau)\right],$$

that extends to normalized systems the first formula in Theorem 1.5 in [11]. This implies that, if $[V, W] \wedge V = 0$, then $\phi_V(t, z)$ preserves the transversality of V and W (see also Corollary 1.6 in [11]).

The following corollary extends Theorem 1.5 and Corollary 1.7 in [11]. See also [2, Theorem 2.4]. In absence of limit cycles, in the plane the stability of an isolated critical point which is not a center concides with its asymptotic stability.

COROLLARY 2.6. *Let* (NT) *hold. Let O is a critical point of* (1.1) *and Ω_1 be a neighbourhood of O; then*

(a) *if $d_V < 0$ in $\Omega_1 \setminus \{O\}$, then O is asymptotically stable,*
(b) *if $d_V \equiv 0$ in Ω_1, then O is a center,*
(c) *if $d_V > 0$ in $\Omega_1 \setminus \{O\}$, then O is negatively asymptotically stable.*

Proof. The formula $\partial_V A = Ad_V$ was proved in [6, Lemma 2(a)]. Since $d_V(O) = 0$ one can use A as a Liapunov function, like in Corollary 1.7 in [11], in order to prove (a) and (c). Point (b) can be proved using A as a first integral.□

We say that a vector field *complete* if ϕ_V is a flow, that is if all solutions to (1.1) exists for all $t \in \mathbb{R}$. The results of Section 2 in [11] about complete commuting systems cannot be extended to normalized systems. Given a function α, every normalizer of V is also a normalizer of αV, so that completeness is not relevant. In fact, for every normalized system (1.1) there exists a second, complete system having the same orbits and the same normalizer,

$$z' = \frac{V(z)}{1 + |V(z)|^2}.$$

Hence it is neither true that a complete normalized system without critical points is parallelizable, nor that a complete normalized system with a center (an asymptotically stable point) has a global center (region of attraction). Counterexamples can be easily constructed starting from hamiltonian systems, for parallelizability or centers, or perturbing a suitable hamiltonian center, for attractivity.

We conclude by extending to surfaces, i.e. two-dimensional compact connected oriented manifolds, part of the results of Section 3 in [11]. In what follows Σ denotes a surface and Ω an open subset of Σ. We recall that every surface Σ is homeomorphic to a sphere with p handles. The Euler characteristic $\chi(\Sigma)$ of a surface homeomorphic to a sphere with p handles is $2 - 2p$.

Theorem 2.7. *Let Σ be a surface of negative characteristic. Then there exist no couples of vector fields satysfying* (NT) *on Σ.*

Proof. Assume by absurd that V and W satysfy (NT), with $\Omega = \Sigma$. By Poincaré–Hopf theorem, the Euler characteristic of Σ is the sum of the indeces of the critical points of V. By Corollary 2.3, every critical point of V has positive index, which is not compatible with $\chi(\Sigma) < 0$. □

We cannot extend Theorem 3.2 of [11] about the torus, since we do not have the parallelizability of complete normalized vector fields. We prove a weaker result.

Theorem 2.8. *Let Σ be a torus, and* (NT) *hold on Σ. Then V has no critical points and no limit cycles. If a cycle exists, then every V-orbit is a cycle.*

Proof. The Euler characteristic of the torus is 0, hence V has no critical points. As for limit cycles, the proof of Theorem 2.2 applies.

Assume that a cycle γ exists. Since a torus is compact, every solution of a smooth vector field exists for all $t \in \mathbb{R}$. Since $\phi_W(s, \cdot)$ is a diffeomorphism, every $Z_\gamma = \{\phi_W(s, \gamma)\}$ is a V-cycle. Let us consider the set $Z = \{\phi_W(s, z)$:

$s \in \mathbb{R},\ z \in \gamma\}$. Z has non-empty interior by the transversality of V and W, it is V-invariant because union of V-orbits, and W-invariant for the same reason. Its boundary ∂Z is closed and both V- and W-invariant. Let z_0 be an arbitrary point of ∂Z. z_0 is non-critical for both vector fields, hence it has a neighbourhood of the form $Z_0 = \{\phi_V(t, (\phi_W(s, z)) : s \in \mathbb{R},\ t \in \mathbb{R}\}$, all contained in ∂Z by its invariance. Hence ∂Z is a subset of Σ, both open and closed, with $\partial Z \neq \Sigma$. This contradicts the connectedness of the torus, unless $\partial Z = \emptyset$ and $Z = \Sigma$. □

In the above theorem, we cannot use the absence of critical points in order to prove the absence of limit cycles, as we could do in the plane, since there could exist non-contractible cycles.

The last result is concerned with ordinary spheres. It is similar to Theorem 3.3 in [11], but we cannot prove that in point (b) all the cycles have the same period.

Theorem 2.9. *Let Σ be a sphere, and* (NT) *hold on Σ. Then one of the following holds.*

(a) *V has just one critical point and every orbit is homoclinic, both for V and for W.*
(b) *V has two critical points $O_1 \neq O_2$; every non-trivial orbit of V is a cycle; every non-trivial orbit of W has O_1 as positive limit set, O_2 as negative limit set, possibly exchanging O_1 and O_2.*
(c) *V has two critical points $O_1 \neq O_2$; every non-trivial orbit of V has O_1 as α-limit, O_2 as ω-limit; the same holds for W, possibly exchanging the words "α-limit" and "ω-limit".*

Proof. As in Theorem 3.3 of [11], considering that V and W have the same index because of trasversality. □

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Marco Sabatini
Dipartimento di Matematica
Universitá di Trento
I-38050 Povo, (TN), ITALY

E-mail address: sabatini@science.unitn.it