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# ON POSITIVE SOLUTIONS OF INDEFINITE INHOMOGENEOUS NEUMANN BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we study a class of inhomogeneous Neumann boundary value problems on a compact Riemannian manifold with boundary where indefinite and critical nonlinearities are included. Applying the fibering approach we introduce a new and, in some sense, more general variational approach to these problems. Using this idea we prove new results on the existence and multiplicity of positive solutions.

## 1. Introduction and Main results

In this paper, we study the existence and multiplicity of positive solutions for the following class of inhomogeneous Neumann boundary value problems with indefinite nonlinearities

$$-\Delta_p u - \lambda k(x)|u|^{p-2}u = K(x)|u|^{\gamma-2}u \quad \text{in } M, \tag{1.1}$$

$$|\nabla u|^{p-2}\frac{\partial u}{\partial n} + d(x)|u|^{p-2}u = D(x)|u|^{q-2}u \quad \text{on } \partial M, \tag{1.2}$$

where $M$ is a smooth connected compact Riemannian manifold of the dimension $n \geq 2$ with metric $g$ and boundary $\partial M$. $\Delta_p$ and $\nabla$, respectively, denotes the $p$-Laplace–Beltrami operator and the gradient in the metric $g$. $\partial/\partial n$ is the normal

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derivative with respect to the outward normal $n$ on $\partial M$ and the metric $g$. When $p = 2$ the problem corresponds to the classical Laplacian. We study the problem (1.1)–(1.2) with respect to the real parameter $\lambda$. In what follows we assume that

$$(1.3)\qquad p < \gamma \leq p^*, \quad \text{where } p^* = \begin{cases} \dfrac{pn}{n-p} & \text{if } p < n, \\ \infty & \text{if } p \geq n, \end{cases}$$

$$(1.4)\qquad p < q \leq p^{**}, \quad \text{where } p^{**} = \begin{cases} \dfrac{p(n-1)}{(n-p)} & \text{if } p < n, \\ \infty & \text{if } p \geq n, \end{cases}$$

and

$$(1.5)\qquad k(\,\cdot\,), K(\,\cdot\,) \in L_\infty(M), \quad d(\,\cdot\,), D(\,\cdot\,) \in L_\infty(\partial M).$$

Here $p^*$ and $p^{**}$ are the critical Sobolev exponents for the embedding $W_p^1(M) \subset L_{p^*}(M)$ and the trace-embedding $W_p^1(M) \subset L_{p^{**}}(\partial M)$, respectively. If $\gamma = p^*$ and/or $q = p^{**}$, then one has a problem with critical exponents. When all non-linear terms are present both in the differential equation (1.1) and in the non-linear Neumann boundary condition (1.2), i.e. when $K \neq 0$ in $M$ and $D \neq 0$ on $\partial M$ one has a inhomogeneous problem. The nonlinearity $K(x)|u|^{\gamma-2}u$ ($D(x)|u|^{q-2}u$) is called indefinite if the function $K$ on $M$ ($D$ on $\partial M$) changes the sign (cf. [1], [2]).

Problems like (1.1)–(1.2) arise in several contexts (see for example [5], [14]). In particular, when $p = 2$, $\gamma = p^*, q = p^{**}$, $n \geq 3$, the problem of the existence a positive solution for (1.1)–(1.2) is equivalent to the classical problem of finding a conformal metric $g'$ on $M$ with the prescribed scalar curvature $K$ on $M$ and the mean curvature $D$ on $\partial M$ [10], [6], [21]. For $p \neq 2$ we refer to [8] for background material and applications.

The case which is best known in the literature is the problem (1.1) with Dirichlet boundary condition when nonlinearity has definite sign. The indefiniteness of the sign of nonlinearity changes essentially the structure of the solutions set. In this case, the dependence of the problem on the parameter $\lambda$ is more complicate, and the problem of finding the bifurcation values of $\lambda$ is not simple (cf. [1]–[3]). The homogeneous cases with indefinite nonlinearity has been treated in several recent papers. The problem was studied in [3], [12], [22] for $p = 2$ and in [9] also for $p \neq 2$ with help of variational methods. In [19], the problem for $p = 2$ on compact manifold without boundary was investigated by bifurcation theory. The homogeneous problems (1.1)–(1.2) (i.e. when $D(x) \equiv 0$ or $K(x) \equiv 0$) involving critical exponent have been studied in [10]–[12], [15]. An additional difficulty occurs if the problem is inhomogeneous or has multiple critical exponents. For instance, in applying the constrained minimization method to the inhomogeneous problem finding of a suitable constraint or finding a suitable modification for the variational problem is not simple. In the case of multiple critical exponents, it is not well-understood in what levels of corresponding Euler functional the Palais-Smale condition holds (cf. [3], [10]–[12], [15]). The inhomogeneous cases of (1.1)–(1.2) for $p = 2$ with definite sign of nonlinearity have been considered in [21], [14].

The main purpose of the present paper is to show that the fibering scheme which we introduce can be easily used to study inhomogeneous Neumann boundary value problems (1.1)–(1.2) with indefinite nonlinearities, including the case of critical exponents.

The fibering scheme enables us to derive for inhomogeneous problem (1.1)–(1.2) two different minimization variational problems with constraints of mixed type that make possible to prove the existence of multiple positive solutions to (1.1)–(1.2). The fundamental property of constrained minimization problems obtained by the fibering scheme is that the solutions of these problems belong to a class of ground states for the associated Euler functional. Another feature of our approach is that the constructive property of the introduced fibering scheme make possible to study the dependence of problem (1.1)–(1.2) with respect to the parameter $\lambda$ in a more natural and constructive way. Therefore we can derive some new bounds for the parameter $\lambda$ which ensure the existence of positive solutions to (1.1)–(1.2) by this constructive method. An additional advantage of our approach is the ease of treatment with different kind of boundary condition (cf. [3], [18]).

Let us state our main results. To illustrate we consider the case $d(x) \equiv 0$. Denote by $d\mu_g$ and $d\nu_g$ the Riemannian measure (induced by the metric $g$) on $M$ and on $\partial M$, respectively. We consider our problem in the framework of the Sobolev space $W = W_p^1(M)$ equipped with the norm

$$||u|| = \left( \int_M |u|^p d\mu_g + \int_M |\nabla u|^p d\mu_g \right)^{1/p}. \tag{1.6}$$

Define

$$\begin{aligned} \lambda^*(K) &= \inf \left\{ \frac{\int_M |\nabla u|^p d\mu_g}{\int_M k(x)|u|^p d\mu_g} \;\middle|\; \int_M K(x)|u|^\gamma d\mu_g \geq 0,\ u \in W \right\}, \\ \lambda^*(D) &= \inf \left\{ \frac{\int_M |\nabla u|^p d\mu_g}{\int_M k(x)|u|^p d\mu_g} \;\middle|\; \int_{\partial M} D(x)|u|^\gamma d\nu_g \geq 0,\ u \in W \right\}. \end{aligned}$$

In the case when the set $\{u \in W_p^1(M) \mid \int_M K(x)|u|^\gamma d\mu_g \geq 0\}$ ($\{u \in W_p^1(M) \mid \int_{\partial M} D(x)|u|^\gamma d\nu_g \geq 0\}$) is empty we put $\lambda^*(K) = \infty$ ($\lambda^*(D) = \infty$).

Let us denote by $I_\lambda$ the Euler functional on $W_p^1(M)$ which corresponds to problem (1.1)–(1.2). Our main results on the existence and multiplicity of positive solutions for (1.1)–(1.2) are summarized in the following theorems.

Theorem 1.1. *Suppose that conditions* (1.5), $k(x) \geq 0$ *on* $M$, *and* $d(x) \equiv 0$ *are satisfied.*

(1) *If* $p < \gamma \leq p^*$ *and* $\int_M K(x)d\mu_g < 0$, *then* $\lambda^*(K) > 0$.
*If* $p < q \leq p^{**}$ *and* $\int_{\partial M} D(x)d\nu_g < 0$, *then* $\lambda^*(D) > 0$.

(2) *Assume* $\int_M K(x)d\mu_g < 0$, $\int_{\partial M} D(x)d\nu_g < 0$ *and* $q < \gamma$.
   (a) *Let* $p < \gamma \leq p^*$, $p < q \leq p^{**}$ . *Then for every* $\lambda \in (0, \min\{\lambda^*(K), \lambda_*(D)\})$ *there exists a weak positive solution* $u_1 \in W_p^1(M)$ *of* (1.1)–(1.2) *such that* $u_1 > 0$ *on* $M$ *and* $I_\lambda(u_1) < 0$.
   (b) *Let* $p < \gamma < p^*$ *and* $p < q < p^{**}$. *Suppose that the set* $\{x \in M \mid K(x) > 0\}$ *is not empty and* $D(x) \leq 0$ *on* $\partial M$. *Then for every* $\lambda < \lambda^*(K)$ *there exists a weak positive solution* $u_2 \in W_p^1(M)$ *of* (1.1)–(1.2) *such that* $u_2 > 0$ *on* $M$ *and* $I_\lambda(u_2) > 0$.

Theorem 1.2. *Suppose that conditions* (1.5), $k(x) \geq 0$ *on* $M$, $d(x) \equiv 0$, $p < \gamma \leq p^*$, $p < q \leq p^{**}$, *and* $q < \gamma$ *are satisfied. Furthermore, we assume*

(a) $\int_M K(x)d\mu_g < 0$,
(b) $D(x) \leq 0$ *on* $\partial M$.

*Then for every* $\lambda \in (0, \lambda^*(K))$ *there exists a ground state* $u_1 \in W_p^1(M)$ *of* $I_\lambda$. *Furthermore,* $u_1 > 0$, $I_\lambda(u_1) < 0$.

Remark 1.3. We refer to the Theorems 4.5, 4.10, 5.1, 5.2, for a more general version of the above results.

Remark 1.4. Similar results like Theorems 1.1, 1.2 (Theorems 4.5, 4.10, 5.1 and 5.2 in more general cases) can be obtained when $\lambda = 0$ ($\lambda \leq 0$) is fixed and the problem of the existence of positive solutions for (1.1) is considered with respect to parameter $\mu \in \mathbb{R}$ in the boundary condition

$$|\nabla u|^{p-2}\frac{\partial u}{\partial n} + \mu d(x)|u|^{p-2}u = D(x)|u|^{q-2}u \quad \text{on } \partial M,$$

instead of (1.2).

The paper is organized as follows. In Section 2, we introduce an explicit process of construction of the constrained minimization problems associated with the given abstract functional on Banach spaces. In Section 3, we give the basic variational formulation for problem (1.1)–(1.2). In Section 4 we prove our main results on the existence and multiplicity of positive solutions in subcritical cases of nonlinearities. Finally, in Section 5 we prove the existence of positive solutions in critical cases of exponents.

## 2. The fibering scheme

A powerful tool of studing the existence of critical points for a functional given on a Banach space is a constrained minimization method [3], [9], [10], [20].

The main difficulty in applying the method is to find suitable constraints on admissible functions and/or to find a suitable modification for the variational problem.

In this section we introduce a fibering scheme which allows us to find constructively a constrained minimization problem which will has property of ground among all constrained minimization problems corresponding to the given functional. Our approach is based on the fibering strategy introduced by Pohozaev in [16].

We assume that $(W, ||\cdot||)$ is a real reflexive Banach space. Furthermore, it will have the property that the norm $||\cdot||$ defines a $C^1$-functional $u \to ||u||$ on $W \setminus \{0\}$. In this case, the subset $W \setminus \{0\}$ is a submanifold of class $C^1$ in $W$. Thus we have the principal fibre bundle $P(W \setminus \{0\}, \mathbb{R}^+)$ over $W \setminus \{0\}$ with the structure group $\mathbb{R}^+$ and the bundle space $\mathbb{R}^+ \times W \setminus \{0\}$.

Let $I$ be a real functional of class $C^1(W \setminus \{0\})$. Corresponding to $I$ we define a functional $\widetilde{I}: \mathbb{R}^+ \times \{W \setminus \{0\} \to \mathbb{R}\}$ by

$$\widetilde{I}(t, v) = I(tv), \quad (t, v) \in \mathbb{R}^+ \times \{W \setminus \{0\}\}. \tag{2.1}$$

We have the following:

Proposition 2.1 (Pohozaev, [16], [17]). *Let $(t_0, v_0) \in \mathbb{R}^+ \times \{W \setminus \{0\}\}$ be a critical point of $\widetilde{I}$. Then $u_0 = t_0 v_0 \in W \setminus \{0\}$ is a critical point of $I$.*

We make an additional condition on the functional $I$ given by

(RD) The first derivative $\partial \widetilde{I}(t, v)/\partial t$ is a $C^1$-functional on $\mathbb{R}^+ \times \{W \setminus \{0\}\}$.

Now we define

$$Q(t, v) = \frac{\partial}{\partial t}\widetilde{I}(t, v), \quad L(t, v) = \frac{\partial^2}{\partial t^2}\widetilde{I}(t, v), \quad (t, v) \in \mathbb{R}^+ \times \{W \setminus \{0\}\}. \tag{2.2}$$

We introduce the following subsets of $\mathbb{R}^+ \times \{W \setminus \{0\}\}$ for the classification of the critical points

$$\Sigma^1 = \{(t, v) \in \mathbb{R}^+ \times \{W \setminus \{0\}\} \mid Q(t, v) = 0,\ L(t, v) > 0\}, \tag{2.3}$$

$$\Sigma^2 = \{(t, v) \in \mathbb{R}^+ \times \{W \setminus \{0\}\} \mid Q(t, v) = 0,\ L(t, v) < 0\}. \tag{2.4}$$

Proposition 2.2. *Let $(t_0, v_0) \in \Sigma^j$, $j = 1, 2$. Then there exist a neighbourhood $\Lambda(v_0) \subset W \setminus \{0\}$ of $v_0 \in W \setminus \{0\}$ and a unique $C^1$-map $t^j: \Lambda(v_0) \to \mathbb{R}$ such that*

$$t^j(v_0) = t_0, \quad (t^j(v), v) \in \Sigma^j, \quad v \in \Lambda(v_0),\ j = 1, 2. \tag{2.5}$$

Proof. Let $j = 1, 2$. We assume that $(t_0, v_0) \in \Sigma^j$ is satisfied. Then

$$\frac{\partial Q(t_0, v_0)}{\partial t} = L(t_0, v_0) \neq 0.$$

It follows from assumption (RD) that we have $Q(\,\cdot\,) \in C^1(\mathbb{R}^+ \times \{W \setminus \{0\}\})$. Hence, by application of the implicit function theorem, we can finish the proof.□

By means of Proposition 2.2 we get the following assertion.

Lemma 2.3. *Assume that* (RD) *holds. Let* $j = 1, 2$. *Then the set* $\Sigma^j$ *is a* $C^1$-*submanifold in* $\mathbb{R}^+ \times \{W \setminus \{0\}\}$ *which is locally* $C^1$-*diffeomorphic to* $W \setminus \{0\}$.

Using the fibering scheme we are able to study the existence of critical points of the functional $I$ defined on $W$.

Let $j = 1, 2$. Then $\widetilde{J}^j$ denotes the restriction of the functional $\widetilde{I}$ to the submanifolds $\Sigma^j$, i.e. we have

$$\widetilde{J}^j(t, v) = \widetilde{I}(t, v), \quad (t, v) \in \Sigma^j, \quad j = 1, 2.$$

Let $d\widetilde{I}(t, v)$ be the differential of $\widetilde{I}\colon \mathbb{R}^+ \times \{W \setminus \{0\}\} \to \mathbb{R}$ at the point $(t, v) \in \mathbb{R}^+ \times \{W \setminus \{0\}\}$, and $T_{(t,v)}(\mathbb{R}^+ \times \{W \setminus \{0\}\})$ denotes the tangent space to $\mathbb{R}^+ \times \{W \setminus \{0\}\}$ at $(t, v)$. Furthermore, let $d\widetilde{J}^j(t, v)$ be the differential of $\widetilde{J}^i\colon \Sigma^j \to \mathbb{R}$ at the point $(t, v) \in \Sigma^j$, and $T_{(t,v)}(\Sigma^j)$ denotes the tangent space to $\Sigma^j$ at $(t, v)$.

Using these definitions we are able to prove the following result which is important for our further considerations in this paper.

Lemma 2.4. *Let* $j = 1, 2$. *We suppose that* (RD) *holds. If* $(t_0, v_0)$ *is a critical point of the functional* $\widetilde{J}^j$ *on the submanifolds* $\Sigma^j$, *i.e. it holds*

$$d\widetilde{J}^j(t_0, v_0)(h) = 0 \quad \text{for all } h \in T_{(t_0,v_0)}(\Sigma^j), \tag{2.6}$$

*then the point* $(t_0, v_0)$ *is also critical point of* $\widetilde{I}$ *on* $\mathbb{R}^+ \times \{W \setminus \{0\}\}$. *Hence we have*

$$d\widetilde{I}(t_0, v_0)(l) = 0 \quad \text{for all } l \in T_{(t_0,v_0)}(\mathbb{R}^+ \times \{W \setminus \{0\}\}). \tag{2.7}$$

Proof. We consider the case $j = 1$. The other one can be handled in the same way. Let $(t_0, v_0)$ be a critical point of $\widetilde{J}^1$ on $\Sigma^1$. At first it holds

$$d\widetilde{I}(t_0, v_0)(\tau, \phi) = \frac{\partial}{\partial t}\widetilde{I}(t_0, v_0)(\tau) + \frac{\delta}{\delta v}\widetilde{I}(t_0, v_0)(\phi) \tag{2.8}$$

for every $\tau \in T_t(\mathbb{R}^+)$ and $\phi \in T_v(W \setminus \{0\})$.

Using (2.3) we have $\partial\widetilde{I}(t_0, v_0)(\tau)/\partial t = 0$. In order to prove (2.7) it is therefore sufficient to show that

$$\frac{\delta}{\delta v}\widetilde{I}(t_0, v_0)(\phi) = 0 \quad \text{for all } \phi \in T_{v_0}(W \setminus \{0\}) \tag{2.9}$$

holds. By Proposition 2.2 there exist a neighbourhood $\Lambda(v_0) \subset W \setminus \{0\}$ of $v_0 \in W \setminus \{0\}$ and a unique $C^1$-map $t^1\colon \Lambda(v_0) \to \mathbb{R}$ such that (2.5) holds. Hence we can introduce the map

$$J^1(v) =: \widetilde{I}(t^1(v), v), \quad v \in \Lambda(v_0).$$

Then we have

$$J^1(v) \equiv \widetilde{J}^1(t^1(v), v), \quad v \in \Lambda(v_0). \tag{2.10}$$

By Lemma 2.3 we know that $\Sigma^1$ is locally $C^1$-diffeomorphic to $W \setminus \{0\}$. Hence condition (2.6) implies that $v_0$ is a critical point of $J^1(v)$ on $\Lambda(v_0)$, i.e. it holds

$$dJ^1(v_0)(\phi) = 0$$

for all $\phi \in T_{v_0}(W \setminus \{0\})$. By our definition we have $J^1(v) = \widetilde{I}(t^1(v), v)$ for all $v \in \Lambda(v_0)$. Hence we get consequently by (2.8) that

$$0 = dJ^1(v_0)(\phi) = \frac{\partial}{\partial t}\overline{I}(t^1(v_0), v_0)(dt^1(v_0))(\phi) + \frac{\delta}{\delta v}\widetilde{I}(t^1(v_0), v_0)(\phi) \tag{2.11}$$

holds for all $\phi \in T_{v_0}(W \setminus \{0\})$. By virtue of (2.3), (2.4) the first term on the right-hand side of (2.11) is equal zero. Thus we have the desired result (2.9), i.e. it holds

$$\frac{\partial}{\partial v}\widetilde{I}(t^1(v_0), v_0)(\phi) = 0$$

for all $\phi \in T_{v_0}(W \setminus \{0\})$. The proof is finished. □

Now we can introduce ground constrained minimization problems associated with the given functional $I$. Assume $I$ of class $C^1(W \setminus \{0\})$, where $(W, ||\cdot||)$ is a real Banach space and the norm $||\cdot||$ is of class $C^1(W \setminus \{0\})$. Suppose that (RD) holds. We call the following problems defined by

$$\widehat{I}^j = \inf\{\widetilde{I}(t, v) \mid (t, v) \in \Sigma^j\}, \quad j = 1, 2, \tag{2.12}$$

the *ground constrained minimization problems* with respect to the fibering scheme. Here we put

$$\widehat{I}^j = \infty \quad \text{if} \quad \Sigma^j = \emptyset, \ j = 1, 2. \tag{2.13}$$

Definition 2.5. Let $j = 1, 2$. The point $(t_0, v_0) \in \Sigma^j$ is said to be a *solution of* (2.12) if

$$-\infty < \widehat{I}^j = \widetilde{I}(t_0, v_0) < \infty.$$

Remark 2.6. It is also meaningful to consider the corresponding maximization problems given by

$$\check{I}^j = \sup\{\widetilde{I}(t, v) \mid (t, v) \in \Sigma^j\}, \quad j = 1, 2. \tag{2.14}$$

Here we replace (2.13) by

$$\check{I}^j = -\infty \quad \text{if } \Sigma^j = \emptyset, \ j = 1, 2. \tag{2.15}$$

However, the substitution $I' = -I$ reduces any maximization problem to a minimization problem again.

Applying Lemma 2.4 and Proposition 2.1 we obtain the following main result.

Theorem 2.7. *Assume that* $I(u) \in C^1(W\backslash\{0\})$ *and* (RD) *hold. Let* $j = 1, 2$. *If there exists a solution* $(t_0^j, v_0^j)$ *of* (2.12), *then*

$$u_0^j = t_0^j v_0^j \in W \setminus \{0\} \tag{2.16}$$

*is a critical point of* $I$.

Let us show that the constrained minimization problems (2.12) possess the ground property.

We denote by $Z$ the set of all (nontrivial) critical points of the functional $I$ on the space $W \setminus \{0\}$. With respect to the fibering scheme we get the following decomposition of $Z$: $Z = Z_- \cup Z_+ \cup Z_0$, where

$$\begin{aligned} Z_+ &= \{u \in Z \mid (||u||, u) \in \Sigma^1\}, \\ Z_- &= \{u \in Z| \mid (||u||, u) \in \Sigma^2\}, \\ Z_0 &= \{u \in Z \mid (||u||, u) \in \mathcal{D}\}, \end{aligned}$$

with

$$\mathcal{D} = \{(t, v) \in \mathbb{R}^+ \times \{W \setminus \{0\}\} \mid Q(t, v) = 0,\ L(t, v) = 0\}. \tag{2.17}$$

From the applications [7] it is known that it is very important to know the ground states of the functionals. By definition (see [7]), the nonzero critical point $u_g \in W \setminus \{0\}$ is said to be the *ground state* if it is a point with the least level of $I$ among all nonzero critical points of $I$, i.e. it holds

$$\inf\{I(u) \mid u \in Z\} = I(u_g). \tag{2.18}$$

We introduce, in addition, the following term.

Definition 2.8. The nonzero critical point $u_g^- \in W$ ($u_g^+ \in W$) is said to be a *ground state of type* $(-1)$ (resp. $(0)$) for $I$ if

$$\min\{I(u) \mid u \in Z_-\} = I(u_g^-), \quad (\min\{I(u) \mid u \in Z_+\} = I(u_g^+)) \tag{2.19}$$

holds.

Thus we have the next result.

Lemma 2.9. *We assume that* $I(u) \in C^1(W \setminus \{0\})$ *and* (RD) *hold. Let* $j = 1, 2$. *If* $(t_0^j, v_0^j)$ *is a solution of the variational problem* (2.12), *then*

$$u^+ = t_0^1 v_0^1 \in W \setminus \{0\}$$

*is a ground state of type* $(0)$ *for* $I$, *and*

$$u^- = t_0^2 v_0^2 \in W \setminus \{0\}$$

*is a ground state of type* $(-1)$ *for* $I$. *Furthermore, if* $Z_0 = \emptyset$ *holds, then one of these solutions* $u^-$ *or* $u^+$ *is a ground state for* $I$, *i.e. we have*

$$\min\{I(u) \mid u \in Z\} = \min\{I(u_g^-), I(u_g^+)\}. \tag{2.20}$$

In the following, let $pr_2$ be the canonical projection from $\mathbb{R}^+ \times \{W \setminus \{0\}\}$ to $W \setminus \{0\}$ and let $\Theta^j = \mathrm{pr}_2(\Sigma^j)$, $j = 1, 2$.

As usual, let $j = 1, 2$. We recall that by Proposition 2.2 for every point $v_0^j \in \Theta^j$ there exist a neighbourhood $\Lambda(v_0^j) \subset \Theta^j$ and a unique $C^1$-map $t^j : \Lambda(v_0^j) \to \mathbb{R}$ such that $(t^j(v), v) \in \Sigma^j$ is satisfied.

Now we give the following definition.

Definition 2.10. Let $j = 1, 2$. The fibering scheme for $I$ on $W$ is said to be *uniquely solvable with respect to* $\Sigma^j$ if for every $v \in \Theta^j$ there exists a unique number $t^j(v) \in \mathbb{R}^+$ such that $(t^j(v), v) \in \Sigma^j$ holds. In the case when the fibering scheme for $I$ on $W$ is uniquely solvable with respect to both $\Sigma^1$ and $\Sigma^2$ then we call it a *solvable scheme*. If, in addition, the functional $t^j(v)$ can be found in an exact form, then the fibering scheme is called an *exactly solvable* one.

We remark that in the papers [3], [9], [20] the constrained minimization method were used for the investigation of homogeneous problems similar to (1.1)–(1.2). These problems can be solved by using the exactly solvable fibering scheme.

We point out that in the present paper we concern with applications where the fibering scheme is uniquely solvable but not necessary exactly solvable.

Observe that by Proposition 2.2 in the case of the uniquely solvable fibering scheme there exist unique global functionals:

$$t^j : \Theta^j \to \mathbb{R}^+, \quad j = 1, 2 \tag{2.21}$$

such that $(t^j(v), v) \in \Sigma^j$, $j = 1, 2$. Moreover, the sets $\Theta^j$, $j = 1, 2$ are submanifolds of class $C^1$ in $W \setminus \{0\}$ and $t_j(\,\cdot\,) \in C^1(\Theta_j)$, $j = 1, 2$. Hence we can define the following global functionals

$$J^1(v) = \widetilde{I}(t^1(v), v), \quad v \in \Theta^1, \tag{2.22}$$

$$J^2(v) = \widetilde{I}(t^2(v), v), \quad v \in \Theta^2. \tag{2.23}$$

Thus the variational problems (2.12) are reduced to the following equivalent ones

$$\widehat{I}^j = \min\{J^j(v) \mid v \in \Theta^j\}, \quad j = 1, 2. \tag{2.24}$$

It follows directly from Theorem 2.7.

Lemma 2.11. *Assume that the fibering scheme applied to the functional $I$ is uniquely solvable. Let $j = 1,2$ and $v_0^j \in \Theta_j$ is a solution of the problem* (2.24)*. Then $u_0^j = t^j(v_0^j)v_0^j$ is a nonzero critical point of the functional $I$.*

The following property is important for our further considerations in this paper

Proposition 2.12. *Let $j = 1,2$. Suppose the fibering scheme for $I$ on $W$ is uniquely solvable with respect to $\Sigma^j$. Then the functional $J^j(v)$ is* 0*-homogeneous, i.e. $J^j(sv) = J^j(v)$, $v \in W \setminus \{0\}$, $s \in \mathbb{R}^+$.*

## 3. The constrained minimization problems associated with (1.1)–(1.2)

In this section, we use the fibering scheme to introduce the constrained minimization problems for (1.1)–(1.2).

Let $(M,g)$ be a connected compact Riemannian manifold with boundary $\partial M$ of dimension $n \geq 2$. Let $g_{i,j}$ be the components of the given metric tensor $g = (g_{ij})$ with inverse matrix $(g^{i,j})$, and let $|g| = \det(g_{i,j})$. If $(x^i)$ is a locally system of coordinates, then we define the divergence operator $\mathrm{div}_g$ on the $C^1$ vector field $X = (X^i)$ by

$$\mathrm{div}_g X = \frac{1}{\sqrt{|g|}} \sum_i \frac{\partial}{\partial x_i}(\sqrt{|g|}X^i),$$

and the $p$-Laplace–Beltrami operator by $\Delta u = \mathrm{div}_g(|\nabla u|^{p-2}\nabla u)$. Here

$$\nabla u = \sum_i g^{i,j}\frac{\partial u}{\partial x_i}$$

denotes the gradient vector field of $u$. Let the Riemannian measure (induced by the metric $g$) on $M$ and $\partial M$, respectively, be denoted by $d\mu_g$ and $d\nu_g$, respectively.

We consider our problems in the framework of the Sobolev space $W = W_p^1(M)$ equipped with the norm

$$||u|| = \bigg(\int_M |u|^p\,d\mu_g + \int_M |\nabla u|^p\,d\mu_g\bigg)^{1/p}. \tag{3.1}$$

Let us introduce the following notations

$$\begin{aligned} f(u) &= \int_M k(x)|u|^p\,d\mu_g, & F(u) &= \int_M K(x)|u|^\gamma\,d\mu_g,\\ b(u) &= \int_{\partial M} d(x)|u|^p\,d\nu_g, & B(u) &= \int_{\partial M} D(x)|u|^q\,d\nu_g,\\ H_\lambda(u) &= \int_M |\nabla u|^p\,d\mu_g + b(u) - \lambda f(u). \end{aligned} \tag{3.2}$$

We recall that there is a continuous embedding $W_p^1(M) \subset L_{p^*}(M)$ and a continuous trace-embedding $W_p^1(M) \subset L_{p^{**}}(\partial M)$, respectively. Using the hypotheses (1.3)–(1.5) and these embedding results it is easy to check that all functionals in (3.2) are well-defined on the Sobolev space $W$ and belong to the class $C^1(W)$. The Euler functional $I$ on $W$ which corresponds to problem (1.1)–(1.2) is defined by

$$I_\lambda(u) = \frac{1}{p} H_\lambda(u) - \frac{1}{q} B(u) - \frac{1}{\gamma} F(u). \tag{3.3}$$

A function $u_0 \in W$ is called the *weak solution* for problem (1.1)–(1.2) if the following identity

$$\frac{\delta}{\delta u} I_\lambda(u_0)(\psi) = 0$$

holds for every function $\psi \in C^\infty(\overline{M})$. Hence the existence of weak solutions of problem (1.1)–(1.2) is equivalent to the existence of critical points for the Euler functional $I_\lambda$ defined above.

Let us apply to functional (3.3) the fibering scheme.

It is easily verified that the norm (3.1) defines a $C^1$-functional $u \to ||u||$ on $W \setminus \{0\}$. Hence the sphere $S^1 = \{v \in W \mid ||v|| = 1\}$ is a closed submanifold of class $C^1$ in $W$ and $S^1 \times \mathbb{R}^+$ is $C^1$-diffeomorphic with $W \setminus \{0\}$.

Following the fibering scheme, we associate with the original functional $I_\lambda$ a new fibering functional $\widetilde{I}_\lambda$ defined by

$$\widetilde{I}_\lambda(t, v) = I_\lambda(tv) = \frac{1}{p} t^p H_\lambda(v) - \frac{1}{q} t^q B(v) - \frac{1}{\gamma} t^\gamma F(v) \tag{3.4}$$

for $(t, v) \in \mathbb{R}^+ \times S^1$.

Next we define for $(t, v) \in \mathbb{R}^+ \times S^1$ the functionals

$$Q_\lambda(t, v) = \frac{\partial}{\partial t} \widetilde{I}_\lambda(t, v) = t^{p-1}(H_\lambda(v) - t^{q-p} B(v) - t^{\gamma-p} F(v)), \tag{3.5}$$

and

$$L_\lambda(t, v) = \frac{\partial^2}{\partial t^2} \widetilde{I}_\lambda(t, v) = t^{p-2}((p-1) H_\lambda(v) - (q-1) t^{q-p} B(v) - (\gamma - 1) t^{\gamma-p} F(v)). \tag{3.6}$$

Thus we can extract in $\mathbb{R}^+ \times S^1$ the sets

$$\Sigma_\lambda^1 = \{(t, v) \in \mathbb{R}^+ \times S^1 \mid Q_\lambda(t, v) = 0, L_\lambda(t, v) > 0\}, \tag{3.7}$$

$$\Sigma_\lambda^2 = \{(t, v) \in \mathbb{R}^+ \times S^1 \mid Q_\lambda(t, v) = 0, L_\lambda(t, v) < 0\}. \tag{3.8}$$

Thus in accordance with the fibering scheme we have the following main variational problems

$$\widehat{I}_\lambda^j = \inf\{\widetilde{I}_\lambda(t, v) \mid (t, v) \in \Sigma_\lambda^j\}, \quad j = 1, 2, \tag{3.9}$$

where

$$\widehat{I}^j_\lambda = \infty \quad \text{if } \Sigma^j_\lambda = \emptyset,\ j = 1, 2. \tag{3.10}$$

From (3.4) it follows that $I_\lambda$ satisfies the condition (RD).

It is easy to verify that the equation $Q_\lambda(t, v) = 0$ may have, in dependence on $H_\lambda(v)$, $B(v)$ and $F(v)$, at most two solutions on $\mathbb{R}^+$. The conditions $L_\lambda(t, v) < 0$ and $L_\lambda(t, v) > 0$ separate them: the equation $Q_\lambda(t, v) = 0$ may have at most one solution $t^1(v) \in \mathbb{R}^+$ such that $Q_\lambda(t^1(v), v) = 0$, $(t^1(v), v) \in \Sigma^1_\lambda$, and at most one solution $t^2(v) \in \mathbb{R}^+$ such that $Q_\lambda(t^2(v), v) = 0$, $(t^2(v), v) \in \Sigma_2$, respectively. Moreover, we have

$$t^j(\,\cdot\,) \in C^1(\Theta^j_\lambda), \quad j = 1, 2 \tag{3.11}$$

where $\Theta^j_\lambda = pr_2(\Sigma^j_\lambda)$, $j = 1, 2$ are submanifolds of class $C^1$ in $S^1$.

Thus we deal with the uniquely solvable fibering scheme and we can define

$$J^1_\lambda(v) = \widetilde{I}_\lambda(t^1(v), v), \quad v \in \Theta^1_\lambda, \tag{3.12}$$

$$J^2_\lambda(v) = \widetilde{I}_\lambda(t^2(v), v), \quad v \in \Theta^2_\lambda. \tag{3.13}$$

Thus the problem (3.9) is reduced to the following

$$\overline{J}^j_\lambda = \min\{J^j_\lambda(v) \mid v \in \Theta^j_\lambda\}, \quad j = 1, 2. \tag{3.14}$$

From Theorem 2.7 we have:

Lemma 3.1. *Let $j = 1, 2$. Assume that $v^j_0 \in \Theta^j_\lambda$ is a solution of the problem* (3.14). *Then $u^j_0 = t^j(v^j_0)v^j_0$ is a nonzero critical point of the functional $I_\lambda$.*

Remark 3.2. In the case when $p = 2$, $\gamma = 2^*, q = p^{**}$, $n \geq 3$, problems of type (1.1)–(1.2) have their root in Riemannian geometry. Let $(M, g)$ be a Riemannian manifold of dimension $n \geq 3$ with boundary $\partial M$, scalar curvature $k(x)$ of $M$ and mean curvature $d(x)$ of $\partial M$. Let $K$ be a given function on $M$ and $D$ be a fixed function on $\partial M$. One may ask the question: Can we find a new metric $\overline{g}$ on $M$ such that $K$ is the scalar curvature of $\overline{g}$ on $M$, $D$ is the mean curvature of $\overline{g}$ on $\partial M$ and $\overline{g}$ is conformal to $g$ (i.e. $\overline{g} = u^{4/(n-2)}g$ holds for some $u > 0$ on $M$)? This is equivalent (see Escobar [10], [11], Taira [21]) to the problem of finding positive solutions $u$ of (1.1)–(1.2) with critical exponents $\gamma = 2^*$ and $q = p^{**}$, where $k$ is the scalar curvature. Thus, by the fibering scheme we have also the variational statements (3.9) for this geometrical problem.

Remark 3.3. Observe that the variational definition (3.14) includes the formulations used by Escobar [10]–[12]. Indeed, let us consider the case $D(x) = 0$. This implies $B(\,\cdot\,) \equiv 0$ in (2.2). It is easy to verify that $L_\lambda(t(v), v) > 0$ and $L_\lambda(t(v), v) < 0$, respectively, holds, if $\operatorname{sgn}(F(v)) < 0$ and $\operatorname{sgn}(F(v)) > 0$, respectively. Hence we have $j = 1$ in the first case and $j = 2$ in the other one.

Observe that, for every $v \in \Theta^j_\lambda$ we can find the solutions $t^j(v)$ in the following explicit form

$$t^j(v) = \left(\frac{H_\lambda(v)}{F(v)}\right)^{1/(\gamma-2)},$$

for $j = 1, 2$, respectively. Thus one gets for every $v \in \Theta^j_\lambda$

$$J^j(v) = \frac{\gamma-2}{2\gamma}\frac{|H_\lambda(v)|^{\gamma/(\gamma-2)}}{|F(v)|^{2/(\gamma-2)}}\operatorname{sgn}(F(v)), \tag{3.15}$$

where $j = 1$ if $\operatorname{sgn}(F(v)) > 0$ and $j = 2$ if $\operatorname{sgn}(F(v)) < 0$. Thus in this case we have an exactly solvable fibering scheme.

Notice that the minimization problem (3.14) is equivalent to the variational formulation (3.14) with

$$\widehat{J}^j(v) = \frac{|H_\lambda(v)|}{|F(v)|^{2/\gamma}}\operatorname{sgn}(F(v)), \quad j = 1, 2,$$

instead of $J^j$, $j = 1, 2$.

In [10] the following variational formulation was considered

$$Q = \min\left\{\frac{H_\lambda(u)}{(\int_M |u|^\gamma d\mu_g)^{2/\gamma}} \;\middle|\; u \in W\right\} \tag{3.16}$$

corresponding to problem (1.1)–(1.2), however under the additional restriction $K(x) = Q$. Thus, we see that if $Q > 0$ holds, then problem (3.16) implies (3.14) in the case $j = 2$ and $K(x) = Q$, and if $Q < 0$ holds, then it implies the same problem with $j = 1$ and $K(x) = Q$.

Remark 3.4. We mention that $Q$ in (3.16) is related to the best Sobolev quotient for the embedding $W^1_2(M) \subset L_{2^*}(M)$ (see [10]–[12]). Thus the quotients $\overline{J}^j$, $j = 1, 2$, introduced by (3.14) are, in some sense, extensions of that concept.

## 4. The results on the existence and multiplicity in subcritical cases

In this section, we prove the main results of the paper, i.e. we show the existence and the multiplicity of positive solutions of (1.1)–(1.2).

Define

$$\lambda^*(K) = \inf\left\{\frac{\int_M |\nabla u|^p d\mu_g + b(u)}{\int_M k(x)|u|^p d\mu_g} \;\middle|\; F(u) \geq 0,\ u \in W\right\}, \tag{4.1}$$

$$\lambda^*(D) = \inf\left\{\frac{\int_M |\nabla u|^p d\mu_g + b(u)}{\int_M k(x)|u|^p d\mu_g} \;\middle|\; B(u) \geq 0,\ u \in W\right\}, \tag{4.2}$$

where in case when the set $\{u \in W^1_p(M) \mid F(u) \geq 0\}$ ($\{u \in W^1_p(M) \mid B(u) \geq 0\}$) is empty we put $\lambda^*(K) = \infty$ ($\lambda^*(D) = \infty$). Remark that

$$\lambda_1 = \inf\left\{\frac{\int_M |\nabla u|^p d\mu_g + b(u)}{\int_M k(x)|u|^p d\mu_g} \;\middle|\; u \in W^1_p(M)\right\} \tag{4.3}$$

and $\lambda_1$ is the simple first eigenvalue of the Neumann boundary problem

$$(4.4)\qquad \begin{aligned} -\Delta_p\phi_1 &= \lambda_1 k(x)|\phi_1|^{p-2}\phi_1 && \text{in } M,\\ |\nabla\phi_1|^{p-2}\frac{\partial\phi_1}{\partial n} + d(x)|\phi_1|^{p-2}\phi_1 &= 0 && \text{on } \partial M,\end{aligned}$$

where $\phi_1 > 0$ is a corresponding principal eigenfunction (see [23], [24]). Suppose that $k(x) \geq 0$ on $M$, $d(x) \geq 0$ on $\partial M$ then it follows immediately from the definitions that $0 \leq \lambda_1 \leq \lambda^*(K)$ and $0 \leq \lambda_1 \leq \lambda^*(D)$.

Let us prove the following main lemma.

LEMMA 4.1. *Assume that condition* (1.5) *holds, $k(x) \geq 0$ on $M$, and $d(x)\geq 0$ on $\partial M$.*

(a) *Suppose that $F(\phi_1) < 0$ and $p < \gamma \leq p^*$. Then $0 < \lambda^*(K)$.*
(b) *Suppose that $B(\phi_1) < 0$ and $p < q \leq p^{**}$. Then $0 < \lambda^*(D)$.*

PROOF. For instance we prove the first assertion (a). For our purpose it is important to prove separately some parts of the lemma in the subcritical cases of exponents and in the critical cases of exponents, respectively.

Let us suppose that $F(\phi_1) < 0$ is satisfied. Assume to the contrary that $0 = \lambda^*(K)$. Hence there exists a minimizing sequence $\{w_m\}$ for the problem (4.2) such that

$$E(w_m) = \frac{\int_M |\nabla w_m|^p d\mu_g + b(u)}{\int_M k(x)|w_m|^p d\mu_g} \to 0 = \lambda^*(K) \quad \text{as } m \to \infty,$$

where $F(w_m) \geq 0$, $m = 1, 2, \ldots$, see (4.1). The functional $E(\,\cdot\,)$ is 0-homogeneous. Therefore we may assume without loss of generality that the sequence $\{w_m\}$ is bounded and that $w_m \rightharpoonup w$ weakly to some $w \in W$.

Since $E$ is lower semi-continuous with respect to $W$ we get $E(w) \leq 0$. But 0 is a minimum of $E$ (see (4.3)) and therefore we get $E(w) = 0$.

Let us consider the subcritical cases, i.e. we assume that $p < \gamma < p^*$ holds. Then since $W$ is compactly embedded in $L_s(M)$ for $p \leq s < p^*$ we may assume that $F(w_m) \to F(w)$ as $m \to \infty$. Hence $F(w) \geq 0$. Note that the eigenvalue $\lambda_1 = 0$ is simple. Hence it follows that $w = r$ for some constant $r > 0$. This implies that we have $F(r) \geq 0$, a contradiction to our assumption $F(r) = r^\gamma F(\phi_1) < 0$.

Now let us consider also the critical case of the exponent. As it has been shown above it suffices to prove that $F(w) \geq 0$. However in these cases, we can not get this in a straightforward way as above. In fact, in this case the embedding $W^1_p(M) \subset L_{p^*}(M)$ is not compact.

However we can show that $w_m \to w$ strongly in $W$. Indeed, as it has been shown above we have $E(w) = \lambda_1$. This implies that

$$\int_M |\nabla w_m|^p \, d\mu_g \to \int_M |\nabla w|^p \, d\mu_g.$$

Now taking into account that $w_m \rightharpoonup w$ weakly in $W$ we get $w_m \to w$ strongly in $W$. Thus we have $F(w) \geq 0$.

Consequently, we have shown that $F(\phi_1) < 0$ implies $\lambda_1 = 0 < \lambda^*(K)$. □

Remark 4.2. The main difficulty in investigating the elliptic equations with critical exponents of nonlinearities is a "lack of compactness" (cf. [4], [20]). From the point of view of overcoming this difficulty, Lemma 4.1 plays the main role in our approach. Generally speaking, in our approach we reduce the problem of the lack of compactness mainly to the investigations at a bifurcation point $\lambda_1 = 0$.

Remark 4.3. Recall that, if the set

$$\{u \in W \mid F(u) \geq 0\} = \emptyset \quad (\{u \in W \mid B(u) \geq 0\} = \emptyset)$$

then $\lambda^*(K) = \infty$ ($\lambda^*(D) = \infty$). Thus in this case Lemma 4.1 is trivial. Remark, that if the conditions $\{u \in W \mid F(u) \geq 0\} = \emptyset$ and $\{u \in W \mid B(u) \geq 0\} = \emptyset$ are satisfied then for $\lambda > 0$ the problem (1.1)–(1.2) become coercive. Observe also the conditions $\{u \in W \mid F(u) \geq 0\} = \emptyset$ and $\{u \in W \mid B(u) \geq 0\} = \emptyset$ mean that $K(x) < 0$ on $M$ and $D(x) < 0$ on $M$, respectively.

Proposition 4.4. *Let* (1.1) *and* $k(x) \geq 0$ *on* $M$ , $d(x) \geq 0$ *on* $\partial M$ *be satisfied. Then the following conditions hold:*

(a) *if* $\lambda < \lambda^*(K)$ ($\lambda < \lambda^*(D)$) *and* $F(u) \geq 0$ ($B(u) \geq 0$) *for some* $u \in W$, *then* $H_\lambda(u) > 0$,
(b) *if* $\lambda < \lambda^*(K)$ ($\lambda < \lambda^*(D)$) *and* $H_\lambda(u) \leq 0$ *for some* $u \in W$, *then* $F(u) < 0$ ($B(u) < 0$).

Proof. The assertions follow immediately from the definitions, see (4.1)–(4.3). □

Let us formulate our main theorem on the existence of positive solutions for the family of problems (1.1)–(1.2) in the subcritical cases.

Theorem 4.5. *Suppose that conditions* (1.5), $k(x) \geq 0$ *on* $M$, *and* $d(x) \geq 0$ *on* $\partial M$, $p < q < p^{**}$, $p < \gamma < p^*$ *and* $q < \gamma$ *are satisfied.*

(a) *Assume that* $F(\phi_1) < 0$ *and* $B(\phi_1) < 0$ *hold. Then for every* $\lambda \in (\lambda_1, \min\{\lambda^*(K), \lambda_*(D)\}$ *there exists a weak positive solution* $u_1$ *of* (1.1)–(1.2) *such that* $u_1 > 0$ *on* $M$ *and* $u_1 \in W^1_p(M)$. *Furthermore, one has* $I_\lambda(u_1) < 0$.

(b) *Suppose that the set* $\{x \in M \mid K(x) > 0\}$ *is not empty and* $D(x) \le 0$ *on* $\partial M$. *Assume* $F(\phi_1) < 0$ *holds. Then for every* $\lambda < \lambda^*(K)$ *there exists a weak positive solution* $u_2$ *of* (1.1)–(1.2) *such that* $u_2 > 0$ *on* $M$ *and* $u_2 \in W^1_p(M)$. *Furthermore, we have* $I_\lambda(u_2) > 0$ *and* $u_2$ *is a ground state of type* $(-1)$ *for* $I_\lambda$.

**Proof of Theorem 4.5.** First let us prove the following

Lemma 4.6. *Let* $k(x) \ge 0$ *on* $M$, $d(x) \ge 0$ *on* $\partial M$, $p < q \le p^{**}$, $p < \gamma \le p^*$ *and* $q < \gamma$.

(a) *Assume* $F(\phi_1) < 0$ *holds. Then for every* $\lambda \in (\lambda_1, \lambda^*(K))$

$$\Theta^0_{1,\lambda} := \{w \in W \mid H_\lambda(w) < 0\} \subseteq \Theta_{1,\lambda} \tag{4.5}$$

*and the set* $\Theta^0_{1,\lambda}$ *is not empty.*

(b) *Suppose that the set* $\{x \in M \mid K(x) > 0\}$ *is not empty and* $D(x) \le 0$ *on* $\partial M$. *Then the set* $\Theta_{2,\lambda}$ *is not empty and, for every* $\lambda < \lambda^*(K)$,

$$\Theta_{2,\lambda} = \{w \in W \mid F(w) > 0\}. \tag{4.6}$$

Proof. We prove the first assertion. Remark, that by Proposition 4.5 we have $\lambda_1 < \min\{\lambda^*(K), \lambda^*(D)\}$). At first we show (4.5).

Let $\lambda \in (\lambda_1, \min\{\lambda^*(K), \lambda^*(D)\})$. We suppose that $w \in \Theta^0_{1,\lambda}$, i.e. $H_\lambda(w) < 0$ holds. Then by Proposition 4.4 we have that $F(w) < 0$ and $B(w) < 0$. These facts and (3.5) imply the existence of a number $t^1(w) > 0$ such that $Q(t^1(w), w) = 0$ and $L(t^1(w), w) > 0$ hold. Thus $w \in \Theta_{1,\lambda}$ and (4.5) is proved.

Let us consider the first eigenvalue $\phi_1 \in S^1$ of problem (4.4). Then for any $\lambda > 0$ we have $H_\lambda(\phi_1) < 0$. Thus $\phi_1 \in \Theta^0_{1,\lambda}$, and therefore the set $\Theta^0_{1,\lambda}$ is not empty for $\lambda \in (\lambda_1, \lambda^*(K))$. The first assertion is proved.

We show the second part. Assume that the set $\{x \in M \mid K(x) > 0\}$ is not empty. Then there exists a function $v_0 \in W$ such that $F(v_0) > 0$. Applying Proposition 4.4 we deduce that $H_\lambda(v_0) > 0$ holds for any $\lambda < \lambda^*(K)$. Recall that we have $p < q < \gamma$. Hence we obtain from (3.5) the existence of a number $t^2(v) > 0$ such that $Q(t^2(v), v) = 0$ and $L(t^2(v), v) < 0$. This implies $v \in \Theta_{2,\lambda}$. Thus the set $\Theta_{2,\lambda}$ is not empty and

$$\{w \in W \mid F(w) > 0\} \subseteq \Theta_{2,\lambda}. \tag{4.7}$$

Suppose $F(w) \le 0$ for some $w \in W$. By assumption we have $B(w) \le 0$. Hence the equation $Q(t, w) = 0$ may have a solution $t^2(w)$ only in the case when $H_\lambda(w) < 0$ is satisfied. However, in this case, we have $L(t^2(w), w) > 0$ by (3.6). This fact yields $w \notin \Theta_{2,\lambda}$ and therefore $\{w \in W \mid F(w) \le 0\} \cap \Theta_{2,\lambda} = \emptyset$. Using this and (4.7) we deduce (4.6). □

In the proof of our theorem we will restrict the functional $J^1_\lambda$ on the set $\Theta^0_{1,\lambda}$. Therefore we consider instead of the minimization problem (2.12) for $j = 1$ the following one

$$\overline{J}^{1,0}_\lambda = \min\{J^1_\lambda(v) \mid v \in \Theta^0_{1,\lambda}\}. \tag{4.8}$$

In order to prove the existence of the solution $u_1 \in W$ and $u_2 \in W$ we apply Lemma 2.11. Therefore, we show that the variational problem (4.8) has a solution $v_1 \in W$ and (3.14) with $j = 2$ has a solution $v_2 \in W$.

We remark that Lemma 4.6 implies also

$$J^{1,0}_\lambda(v) < 0 \quad \text{if } v \in \Theta^0_{1,\lambda}, \tag{4.9}$$

$$J^2_\lambda(v) > 0 \quad \text{if } v \in \Theta_{2,\lambda}. \tag{4.10}$$

Now we prove a mapping property of the functionals $J^j_\lambda$, $j = 1, 2$.

Lemma 4.7. *Let $k(x) \geq 0$ on $M$, $d(x) \geq 0$ on $\partial M$, $p < q < p^{**}$, $p < \gamma < p^*$ and $q < \gamma$.*

(a) *Assume that $F(\phi_1) < 0$, $B(\phi_1) < 0$. Let $\lambda \in (\lambda_1, \min\{\lambda^*(K), \lambda^*(D)\})$. Then the functional $J^1_\lambda(\,\cdot\,)$ defined on $\Theta^0_{1,\lambda}$ is bounded below, i.e.*

$$-\infty < \inf_{\Theta^0_{1,\lambda}} J^1_\lambda(w).$$

(b) *Suppose that the set $\{x \in M \mid K(x) > 0\}$ is not empty and $D(x) \leq 0$ on $\partial M$. Let $\lambda < \lambda^*(K)$. Then the functional $J^2_\lambda(\,\cdot\,)$ defined on $\Theta_{2,\lambda}$ is bounded below, i.e.*

$$-\infty < \inf_{\Theta_{2,\lambda}} J^2_\lambda(w).$$

Proof. Let us prove the first assertions. Observe that $\sup_{\Theta^0_{1,\lambda}} |J^1_\lambda(w)| = \infty$ if and only if there exists a sequence $v_m \in \Theta^0_{1,\lambda}$ , $m = 1, 2, \ldots$ such that $t^1(v_m) \to \infty$ as $m \to \infty$.

By Proposition 4.4, if $H_\lambda(v) \leq 0$ and $\lambda \in (\lambda_1, \min\{\lambda^*(K), \lambda^*(D)\})$ then we have $F(v) < 0$ and $B(v) < 0$. Since $H_\lambda(w)$ is bounded on $\Theta^0_{1,\lambda} \subset S^1$ we deduce from the equation $Q_\lambda(t^1(v), v) = 0$ (cf. (3.5)) that is impossible if $t^1(v) \to \infty$.

To prove the assertion (b) observe that from equation $Q_\lambda(t^2(v), v) = 0$ it follows that

$$\widetilde{I}_\lambda(t^2(v), v) = (t^2(v))^p \left[\left(\frac{1}{p} - \frac{1}{\gamma}\right) H_\lambda(v) - \left(\frac{1}{q} - \frac{1}{\gamma}\right)(t^2(v))^{q-p} B(v)\right]. \tag{4.11}$$

From Proposition 4.4 it follows that if $v \in \Theta_{2,\lambda}$ and $\lambda < \lambda^*(K)$ then $H_\lambda(v) > 0$ holds. Since by assumption $B(v) \leq 0$ we deduce from (4.11) that $J^2_\lambda(v) = \widetilde{I}_\lambda(t^2(v), v) > 0$ for $v \in \Theta_{2,\lambda}$ and therefore the assertion (b) holds. □

Lemma 4.8. *Let $k(x) \geq 0$ on $M$, $d(x) \geq 0$ on $\partial M$, $p < q < p^{**}$, $p < \gamma < p^*$ and $q < \gamma$.*

(a) *Assume that $F(\phi_1) < 0$, $B(\phi_1) < 0$, $\lambda \in (\lambda_1, \min\{\lambda^*(K), \lambda^*(D)\})$. Then the functional $J^1_\lambda(\,\cdot\,)$ defined on $\Theta^0_{1,\lambda}$ is weakly lower semi-continuous with respect to $W$.*

(b) *Suppose that the set $\{x \in M \mid K(x) > 0\}$ is not empty and $D(x) \leq 0$ on $\partial M$. Let $\lambda < \lambda^*(K)$. Then the functional $J^2_\lambda(\,\cdot\,)$ defined on $\Theta_{2,\lambda}$ is weakly lower semi-continuous with respect to $W$.*

Proof. Let $j = 1$ or $j = 2$ be fixed. We assume that $v_m \rightharpoonup v$ weakly with respect to $W$ for some $v \in \Theta_j$. Recall that $\Theta_j \subset S^1$ and therefore $\{v_m\}$ is bounded in $W$. Thus we may assume that

$$B(v_m) \to B(v), \quad F(v_m) \to F(v) \quad \text{as } m \to \infty, \tag{4.12}$$

$$H_\lambda(v_m) \to \widehat{H} \quad \text{as } m \to \infty, \tag{4.13}$$

where $\widehat{H}$ is finite. Since $H_\lambda(\cdot)$ is weakly lower semi-continuous with respect to $W$ we get

$$H_\lambda(v) \leq \widehat{H}. \tag{4.14}$$

From (4.12), (4.13) it follows that $\{t^j(v_m)\}$ is a convergent sequence. Furthemore one has $t^j(v_m) \to \widehat{t} < \infty$ as $m \to \infty$. Indeed, in both cases of assertions (a), (b) we have $F(v) \neq 0$ and $B(v) \neq \infty$, $|\widehat{H}| \neq \infty$ for $v \in \Theta_j$, $j = 1, 2$, respectively. From (3.5) it follows that supposing $t^j(v_m) \to \widehat{t} = +\infty$ as $m \to +\infty$ is impossible. Thus $t^j(v_m) \to \widehat{t} < \infty$ as $m \to \infty$.

Now we define

$$\widehat{I}(t) = \frac{1}{p} t^p \widehat{H} - \frac{1}{q} t^q B(v) - \frac{1}{\gamma} t^\gamma F(v)$$

for $t \in \mathbb{R}^+$. Then

$$J^j_\lambda(v_m) \to \widehat{I}(\widehat{t}) \quad \text{as } m \to \infty. \tag{4.15}$$

Let us prove the assertion (b). It follows from (4.14) that $\widehat{I}(\widehat{t}) \geq \widetilde{I}_\lambda(\widehat{t}, v)$. It is easy to see that $t^1(v)$ is the minimization point of the function $\widetilde{I}_\lambda(t, v)$ on $\mathbb{R}^+$. Therefore we have $\widetilde{I}_\lambda(\widehat{t}, v) \geq \widetilde{I}_\lambda(t_1(v), v)$ and, consequently,

$$\lim_{m \to \infty} J^1_\lambda(v_m) = \widehat{I}(\widehat{t}) \geq J^1_\lambda(v).$$

Hence $J^1_\lambda(v)$ is weakly lower semi-continuous on $\Theta^0_{1,\lambda}$ with respect to $W$.

Now Let us prove the assertion (b). Let us define

$$\widehat{Q}(t) = \frac{1}{t^{p-1}} \frac{\partial}{\partial t} \widehat{I}(t) \quad \text{and} \quad \widehat{L}(t) = \frac{1}{t^{p-2}} \frac{\partial^2}{\partial t^2} \widehat{I}(t)$$

for all $t \in \mathbb{R}^+$. Then it follows from (4.12), (4.13), (3.5) and (3.6) that

$$\widehat{Q}(\widehat{t}) = \widehat{H} - \widehat{t}^{q-p}B(v) - \widehat{t}^{\gamma-p}F(v) = 0, \tag{4.16}$$

and

$$\widehat{L}(\widehat{t}) = (p-1)\widehat{H} - (q-1)\widehat{t}^{q-p}B(v) - (\gamma-1)\widehat{t}^{\gamma-p}F(v) \leq 0. \tag{4.17}$$

Assume that we have equality in (4.17). Then by (4.16) and (4.17) we get

$$(\gamma - p)\widehat{H} - (\gamma - q)\widehat{t}^{q-p}B(v) = 0.$$

Recall that $B(v) \leq 0$ and $p < q < \gamma$ hold. Therefore, $\widehat{H} \geq 0$ is only possible in the case when $\widehat{H} = 0$. Then we deduce from (4.14) that $H_\lambda(v) \leq 0$. By (4.6) we have $F(v) > 0$ for $v \in \Theta^2_\lambda$. Hence, since $\lambda < \lambda^*(K)$ we obtain by Proposition 4.4 a contradiction. Thus we have in (4.17) a strong inequality. This implies that the function $\widehat{I}(t)$ defined on $\mathbb{R}^+$ attains a maximum at the point $\widehat{t}$. Using (4.14) we infer that

$$\lim_{m\to\infty} J^2_\lambda(v_m) = \widehat{I}(\widehat{t}) \geq \widehat{I}(t^2(v)) \geq \widetilde{I}_\lambda(t^2(v), v) = J^2_\lambda(v),$$

i.e. the second case is proved. □

Now we finish the proof of our theorem. We start with the first part of Theorem 4.5. Therefore we suppose that all corresponding assumptions are satisfied. We consider the minimization problem (4.8). Let $\{v_m\}$ be a minimizing sequence for this problem, i.e. we have $v_m \in \Theta^0_1$ and $J^1_\lambda(v_m) \to \overline{J}^{1,0}_\lambda$. Recall that

$$||v_m|| = 1 \quad \text{for} \quad m = 1, 2, \ldots \tag{4.18}$$

Thus $v_m$ is bounded in $W$. Hence since $W$ is reflexive we may assume $v_m \rightharpoonup v^1$ weakly for some $v^1 \in W$. Let us suppose, for the moment, that

$$v^1 \in \Theta^0_1 \tag{4.19}$$

holds. Then the boundedness and weakly lower semi-continuity of $J^1_\lambda$ shows that

$$-\infty < J^1_\lambda(v^1) \leq \overline{J}^1_\lambda.$$

Thus $v^1$ is solution of the problem (2.12).

Now we prove (4.19). First of all we observe from (4.18) that $v^1 \neq 0$. Indeed, assume to the contrary that $v^1 = 0$. Since $W^1_p(M)$ is compactly embedded in the space $L_p(M)$ and also compactly trace-embedded in the space $L_p(\partial M)$, we may assume $b(v_m) \to 0$ and $f(v_m) \to 0$ as $m \to \infty$. These and (4.18) imply $H_\lambda(v_m) > 0$ for $m$ large enough. Therefore we get a contradiction to the fact that $H_\lambda(v_m) < 0$ for $v_m \in \Theta^0_1$.

Now we show $v^1 \notin \partial\Theta_1^0$. It is sufficient to prove that the following strong inequality

$$H_\lambda(v^1) < 0 \tag{4.20}$$

holds. Using the weakly lower semi-continuity of $H_\lambda$ it follows from the definition of $v^1$ that $H_\lambda(v^1) \leq 0$. Assume to the contrary that $H_\lambda(v^1) = 0$. Since $\lambda < min\{\lambda^*(K), \lambda^*(D)\}$ we conclude by Proposition 4.4(b) that $F(v^1) < 0$, $B(v^1) < 0$. This fact and the continuity of $F$ on $L_\gamma(M)$ and $B$ on $L_q(\partial M)$ imply that $t^1(v_m) \to 0$ as $m \to \infty$. Applying now (3.4) we obtain that $\overline{I}_\lambda(t^1(v_m), v_m) \to 0$ as $m \to \infty$. On the other hand it is easy to see that $J^1_\lambda(v) < 0$ for all $v \in \Theta^0_{1,\lambda}$. Therefore we have a contradiction to the assumption that $\{v_m\}$ is minimizing sequence. Thus we have proved (4.20). Hence (4.19) is true.

Now let us prove the second statement of Theorem 4.5. Suppose that the corresponding assumptions of Theorem 4.5 hold. We consider the minimization problem (2.12) with $j = 2$. Let $\{v_m\}$ be a minimizing sequence for this problem, i.e. we have $v_m \in \Theta_2$ and $J^2_\lambda(v_m) \to \overline{J}^2_\lambda$. As above in the proof of the first part of Theorem 4.5 it can be shown that $v_m \rightharpoonup v^2$ weakly with some $v^2 \in W$. Therefore the proof is finished if

$$v^2 \in \Theta_2 \tag{4.21}$$

holds. By the second part of Lemma 4.6 it is sufficient to show that the following strong inequality

$$F(v^2) > 0 \tag{4.22}$$

holds. Assume to the contrary that $F(v^2) = 0$. Since $\lambda < \lambda^*(K)$ we conclude by Proposition 4.4(a) that $H_\lambda(v^2) > 0$. Hence using the continuity of $F$ on $L_\gamma(M)$, supposing $B(v_m) \leq 0$ we derive that $t^2(v_m) \to \infty$ as $m \to \infty$. Observe that by (3.8), (3.4), (3.5) we have

$$\widetilde{I}_\lambda(t^2(v_m)v_m) = (t^2(v_m))^p\left[\left(\frac{1}{p} - \frac{1}{\gamma}\right)H_\lambda(v_m) - \left(\frac{1}{q} - \frac{1}{\gamma}\right)(t^2(v_m))^{q-p}B(v_m)\right].$$

This fact, the lower semi-continuity of $H_\lambda$ and the inequalities $B(v_m) \leq 0$, $m = 1, 2, \ldots$ imply that $\widetilde{I}_\lambda(t^2(v_m), v_m) \to \infty$ as $m \to \infty$. Therefore we get a contradiction to the assumption that $\{v_m\}$ is minimizing sequence. Thus (4.21) is proved.

By Lemma 3.1 the functions $u_j = t^j(v^j)v^j$, $j = 1, 2$, are weak solutions of (1.1) and (1.2). Since the functional $I_\lambda$ is even then $u_j \geq 0$ in $M$. By the maximum principle and the Hopf lemma, since $u_j \not\equiv 0$, we see that $u_j > 0$ in $M$. Finally, it follows from (4.9) and (4.10), respectively, that $I_\lambda(u_1) > 0$ and $I_\lambda(u_2) < 0$. By Lemma 2.9 we have that $u_2$ is a ground state of type $(-1)$ for $I_\lambda$. The proof of Theorem 4.5 is finished. □

Let us prove the lemma on the existence of a ground state.

Lemma 4.9. *Suppose that conditions* (1.5), $k(x) \geq 0$ *on* $M$, $d(x) \geq 0$ *on* $\partial M$, $p < \gamma < p^*$, $p < q < p^{**}$ *and* $q < \gamma$ *are satisfied. Furthermore, assume*

(a) $F(\phi_1) < 0$,

(b) $D(x) \leq 0$ *on* $\partial M$.

*Then for every* $\lambda \in (\lambda_1, \lambda^*(K))$ *there exists a ground state* $u_1 \in W^1_p(M)$ *of* $I_\lambda$. *Furthermore, we have* $u_1 > 0$ *and* $I_\lambda(u_1) < 0$.

Proof. First let us remark that under the additional assumption $D(x) \leq 0$ on $\partial M$ we have

$$\Theta^0_{1,\lambda} = \Theta_{1,\lambda}. \tag{4.23}$$

Indeed, suppose $H_\lambda(w) \geq 0$ for some $w \in W$. By assumption we have $B(w) \leq 0$. Hence the equation $Q(t,w) = 0$ may have a solution $t^1(w) \neq 0$ only in the case when $F(w) > 0$ is satisfied. However, in this case, we have $L(t^1(w), w) < 0$ by (3.6). This fact yields $w \notin \Theta_{1,\lambda}$ and therefore $\{w \in W \mid H_\lambda(w) \leq 0\} \cap \Theta_{1,\lambda} = \emptyset$. Using this and Lemma 4.6 we deduce (4.23).

It follows from the proof of Theorem 4.5 and from (4.23) that there exists a positive solution $u_1 \in W^1_p(M)$ of variational problem (3.14), $j = 1$ such that $I_\lambda(u_1) < 0$.

Now let us show that $u_1$ is a ground state for $I_\lambda$. First note that for the solution $u_2$ of (3.14), $j = 2$ we have $I_\lambda(u_2) > 0$. Hence

$$\min\{I_\lambda(u_1), I_\lambda(u_2)\} = I_\lambda(u_1).$$

Therefore by Lemma 2.9 to prove our assertion it remains to show that the set

$$\partial\sigma = \{(t,v) \in \mathbb{R}^+ \times S^1 \mid Q(t,v) = 0,\ L(t,v) = 0\},$$

is empty. Assume the converse. Then by (3.5), (3.6) there exists $(t,v) \in \mathbb{R}^+ \times S^1$ such that the following system of equations holds

$$\begin{cases} H_\lambda(v_0) - t^{q-p}B(v_0) - t^{\gamma-p}F(v_0) = 0, \\ (p-1)H_\lambda(v_0) - (q-1)t^{q-p}B(v_0) - (\gamma-1)t^{\gamma-p}F(v_0) = 0. \end{cases} \tag{4.24}$$

From here we derive

$$(q-p)H_\lambda(v) + (\gamma-q)t^{\gamma-p}F(v) = 0. \tag{4.25}$$

However this is impossible since by Proposition 4.4 we have for $\lambda < \lambda^*(K)$, if $F(v) \geq 0$ then $H_\lambda(v) > 0$ and if $H_\lambda(u) \leq 0$ then $F(u) < 0$. The contradiction proves the lemma. □

From Theorem 4.5 and Lemma 4.9 we can derive the following multiplicity results.

Theorem 4.10. *Suppose that* (1.5), $k(x) \geq 0$ *on* $M$, $d(x) \geq 0$ *on* $\partial M$, $p < \gamma < p^*$, $p < q < p^{**}$ *and* $q < \gamma$ *are satisfied. Furthermore, assume*

(a) $F(\phi_1) < 0$,
(b) *the set* $\{x \in M \mid K(x) > 0\}$ *is not empty,*
(c) $D(x) \leq 0$ *on* $\partial M$.

*Then, for every* $\lambda \in (\lambda_1, \lambda^*(K))$, *there exists at least two different weak positive solutions* $u_1$ *and* $u_2$ *of* (1.1)–(1.2) *such that* $u_1 > 0$ *and* $u_2 > 0$ *on* $M$. *Furthermore, we have* $u_1, u_2 \in W_p^1(M)$, $I_\lambda(u_1) < 0$, $I_\lambda(u_2) > 0$. $u_1$ *is a ground state and* $u_2$ *is a ground state of type* $(-1)$ *for* $I_\lambda$.

## 5. The results on the existence in critical cases of exponents

In this section, we prove the result on the existence of positive solutions of (1.1)–(1.2) in the cases, where the exponents may be critical.

The main theorem in this section is the following

Theorem 5.1. *Suppose that* $k(x) > 0$ *on* $M$, $d(x) \geq 0$ *on* $\partial M$, $q < \gamma$, $p < \gamma \leq p^*$ *and* $p < q \leq p^{**}$ *are satisfied. Assume that* $F(\phi_1) < 0$ *and* $B(\phi_1) < 0$ *hold. Then for every* $\lambda \in (\lambda_1, \min\{\lambda^*(K),\ \lambda^*(D)\})$ *there exists a weak positive solution* $u_1$ *of* (1.1)–(1.2) *such that* $u_1 > 0$ *on* $M$ *and* $u_1 \in W_p^1(M)$. *Furthermore, one has* $I_\lambda(u_1) < 0$.

Proof. The proof in the cases $p < \gamma < p^*$ and $p < q < p^{**}$ follows from Theorem 4.5. The result for critical exponents $\gamma = p^*$ and $q = p^{**}$ will be obtained by limiting arguments from the subcritical cases. As an example, let us suppose that $\gamma = p^*$ and $p < q < p^{**}$ hold. The other cases can be done analogously.

Let $p < \beta \leq p^*$. Then we define

$$F_\beta(u) = \int_M K(x)|u|^\beta \, dm_g, \quad u \in W.$$

Analogously one defines $\lambda^*_\beta(K)$. We assume that $F_{p^*}(\phi_1) < 0$. Then it follows from Lemma 4.1 that $\lambda_1 < \min\{\lambda^*_{p^*}(K), \lambda^*(D)\})$. Furthermore, let $\lambda_0 \in (\lambda_1, \min\{\lambda^*(p^*), \lambda^*(D)\})$. Then it is easy to see that one can find a number $\varepsilon > 0$ such that $F_\beta(\phi_1) < 0$, $|p^* - \beta| < \varepsilon$ and $\lambda^*_\beta(K) \to \lambda^*_{p^*}(K)$ as $\beta \to p^*$. Hence we have $\lambda_0 \in (\lambda_1, \min\{\lambda^*_\beta(K), \lambda^*(D)\}$ if $|p^* - \beta| < \varepsilon_0$ for some $\varepsilon_0 > 0$. Applying now Theorem 4.5 we obtain the existence of a weak positive solution $u_{1,\beta}$ of (1.1)–(1.2) with $\gamma = \beta$ such that

$$\begin{aligned}(5.1)\quad & \int_M |\nabla u_{1,\beta}|^{p-2} \nabla u_{1,\beta} \nabla \psi \, d\mu_g - \lambda_0 \int_M k(x)|u_{1,\beta}|^{p-2} u_{1,\beta} \psi \, d\mu_g \\ & - \int_M K(x)|u_{1,\beta}|^{\beta-2} u_{1,\beta} \psi \, d\mu_g - \int_{\partial M} D(x)|u_{1,\beta}|^{q-2} u_{1,\beta} \psi \, d\nu_g = 0\end{aligned}$$

holds for any $\psi \in C^\infty(\overline{M})$.

We show that the functions $u_{1,\beta}$ are uniformly bounded in the $W$-norm. Suppose to the contrary that $||u_{1,\beta_i}|| \to \infty$ for some sequence $\beta_i$ such that $\beta_i \to p^*$ as $i \to \infty$. Let $v_{1,\beta_i} = u_{1,\beta_i}/||u_{1,\beta_i}||$ for $i = 1, 2, \dots$ Then we have $u_{1,\beta_i} = t^1(v_{1,\beta_i})v_{1,\beta_i}$, where $||v_{1,\beta_i}|| = 1$ and by assumption $t^1(v_{1,\beta_i}) \to \infty$.

Since the functions $v_{1,\beta_i}$ are uniformly bounded in the $W$-norm then, by weak compactness, we can find a weak convergent subsequence of $\{v_{1,\beta_i}\}$ (again denoted by $\{v_{1,\beta_i}\}$) which converges weakly to some point $w \in W$.

Suppose that $w = 0$. Since $W$ is compactly embedded in $L_p(M)$ and compactly trace-embedded in $L_p(\partial M)$ we may assume that $\int_M k(x)u_{1,\beta_i}v_{1,\beta_i}\, d\mu_g \to 0$ as $i \to \infty$. This implies $H_{\lambda_0}(v_{1,\beta_i}) > 0$ for $\beta_i$ near $p^*$. Therefore we get a contradiction to the fact that $H_{\lambda_0}(v_{1,\beta_i}) < 0$ for $v_{1,\beta_i} \in \Theta^*_{1,\beta_i}$. Thus $w \neq 0$ and therefore we can find $\psi_0 \in C^\infty(\overline{M})$ such that

$$\int_M K(x)|w|^{p^*-2}w\psi_0\, d\mu_g \neq 0. \tag{5.2}$$

It follows from (5.1) that

$$\begin{aligned}\int_M |\nabla v_{1,\beta_i}|^{p-2}\nabla v_{1,\beta_i}\nabla\psi_0\, d\mu_g - \lambda_0 \int_M k(x)|v_{1,\beta_i}|^{p-2}v_{1,\beta_i}\psi_0\, d\mu_g \\ = t^1(v_{1,\beta_i})^{\beta_i-2}\int_M K(x)|v_{1,\beta_i}|^{\beta_i-2}v_{1,\beta_i}\psi_0\, d\mu_g \\ + t^1(v_{1,\beta_i})^{q-2}\int_{\partial M} D(x)|v_{1,\beta_i}|^{q-2}v_{1,\beta_i}\psi_0 d\nu_g.\end{aligned} \tag{5.3}$$

Since $W^1_p$ is compactly embedded in $L_s(M)$ for $p < s < p^*$ and trace-embedded in $L_q(\partial M)$ for $p < q < p^{**}$, it follows that $v_{1,\beta_i} \to w$ in $L_s(M)$, $p < s < p^*$ and in $L_q(\partial M)$, $p < q < p^{**}$. Hence and by (5.2) it follows that the right hand side of (5.3) converges to infinity as $i \to \infty$ in contrast to the fact that the left hand side of this equality is bounded. Thus we get a contradiction and the functions $u_{1,\beta}$ are uniformly bounded in the $W$-norm.

Therefore, by weak compactness, we can find a weak convergent subsequence of $\{u_{1,\beta}\}$ (again denoted by $\{u_{1,\beta}\}$). Since $W^1_p$ is compactly embedded in $L_s(M)$ for $p < s < p^*$ and trace-embedded in $L_q(\partial M)$ for $p < q < p^{**}$, it follows easily that the weak $W^1_p$-limit $u_{1,p^*}$ of the sequence $u_{1,\beta}$ satisfies also (5.1). To prove our theorem it remains to show that $u_{1,p^*}$ is nonzero. Suppose to the contrary that $u_{1,p^*} = 0$. Let $v_{1,\beta} = u_{1,\beta}/||u_{1,\beta}||$. Then $u_{1,\beta} = t^1(v_{1,\beta})v_{1,\beta}$ where $||v_{1,\beta}|| = 1$. Hence $t^1(v_{1,\beta}) \to 0$ and/or $v_{1,\beta} \rightharpoonup 0$ weakly with respect to $W$ as $\beta \to p^*$.

Suppose the second case holds: $v_{1,\beta} \rightharpoonup 0$ weakly as $\beta \to p^*$. Since $W^1_p$ is compactly embedded in $L_s(M)$ for $p < s < p^*$, we may assume $f(v_{1,\beta}) \to 0$ as $\beta \to p^*$. This implies $H_{\lambda_0}(v_{1,\beta}) > 0$ for $\beta$ near $p^*$. Therefore we have a contradiction to the fact that $H_{\lambda_0}(v_{1,\beta})) < 0$ for $v_{1,\beta} \in \Theta^0_{1,\beta}$.

Thus $v_{1,p^*} \neq 0$. Suppose now that $t^1(v_{1,\beta}) \to 0$ as $\beta \to 0$. By virtue of (5.1) we have

$$\int_M |\nabla v_{1,\beta}|^{p-2}\nabla v_{1,\beta}\nabla\psi\, d\mu_g - \lambda_0 \int_M k(x)|v_{1,\beta}|^{p-2}v_{1,\beta}\psi\, d\mu_g = t^1(v_{1,\beta})^{\beta-1}\int_M K(x)|v_{1,\beta}|^{\beta-2}v_{1,\beta}\psi\, d\mu_g + t^1(v_{1,\beta})^{q-1}\int_{\partial M} D(x)|v_{1,\beta}|^q v_{1,\beta}\psi\, d\nu_g. \tag{5.4}$$

Passing to the limit in (5.4) as $\beta \to p^*$ we get

$$\int_M |\nabla v_{1,p^*}|^{p-2}\nabla v_{1,p^*}\nabla\psi\, d\mu_g - \lambda_0 \int_M k(x)|v_{1,p^*}|^{p-2}v_{1,p^*}\psi\, d\mu_g = 0. \tag{5.5}$$

By the maximum principle and the Hopf lemma, since $v_{1,p^*} \not\equiv 0$, we see that $v_{1,p^*} > 0$ in $\overline{M}$. Substitute $\psi = 1$ in (5.5)

$$\lambda_0 \int_M k(x)v_{1,p^*}\, d\mu_g = 0. \tag{5.6}$$

Hence, since $v_{1,p^*} > 0$, $k(x) \geq 0$ and by assumption $\lambda_0 > 0$ we get a contradiction. Thus there exists a weak solutions $u_{1,p^*} \geq 0$ of problem (1.1)–(1.2) with $\gamma = p^*$ and $p < q < p^{**}$.

Since the functional $H_\lambda$ is a weakly lower semi-continuous on $W_p^1$ we have $H_\lambda(u_{1,p^*}) \leq \liminf_{\beta\to p^*} H_\lambda(u_{1,\beta}) < 0$. Hence for $\lambda \in (\lambda_1, \min\{\lambda^*(p^*), \lambda^*(D)\})$ it follows $F(u_{1,p^*}) < 0$,$B(u_{1,p^*}) < 0$, by Proposition 4.4. It means that $I_\lambda(u_{1,p^*}) < 0$. By the maximum principle and the Hopf lemma, since $u_{1,p^*} \not\equiv 0$, we see that $u_{1,p^*} > 0$ in $\overline{M}$. The proof of the Theorem 5.1 is finished. □

Let us prove the following result on the existence of ground state in critical cases.

Theorem 5.2. *Suppose that* (2.1), $k(x) \geq 0$ *on* $M$, $d(x) \geq 0$ *on* $\partial M$, $p < \gamma \leq p^*$, $p < q \leq p^{**}$ *and* $q < \gamma$ *are satisfied. Furthermore, assume*

(a) $F(\phi_1) < 0$,
(b) $D(x) \leq 0$ *on* $\partial M$.

*Then for every* $\lambda \in (\lambda_1, \lambda^*(K))$ *there exists a ground state* $u_1 \in W_p^1(M)$ *of* $I_\lambda$. *Furthermore, we have* $u_1 > 0$ *and* $I_\lambda(u_1) < 0$.

Proof. The existence of ground state in subcritical cases of exponents $p < \gamma < p^*$, $p < q < p^{**}$ follows from Lemma 4.9. As an example, let us prove the assertion of the theorem for the following critical case $p < q < p^{**}$, $\gamma = p^*$. The other cases can be done analogeously.

Suppose $\lambda \in (\lambda_1, \lambda^*(K))$ and let $u_{1,\beta}$ be a ground state of $I_{\lambda,\beta}$ when $p < \beta < p^*$. Using the same arguments as in proving of Theorem 5.1 one can show the existence of weak convergent subsequence $u_{1,\beta_i} \rightharpoonup u_{1,p^*}$ with respect to $W$

as $\beta_i \to p^*$ where $u_{1,p^*}$ is a positive solution of (1.1)–(1.2). Let us show that $u_{1,p^*}$ is a ground state.

First note that the functional $J^1_{\lambda,\beta}(\,\cdot\,)$ defined on $\Theta^0_{1,\lambda,\beta}$ is bounded below, i.e.

$$-\infty < \overline{J}^1_{\lambda,\beta} = \inf\{J^1_{\lambda,\beta}(w) \mid w \in \Theta_{1,\lambda}\}$$

for $\lambda \in (\lambda_1, \lambda^*(K))$ and $p < \beta \le p^*$ (see Lemma 4.7). Next we remark that for every $w \in \Theta_{1,\lambda}$ the function $J^1_{\lambda,\beta}(w)$ is continuous with respect to $\beta \in (p, p^*]$. Hence it follows that $\overline{J}^1_{\lambda,\beta}$ is also continuous with respect to $\beta \in (p, p^*]$ and

$$\overline{J}^1_{\lambda,\beta} \to \overline{J}^1_{\lambda,p^*} \quad \text{as } \beta \to p^*. \tag{5.7}$$

Thus to prove the claim it is sufficient to show that

$$J^1_{\lambda,p^*}(v_{1,p^*}) \le \overline{J}^1_{\lambda,p^*} = \lim_{\beta \to p^*} J^1_{\lambda,\beta}(v_{1,\beta}). \tag{5.8}$$

where $v_{1,\beta} = u_{1,\beta}/||u_{1,\beta}||$.

Observe that from the convergence $u_{1,\beta_i} \rightharpoonup u_{1,p^*}$ it follows that

$$B(u_{1,\beta_i}) \to \widehat{B}, \quad F_{\beta_i}(u_{1,\beta_i}) \to \widehat{F} \quad \text{as } i \to \infty, \tag{5.9}$$

$$H_\lambda(u_{1,\beta_i}) \to \widehat{H} \quad \text{as } i \to \infty, \tag{5.10}$$

where $\widehat{H}$, $\widehat{F}$, $\widehat{B}$ are finite. Since $H_\lambda(\,\cdot\,)$ is weakly lower semi-continuous with respect to $W$ we have

$$H_\lambda(u_{1,p^*}) \le \widehat{H}. \tag{5.11}$$

Let us show that

$$F_{p^*}(u_{1,p^*}) \le \widehat{F}. \tag{5.12}$$

Consider a finite partition of unity for $M$: $\psi_j\colon M \to \mathbb{R}$, $\operatorname{supp}(\psi_j) \subset M$, $0 \le \psi_j \le 1$, $\sum_j \psi_j(x) \equiv 1$ on $M$. Let $p < \beta < p^*$. Then testing (1.1) by $\psi_j u_{1,\beta_i}$ we obtain

$$\begin{aligned} &\int_M |\nabla u_{1,\beta_i}|^p \psi_j \, d\mu_g + \int_M |\nabla u_{1,\beta_i}|^{p-2} (\nabla u_{1,\beta_i}, \nabla \psi_j) \, d\mu_g \\ &\qquad - \lambda \int_M k(x) |u_{1,\beta_i}|^p \psi_j \, d\mu_g - \int_M K(x) |u_{1,\beta_i}|^{\beta_i} \psi_j \, d\mu_g = 0. \end{aligned} \tag{5.13}$$

From the weak convergence $u_{1,\beta_i} \rightharpoonup u_{1,p^*}$ with respect to $W$ and strong convergence $u_{1,\beta_i} \to u_{1,p^*}$ in $L_s(M)$, $p < s < p^*$ it follows that

$$H_\lambda(u_{1,\beta_i}\psi_j) \to \widehat{H}_j, \quad F_{\beta_i}(u_{1,\beta_i}\psi_j) \to \widehat{F}_j \quad \text{as } i \to \infty, \tag{5.14}$$

and

$$\int_M |\nabla u_{1,\beta_i}|^{p-2} (\nabla u_{1,\beta_i}, \nabla \psi_j) \, d\mu_g \to \int_M |\nabla u_{1,p^*}|^{p-2} (\nabla u_{1,p^*}, \nabla \psi_j) \, d\mu_g$$

as $i \to \infty$. Hence passing to the limit in (5.13) we deduce

$$\widehat{H}_j + \int_M |\nabla u_{1,p^*}|^{p-2}(\nabla u_{1,p^*}, \nabla \psi_j)\, d\mu_g = \widehat{F}_j. \tag{5.15}$$

On the other hand from (1.1) in critical case $\gamma = p^*$ we have

$$H_\lambda(u_{1,p^*}\psi_j) + \int_M |\nabla u_{1,p^*}|^{p-2}(\nabla u_{1,p^*}, \nabla \psi_j) d\mu_g = F_{p^*}(u_{1,p^*}\psi_j). \tag{5.16}$$

Hence since $\widehat{H}_j \geq H_\lambda(u_{1,p^*}\psi_j)$ it follows that $F_{p^*}(u_{1,p^*}\psi_j) \leq \widehat{F}_j$. Thus by summing these inequalities we obtain (5.12).

Observe that from the equation $Q_\lambda(t^1(v_{1,\beta}), v_{1,\beta}) = 0$, $\beta \in (p, p^*]$ it follows that

$$J^1_{\lambda,p^*}(v_{1,\beta}) = \frac{q-p}{pq}(t^1(v_{1,\beta}))^p H_\lambda(v_{1,\beta}) + \frac{\gamma - q}{\gamma q}(t^1(v_{1,\beta}))^\gamma F_\beta(v_{1,\beta}).$$

Hence from (5.9)–(5.12) we deduce

$$J^1_{\lambda,p^*}(v_{1,p^*}) \leq \frac{q-p}{pq}\widehat{H} + \frac{\gamma - q}{\gamma q}\widehat{F} = \lim_{\beta \to p^*} J^1_{\lambda,\beta}(v_{1,\beta}).$$

Thus we obtain (5.8) and the proof of theorem completes. □

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