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ASYMPTOTICALLY CRITICAL POINTS AND MULTIPLE ELASTIC BOUNCE TRAJECTORIES

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Abstract. We study multiplicity of elastic bounce trajectories (e.b.t.'s) with fixed end points A and B on a nonconvex "billiard table" Ω. As well known, in general, such trajectories might not exist at all. Assuming the existence of a "bounce free" trajectory γ_0 in Ω joining A and B we prove the existence of multiple families of e.b.t.'s γ_λ bifurcating from γ_0 as a suitable parameter λ varies. Here λ appears in the dynamics equation as a multiplier of the potential term.

We use a variational approach and look for solutions as the critical points of the standard Lagrange integrals on the space $X(A,B)$ of curves joining A and B. Moreover, we adopt an approximation scheme to obtain the elastic response of the walls as the limit of a sequence of repulsive potentials fields which vanish inside Ω and get stronger and stronger outside. To overcome the inherent difficulty of distinct solutions for the approximating problems covering to a single solutions to the limit one, we use the notion of "asymptotically critical points" (a.c.p.'s) for a sequence of functional. Such a notion behaves much better than the simpler one of "limit of critical points" and allows to prove multiplicity theorems in a quite natural way.

A remarkable feature of this framework is that, to obtain the e.b.t.'s as a.c.p.'s for the approximating Lagrange integrals, we are lead to consider the L^2 metric on $X(A,B)$. So we need to introduce a nonsmooth version of the definition of a.c.p. and prove nonsmooth versions of the multliplicity theorems, in particular of the "∇-theorems" used for the bifurcation result. To this aim we use several results from the theory of φ-convex functions.

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1. Introduction

When studying multiple trajectories of a point ball going from a point A to a point B in a billiard table Ω, having rigid and perfectly elastic walls, it seems spontaneous to ask oneself whether there exists a functional having the same properties of the integral of the Lagrangian, that is a functional such that its critical points (in some sense) on a suitable space of curves joining A and B, are the expected elastic bounce trajectories. Roughly speaking one could wonder whether there exists a Hamilton-like principle for the elastic bounce trajectories from A to B in Ω.

If Ω is convex, a variational approach has been possible and fruitful, and in [7] has allowed to prove the existence of infinitely many elastic bounce trajectories joining two arbitrarily chosen points A and B in an arbitrarily chosen time interval $[0,T]$, also in presence of a conservative force field. We can say that the main idea for proving such a result is looking at the billiard table as a plate "with two faces" Ω^+ and Ω^-, each one being a copy of Ω, joined through the boundary $\partial\Omega$, and noticing that the elastic bounce trajectories in Ω correspond to the geodesics turnig around the plate, provided Ω is convex. In fact in [7] an approximation approach is used to prove the result: the "plate" is approximated by a sequence of "biconvex lens like" manifolds, whose edges coincide with $\partial\Omega$, getting flatter and flatter, and the desired trajectories are found as limits of geodesics on such lenticular manifolds. In this case, for the approximating manifolds, the functional is the usual integral of the Lagrangian.

If the billiard table is not convex this appoach fails since the curves obtained with this method are not, in general, bounce trajectories anymore. In this case it should be first pointed out that in general it is not true that even one trajectory exists. As well known, (see [18]), if Ω is the "Penrose mushroom", then no elastic bounce trajectories exist for A and B chosen in an appropriate way.

For the nonconvex billiard table some interesting results were proved in [2], [10], concerning the existence of bounce trajectories with few bounce points. Other results, concerning the Cauchy problem, even in the case of Ω being a manifold, possibly with nonsmooth boundary, are treated in [4], [19], [3].

In both these sets of papers the walls of the billiard table are approximated by a suitable sequence of repulsive force fields, having potentials which are zero in Ω and tend to ∞ outside Ω. The bounce trajectories are then found as limits of solutions of the approximating dynamics equations. Notice that some care is needed in the choice of the approximation, otherwise the resulting limits may not be elastic bounce trajectories.

It must be pointed out that a difficulty arises whenever looking for multiple solutions of some problem as limits of solutions of approximating problems. It can indeed happen that distinct sequences of solutions of the approximating problems have the same limit. In the case where the approximating solutions are obtained as critical points of a sequence of functionals converging to a limit one, it is sometimes possible to individuate distinct sequences of solutions whose critical values have distinct limits, making thus possible to distinguish the corresponding limit solutions. This is actually the case in [7], [2], [10].

We as well have considered a sequence of potentials to approximate the elastic reactions of the table walls, but in our main result (see (c) of Theorem 2.13), it can happen that the critical values corresponding to distinct approximating solutions converge to the same limit, and nevertheless more than one solution is expected at that level. To get rid of this fact, following a method introduced in [11], which was inspired by [9], [1], we have studied the "asymptotically critical points" γ for the sequence of the approximating Lagrangian integrals f_n, that is the points obtained as limits of sequences (γ_n) such that $\gamma_n \to \gamma$ and the "gradients" of f_n at γ_n (are not necessarily zero, but) tend to zero. From [11] we know that, under suitable assumptions, the multiplicity of such points is precisely what the topological features of the sublevels of f_n suggest.

Actually with this method another difficulty arises, since the asymptotically critical points of f_n with respect to the metric of $W^{1,2}$, which seems to fit naturally the Lagrange integral, turn out not to be necessarily elastic bounce trajectories, but more generally all those bounce trajectories obtained in presence of an inelastic reaction of the billiard table wall (even plastic or hyperelastic).

But an interesting fact allows to overcome this difficulty: if the L^2 metric is considered, with respect to which, by contrast, the functionals are not smooth, then the asymptotically critical points are really elastic bounce trajectories. Notice that the "L^2-gradient" of f_n has a much bigger L^2 norm (possibly ∞) than the corresponding $W^{1,2}$ norm of the $W^{1,2}$-gradient of f_n.

We obtain some multiplicity results, of bifurcation type, which are contained in Theorem 2.13. For this theorem we have employed, as we already did in [14], [15], the "nabla theorems", which we introduced in [14] and which we have now extended to some classes of nonsmooth functions. Roughly speaking, these theorems exploit certain properties the gradient of the functional has in some problems. Such properties make it possible to introduce a "fictitious" constraint which does not add critical points, but nevertheless makes the topology of the sublevels richer, thus allowing to get the expected multiplicity of solutions. Since the involved functionals are not smooth with respect to the L^2 metric, to perform such analysis on the constraints we have used the theory of ϕ-convex functions (see [8], [17], [13]).

To conclude we wish to recall a nice problem which to our knowdlege is completely open: if we assume for instance that there is no external force field and if the segment between the two points entirely lies in Ω, can one conclude that there exist infinitely many elastic bounce trajectories joining A and B?

We finally describe briefly the layout of the sections of the paper. In Section 2 we introduce the problem and state the main results. In Section 3 we extend the theory of the asymptotically critical points to a class of ϕ-convex functions and we prove a multiplicity theorem. In Section 4 we study the properties of the Lagrange integrals associated with the approximating potentials, in connection with the properties required in Section 3. In Section 5 we extend a ∇-theorem to the case of asymptotically critical points, for the class of ϕ convex functions which is involved in our problem, while in Section 6 we study the conditions required by that theorem in the concrete case. In Section 7 we perform the proofs of the main theorems. Finally in the Appendix 8 we recall some concepts and some properties of the ϕ convex functions used throughout the paper.

2. Assumptions and main results

Let Ω be a bounded open subset of $\mathbb{R}^N$ with C^2 boundary, A, B two points in Ω and let us denote by $\nu: \partial\Omega \to \mathbb{R}^N$ the unit normal vector to Ω pointing inwards. For what follows it is convenient to extend ν to the whole $\mathbb{R}^N$ in such a way that ν is of class C^2, $\|\nu(x)\| \leq 1$ and $\nu(A) = \nu(B) = 0$.

Moreover, let $a < b$ be two real numbers, and $V(t,x)$ a given potential: $V: [0,T] \times \overline{\Omega} \to \mathbb{R}$ (as smooth as needed — we will be more precise in the specific cases of the main results).

In the following $\nabla V(t,x)$ will denote the gradient of $V(t,x)$ with respect to x.

Definition 2.1. Let $\gamma \in W^{1,2}(0,T;\mathbb{R}^N)$. We say that γ is an elastic bounce trajectory from A to B in $\overline{\Omega}$ with respect to the potential V (briefly an elastic bounce trajectory in the following), if $\gamma(t) \in \overline{\Omega}$ for all t in $[0,T]$, $\gamma(0) = A$, $\gamma(T) = B$, and

(a) there exists a Radon measure μ on $]0,T[$ such that $\mu \geq 0$, the support of μ is contained in the contact set $C(\gamma) = \{t \in [0,T] \mid \gamma(t) \in \partial\Omega\}$, and

$$\ddot{\gamma} + \nabla V(t,\gamma) = \mu\,\nu(\gamma); \tag{2.1}$$

(b) (energy conservation λw) the energy

$$E(t) = \frac{1}{2}|\dot{\gamma}(t)|^2 + V(t,\gamma(t))$$

veryfies:

$$\int_0^T E(t)\dot{\varphi}(t)\,dt = \int_0^T (\nabla V(t), \gamma(t))\dot{\gamma}(t) + V(t,\gamma(t))\dot{\varphi}(t)\,dt$$

for all φ in $C^\infty(0,T,\mathbb{R})$, that is:

$$\int_0^T \frac{1}{2}|\dot\gamma|^2\dot\varphi\,dt = \int_0^T \nabla V(t,\gamma)\dot\gamma\varphi\,dt \quad \text{for all } \varphi \in C^\infty(0,T,\mathbb{R}).$$

We say that μ is the *distribution of the scalar constraint reaction* associated with γ. If $\mu \neq 0$ we say that γ is a true elastic bounce trajectory.

Remark 2.2. It is easy to see that the energy conservation (b) is not a consequence of (a). We recall that (2.1) corresponds to

$$\int_0^T \dot\gamma\,\dot\delta\,dt - \int_0^T \nabla V(t,\gamma)\delta\,dt + \int_0^T \nu(\gamma)\delta(\gamma)\,d\mu = 0 \tag{2.2}$$

for all δ in $C_0^\infty(0,T;\mathbb{R}^N)$. Moreover, it is not difficult to see that (2.2) is equivalent to the following *reversed* variational inequality:

$$\int_0^T \dot\gamma\,\dot\delta\,dt - \int_0^T \nabla V(t,\gamma)\delta\,dt \leq 0 \tag{2.3}$$

for all δ in $W_0^{1,2}(0,T;\mathbb{R}^N)$ such that $\nu(\gamma(t))\delta(t) \geq 0$ for all t in $C(\gamma)$.

The following characterization is easy to prove.

Remark 2.3. Let γ be a curve in $W^{1,2}(0,T;\mathbb{R}^N)$ with $\gamma([0,T]) \subset \overline{\Omega}$.

(a) If (a) of Definition 2.1 holds, then γ is of class C^2 in $]0,T[\setminus C(\gamma)$, $\dot\gamma$ has bounded variation on (every compact subset of) $]0,T[$, and $\dot\gamma - (\dot\gamma\,\nu(\gamma))\,\nu(\gamma)$ is absolutely continuous on $]0,T[$. We may think of γ as a "bounce trajectory of either elastic or inelastic type".

(b) If γ is an elastic bounce trajectory, then for any t_0 in $C(\gamma)$

$$\lim_{t\to t_0^+} \dot\gamma(t)\nu(\gamma(t)) = -\lim_{t\to t_0^-} \dot\gamma(t)\nu(\gamma(t)) = \frac{1}{2}\mu(\{t_0\});$$

in particular, if $\mu(\{t_0\}) > 0$, then t_0 is isolated in $C(\gamma)$.

We also point out a compactness result.

Remark 2.4. Let $(\gamma_n)_n$ be a sequence of elastic bounce trajectories from A to B. Then the following two facts are equivalent:

(a) $(\gamma_n)_n$ is bounded in $W^{1,2}(0,T;\mathbb{R}^N)$;

(b) the sequence $(\mu_n)_n$ of the constraint reactions associated with $(\gamma_n)_n$ (as in (a) of Definition 2.1) is bounded, that is $(\mu_n(]0,T[))_n$ is bounded.

Moreover, if (a) (or (b)) holds, then $(\gamma_n)_n$ admits a subsequence which converges in $W^{1,2}(0,T;\mathbb{R}^N)$ to an elastic bounce trajectory from A to B.

Proof. (a) $\Rightarrow$ (b) Notice that $\delta = \nu(\gamma_n)$ is in $W_0^{1,2}(0,T;\mathbb{R}^N)$ since $\nu(A) = \nu(B) = 0$. Using such a δ in (2.2) gives

$$\int_0^T (d\nu(\gamma_n)\dot\gamma_n)\dot\gamma_n\,dt - \int_0^T \nabla V(t,\gamma_n)\nu(\gamma_n)\,dt + \mu_n(]0,T[) = 0$$

so $(\mu_n(]0,T[))_n$ is bounded, whenever $(\gamma_n)_n$ is bounded in $W^{1,2}(0,T;\mathbb{R}^N)$.

(b) $\Rightarrow$ (a). From (2.2) one gets:

$$\left|\int_0^T \dot\gamma_n \dot\delta\, dt\right| \leq (\|\nabla V(t,\gamma_n)\|_{L^1} + \mu_n(]0,T[))\|\delta\|_{L^\infty}$$

for all δ in $\mathcal{C}_0^\infty(]0,T[;\mathbb{R}^N)$. So if $(\mu_n(]0,T[))_n$ is bounded it follows that $(\dot\gamma_n)_n$ is bounded in BV, hence it is relatively compact in L^p for any $p \geq 1$. In particular $(\gamma_n)_n$ is bounded and relatively compact in $W^{1,2}(0,T;\mathbb{R}^N)$. It is then clear that every limit curve of $(\gamma_n)_n$ is an elastic bounce trajectory.

Now we consider an open nonempty interval of parameters Λ_0 and a family of potentials $V_\lambda(t,x) = V(\lambda,t,x)$ where $V\colon \Lambda_0 \times [0,T] \times \overline{\Omega} \to \mathbb{R}^N$, which we assume as smooth as needed in the following definitions. □

Definition 2.5. We say that λ in Λ_0 is a *transition value* for the elastic bounce problem in $\overline{\Omega}$, from A to B, with respect to V, if there exist a sequence $(\lambda_n)_n$ in Λ_0 converging to λ and a sequence $(\gamma_n)_n$ of elastic bounce trajectories with respect to the potentials V_{λ_n}, such that the corresponding constraint reactions μ_n converge to 0: $\mu_n(]0,T[) \to 0$.

The following remark is a consequence of Remark 2.4.

Remark 2.6. If λ is a transition value, then there exists a solution γ of the problem

$$\begin{cases} \ddot\gamma + \nabla V(\lambda,t,\gamma) = 0, \\ \gamma(0) = A,\ \gamma(T) = B, \end{cases} \tag{2.4}$$

such that $\gamma([0,T]) \subset \overline{\Omega}$, $\gamma([0,T]) \cap \partial\Omega \neq \emptyset$. More precisely, given any sequences $(\lambda_n)_n$ in $\mathbb{R}$ converging to λ, $(\gamma_n)_n$ in $W^{1,2}(0,T;\mathbb{R}^N)$ where γ_n are elastic bounce trajectories with respect to V_{λ_n} such that the corresponding constraint reactions μ_n tend to zero, there exists a subsequence $(\gamma_{n_k})_k$ which converges in $W^{1,2}(0,T;\mathbb{R}^N)$ to a trajectory γ with the properties stated above.

According to what we said in the introduction we now introduce the main assumptions on Ω and on V.

(V) $V(\lambda,t,x) = (\lambda/2)(\beta(t)x)x + x_0(t)x$ for all λ in Λ_0, all t in $[0,T]$ and all x in $\overline{\Omega}$ where $\beta = (\beta_{ij})$ is a symmetric $N \times N$ matrix, $\beta_{ij} \in L^2(0,T;\mathbb{R})$, $\beta \not\equiv 0$ and $x_0 = (x_{0i})$ is an N vector, $x_{0i} \in L^2(0,T;\mathbb{R})$;

(Λ_0) for every λ in Λ_0 there exists a curve $\gamma_{0,\lambda}$ in $W^{1,2}(0,T;\mathbb{R}^N)$ with

$$\ddot\gamma_{0,\lambda} + \lambda\beta(t)\gamma_{0,\lambda} + x_0(t) = 0,$$
$$\gamma_{0,\lambda}(0) = A, \quad \gamma_{0,\lambda}(T) = B, \quad \gamma_0([0,T]) \subset \Omega$$

and the map $\lambda \mapsto \gamma_{0,\lambda}$ is continuous from Λ_0 to $W^{1,2}(0,T;\mathbb{R}^N)$.

The following remark is a simple consequence of Remark 2.6.

Remark 2.7. Assume that (V) and (Λ_0) hold. If λ in Λ_0 is a transition value for the elastic bounce problem, then λ is an eigenvalue of the problem

$$\begin{cases} \ddot{\delta} + \lambda\beta(t)\delta = 0, \\ \delta \in W_0^{1,2}(0,T;\mathbb{R}^N), \ \delta \neq 0. \end{cases} \tag{2.5}$$

The following remark follows using standard arguments.

Remark 2.8. There exists an unbounded interval I in $\mathbb{Z}$ such that for any i in I there exist an eigenvalue λ_i and an eigenfunction e_i of (2.5), that is:

$$\begin{cases} \ddot{e}_i + \lambda_i\beta(t)e_i = 0, \\ e_i \in W_0^{1,2}(0,T;\mathbb{R}^N), \ e_i \neq 0, \end{cases}$$

with the properties $\lambda_i \leq \lambda_{i+1}$ for all i, $\lambda_i < 0$ if $i < 0$ and $\lambda_i > 0$ if $i \geq 0$, $\lambda_i \to \infty$ as $i \to \infty$ (provided $\sup I = \infty$), $\lambda_i \to -\infty$ as $i \to -\infty$ (provided $\inf I = -\infty$); moreover,

$$\int_0^T \dot{e}_i\dot{e}_j\,dt = \delta_{ij}, \qquad W_0^{1,2}(0,T;\mathbb{R}^N) = \overline{\operatorname{span}\{e_i, i \in I\}} \oplus E_0,$$

where

$$E_o = \{\delta \in W_0^{1,2}(0,T;\mathbb{R}^N) \mid \beta(t)\delta(t) = 0 \text{ q.o. } t\}. \tag{2.6}$$

In the following we use the notation: for any eigenvalue λ_i

$$E_{\lambda_i} = \{\delta \in W_0^{1,2}(0,T;\mathbb{R}^N) \mid \ddot{\gamma} + \lambda_i\beta(t)\gamma = 0\}.$$

Now we want to discuss the previous assumption (Λ_0).

Remark 2.9. Let λ_i be an eigenvalue of (2.5).

(a) There exists a solution γ in $W^{1,2}(0,T;\mathbb{R}^N)$ of

$$\begin{cases} \ddot{\gamma} + \lambda_i\beta(t)\gamma + x_0(t) = 0, \\ \gamma(0) = A, \ \gamma(T) = B \end{cases} \tag{2.7}$$

if and only if

$$\int_0^T x_0 e\,dt = \dot{e}(T)B - \dot{e}(0)A \quad \text{for all } e \text{ in } E_{\lambda_i} \tag{2.8}$$

(b) If a solution γ of (2.7) exists and if γ_1 solves

$$\begin{cases} \ddot{\gamma}_1 + \lambda\beta(t)\gamma_1 + x_0(t) = 0, \\ \gamma_1(0) = A, \ \gamma_1(T) = B, \end{cases} \tag{2.9}$$

with $\lambda \neq \lambda_i$, then

$$\int_0^T \dot\gamma_1 \dot e\,dt - \int_0^T x_0 e\,dt = \int_0^T (\beta(t)\gamma_1(t))e(t)\,dt = 0 \quad \text{for all } e \text{ in } E_{\lambda_i} \tag{2.10}$$

Proof. (a) Let γ_0 be a smooth curve joining A to B. Then γ is a solution of (2.7) if and only if $\delta = \gamma - \gamma_0$ is a solution of

$$\begin{cases} \ddot\delta + \lambda_i\beta(t)\delta = \ddot\gamma_0 + \lambda_i\beta(t)\gamma_o + x_0, \\ \delta \in W_0^{1,2}(0,T;\mathbb{R}^N). \end{cases}$$

Such a solution may exists if and only if, for all e in E_{λ_i}

$$0 = \int_0^T (\ddot\gamma_0 + \lambda_i\beta(t)\gamma_0 + x_0)e\,dt \;\Leftrightarrow\; 0 = -[\gamma_0(t)\dot e(t)]_a^b + \int_0^T x_0 e\,dt$$

which gives the conclusion.

(b) Let $e \in E_{\lambda_i}$. Multiplying (2.9) by e and integrating yields

$$\begin{aligned} 0 &= \int_0^T \dot\gamma_1\dot e\,dt - \lambda\int_0^T (\beta(t)\gamma_1(t))e(t)\,dt - \int_0^T x_0 e\,dt \\ &= \int_0^T \dot\gamma_1\dot e\,dt - \lambda_i\int_0^T (\beta(t)\gamma_1(t))e(t)\,dt - \int_0^T x_0 e\,dt \\ &\quad + (\lambda_i - \lambda)\int_0^T (\beta(t)\gamma_1(t))e(t)\,dt \\ &= [\gamma_1(t)\dot e(t)]_a^b - \int_0^T x_0 e\,dt + (\lambda_i - \lambda)\int_0^T (\beta(t)\gamma_1(t))e(t)\,dt. \end{aligned} \tag{2.11}$$

Using (2.8) we get that $\int_0^T (\beta(t)\gamma_1(t))e(t)\,dt = 0$; plugging such an equality in (2.11) gives the whole (2.10). □

The following proposition explains the meaning of the assumpion (Λ_0).

Proposition 2.10. *Let λ_i be an eigenvalue of* (2.5).

(a) *Let $(\lambda^{(n)})_n$ be a sequence in $\mathbb{R}$ such that $\lambda^{(n)} \neq \lambda_i$, $\lambda^{(n)} \to \lambda_i$. If $(\gamma_n)_n$ is a sequence in $W^{1,2}(0,T;\mathbb{R}^N)$ such that for all n*

$$\begin{cases} \ddot\gamma_n + \lambda^{(n)}\beta(t)\gamma_n + x_0(t) = 0, \\ \gamma_n(0) = A,\ \gamma_n(T) = B, \end{cases}$$

and $\gamma_n \to \overline\gamma$ in $W^{1,2}(0,T;\mathbb{R}^N)$. Then $\overline\gamma$ is a solution of (2.7) *and*

$$\int_0^T \dot{\overline\gamma}\dot e\,dt - \int_0^T x_0 e\,dt = \int_0^T (\beta(t)\overline\gamma(t))e(t)\,dt = 0 \quad \textit{for all } e \textit{ in } E_{\lambda_i}. \tag{2.12}$$

Notice that the last property means that $\overline\gamma$ minimizes the expression

$$\gamma \mapsto \frac{1}{2}\int_0^T |\dot\gamma|^2\,dt - \int_0^T x_0\gamma\,dt \tag{2.13}$$

(or alternatively $\gamma \mapsto \lambda\int_0^T (\beta(t)\gamma)\gamma\,dt$*) among all solutions* γ *of* (2.7)*. Such a condition individuates a unique* $\overline{\gamma}$ *since the above expression is strictly convex and coercive in* $W^{1,2}(0,T;\mathbb{R}^N)$*.*

(b) *Let* β *and* x_0 *be such that a solution of* (2.7) *exists, i.e. let condition* (2.8) *be fulfilled. Let* $\overline{\gamma}$ *be the minimal solution of* (2.7)*, that is the solution which minimizes the expression in* (2.13)*. If* $\overline{\gamma}([0,T]) \subset \Omega$*, then assumption* (Λ_0) *holds.*

Proof. (a) It is clear that $\overline{\gamma}$ solves (2.7). Moreover, from Remark 2.9(b):

$$\int_0^T \dot{\gamma}^{(n)}\dot{e}\,dt - \int_0^T x_0 e\,dt = \int_0^T (\beta(t)\gamma^{(n)}(t))e(t)\,dt = 0 \quad \text{for all } e \text{ in } E_{\lambda_i}$$

and going to the limit as $n \to \infty$ the conclusion follows.

(b) For $\lambda \neq \lambda_i$, λ close to λ_i there exist a unique solution γ_λ of

$$\begin{cases} \ddot{\gamma} + \lambda\beta(t)\gamma + x_0(t) = 0, \\ \gamma_n(0) = A,\ \ \gamma_n(T) = B, \end{cases}$$

and γ_λ verifies (2.12). By Remark 2.4 and by the uniqueness of $\overline{\gamma}$ it follows that γ_λ converges to $\overline{\gamma}$ as $\lambda \to \lambda_i$ in $W^{1,2}(0,T;\mathbb{R}^N)$, hence in $\mathrm{C}^0(0,T;\mathbb{R}^N)$. This allows to define $\gamma_{0,\lambda}$ as in (Λ_0). □

Definition 2.11. Given a continuous curve $\gamma_0\colon [0,T] \to \Omega$ we say that Ω is *uniformly star-shaped with respect to* Ω, if there exists $\varepsilon > 0$ such that

$$-\nu(x)(x-z) \geq \varepsilon \quad \text{for all } x \text{ in } \partial\Omega \text{ and all } z \text{ in } \gamma_0([0,T]).$$

Remark 2.12. Assume that (V) and (Λ_0) hold. Let λ_i be an eigenvalue with $\lambda_i \in \Lambda_0$ and suppose that Ω is uniformly star-shaped with respect to γ_{0,λ_i}. Let $\lambda^{(n)} \to \lambda_i$, let γ_n be elastic bounce trajectories with respect to the potentials $V_n(t,x) := \lambda^{(n)}\beta(t)(x) + x_0(t)$ and let μ_n be the corresponding constraint reactions. Let $Q_\lambda(\delta) := (1/2)\int_0^T (\dot{\delta}^2 - \lambda\beta(t)\delta\delta)\,dt$. If $Q_{\lambda^{(n)}}(\gamma_{-}\gamma_{0,\lambda^{(n)}}) \to 0$, then $\mu_n \to 0$.

Proof. Setting $\delta_n := \gamma_n - \gamma_{0,\lambda^{(n)}}$ we have, for all n,

$$\ddot{\delta}_n + \lambda^{(n)}\beta(t)\delta_n = \mu_n\nu(\gamma_n).$$

Multiplying by δ_n and integrating over $]0,T[$:

$$2Q_{\lambda^{(n)}}(\delta_n) = -\int_{]0,T[} \delta_n\nu(\gamma_n)\,d\mu_n \geq \frac{\varepsilon}{2}\mu_n(]0,T[)$$

for n large, hence the conclusion. □

We state now our main result.

Theorem 2.13. *Assume that* (V) *and* (Λ_0) *hold. Then the following facts are true.*

(a) *For every* λ *in* Λ_0 *there exists a true elastic bounce trajectory* γ_λ *in* Ω *joining* A *to* B.

(b) *If* λ_i *is an eigenvalue of* (2.5) *and* $\lambda_i \in \Lambda_0$, *then there exists* $\varepsilon > 0$ *such that for every* λ *in* $[\lambda_i - \varepsilon, \lambda_i[\cap \Lambda_0$, *in the case* $\lambda_i > 0$ (*resp. for every* λ *in* $]\lambda_i, \lambda i + \varepsilon] \cap \Lambda_0$, *in the case* $\lambda_i < 0$) *there exists a second true elastic bounce trajectory* $\eta_\lambda \neq \gamma_\lambda$, *joining* A *to* B. *Moreover, we can say that*

$$(2.14)\quad \frac{1}{2}\int_0^T |\dot\gamma_\lambda|^2\,dt - \frac{\lambda}{2}\int_0^T \beta(t)(\gamma_\lambda)\gamma_\lambda\,dt - \int_0^T x_0(t)\gamma_\lambda\,dt < \frac{1}{2}\int_0^T |\dot\eta_\lambda|^2\,dt - \frac{\lambda}{2}\int_0^T \beta(t)(\eta_\lambda)\eta_\lambda\,dt - \int_0^T x_0(t)\eta_\lambda\,dt.$$

(c) *If* λ_i *is an eigenvalue of* (2.5) *and* $\lambda_i \in \Lambda_0$, *and if* Ω *is uniformly star-shaped with respect to* γ_{0,λ_i}, *then there exists* $\varepsilon > 0$ *such that for every* λ *in* $[\lambda_i - \varepsilon, \lambda_i[\cap \Lambda_0$, *in the case* $\lambda_i > 0$ (*resp. for every* λ *in* $]\lambda_i, \lambda_i + \varepsilon] \cap \Lambda_0$, *in the case* $\lambda_i < 0$) *there exist three distinct true elastic bounce trajectories* $\gamma_{1,\lambda}$, $\gamma_{2,\lambda}$, *and* η_λ *such that* (2.14) *holds with* $\gamma_\lambda = \gamma_{1,\lambda}$ *and* $\gamma = \gamma_{2,\lambda}$. *Moreover,* λ_i *is a transition value: more precisely,* $\mu_{h,\lambda}(]0,T[) \to 0$ *as* $\lambda \to \lambda_i$, *where* $\mu_{h,\lambda}$ *denotes the scalar constraint reaction associated with* $\gamma_{h,\lambda}$, $h = 1, 2$.

Actually (c) is the most interesting point. Notice that (a) is contained in the results of [10], but we state it here for completeness, as a simple consequence of the proofs. The proof of Theorem 2.13 is accomplished in Section 5.

3. Asymptotically critical points and their multiplicity

As we said in the Introduction we are going to study the problem of the elastic bounce in Ω by means of a sequence of approximating variational problems. In this section we introduce the theoretical tools which will allow us to obtain multiplicity results in spite of the fact that distinct solutions of the approximating problems could, in principle, have the same limit. These tools are the concept of asymptotical critical point for a sequence of functionals and a related multiplicity theorem for such points.

These notions, which were inspired by [9], [1] were introduced in [11], [12] for sequences of smooth functionals, however we need to extend them to the case of sequences of nonsmooth functionals in a suitable class, more precisely in the class of φ-convex function. For the reader's convenience the definitions of φ-convexity and subdifferential are recalled in the Appendix.

Let H be a Hilbert space with inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ and norm $\|\cdot\|$. In the sequel we consider a sequence $(W_n)_N$ of open subsets of H and a sequence of functions $(f_n)_n$ with $f_n: W_n \to \mathbb{R} \cup \{\infty\}$. We also consider a function $f: \mathcal{D} \to \mathbb{R}$, where $\mathcal{D}$ is a subset of H.

We remind that $\mathcal{D}(f_n) := \{u \in W_n \mid f_n(u) < \infty\}$ and that f_n^c denotes the set $\{u \in W_n \mid f_n(u) \le c\}$, for any c in $\mathbb{R}$.

Definition 3.1. We say that a point u in $\mathcal{D}$ is *asymptotically critical* for $((f_n)_n, f)$, if there exists a strictly increasing sequence $(k_n)_n$ in $\mathbb{N}$, a sequence $(\alpha_n)_n$ in H, and a sequence α_k in H such that

$$\begin{aligned} u_n &\to u, & u_n &\in \mathcal{D}(f_{k_n}) && \text{for all } n, \\ f_{k_n}(u_n) &\to f(u), & \alpha_n &\in \partial^- f_{k_n}(u_n) && \text{for all } n,\ \alpha_n \to 0. \end{aligned}$$

We also say that c is an *asymptotically critical value for* $((f_n)_n, f)$ if there exists an asymptotically critical point u such that $f(u) = c$.

Definition 3.2. Let c be a real number. We say that a sequence $(u_n)_n$ in H is a *nabla sequence* for $((f_n)_n, f)$ at level c, briefly a $\nabla(f_n, f, c)$-*sequence*, if there exists a strictly increasing sequence $(k_n)_n$ in $\mathbb{N}$ and a sequence $(\alpha_n)_n$ in H such that:

$$\begin{aligned} u_n &\in \mathcal{D}(f_{k_n}) && \text{for all } n, & f_{k_n}(u_n) &\to c, \\ \alpha_n &\in \partial^- f_{k_n}(u_n) && \text{for all } n, & \alpha_n &\to 0. \end{aligned}$$

We say that $((f_n)_n, f)$ verifies the *nabla property* at level c, briefly $\nabla(f_n, f, c)$ holds, if every $\nabla(f_n, f, c)$-sequence admits a subsequence which converges to some point u in $\mathcal{D}$ such that $f(u) = c$.

Notice that, by definition, such a u is an asymptotically critical point for $((f_n)_n, f)$.

The following remark is very easy to prove.

Remark 3.3. Let c be a real number, let $\nabla(f_n, f, c)$ hold, and assume c not to be a critical value for $((f_n)_n, f)$. Then there exists $\varepsilon > 0$ such that every c' in $[c - \varepsilon, c + \varepsilon]$ is not a critical value for $((f_n)_n, f)$.

For the multiplicity theorem we are going to prove, we also take an additional sequence $(C_n)_n$ of subsets of H, and two real numbers $a < b$. We suppose that the following assumptions hold:

- (A) f_n is lower semicontinuous and φ_n-convex of order 2 in W_n, for all n in $\mathbb{N}$,
- (B) $\overline{f_n^{-1}([a, b])} \subset W_n$, $f_n^b \subset C_n$ for all n in $\mathbb{N}$,
- (C) for every u_0 in $\mathcal{D}$ such that $f(u_0) \in [a, b]$ and u_0 is an asymptotically critical point for $((f_n)_n, f)$, there exist $\rho > 0$ and $\overline{n}$ in $\mathbb{N}$ such that $\overline{B(u_0, \rho)} \cap f_n^b$ is contractible in C_n for all $n \ge \overline{n}$.

We recall now the definition of category, actually one of the possible definitions, which is the most suited to our needs.

Definition 3.4. Let X be a topological space and (B, A) a topogical pair in X, that is $A \subset B \subset X$, A and B are endowed with the topology induced by X. We define the category of (B, A) in X, denoted by $\mathrm{cat}_X(B, A)$, as the smallest integer n such that there exist $n+1$ closed subsets $U_0, U_1, \dots, U_n$ in X with the properties

(a) $B \subset \bigcup_{i=0}^n U_i$,
(b) $U_1, \dots, U_n$ are contractible in X,
(c) $A \subset U_0$ and A is a strong deformation retract in X of U_0.

If there exist no n with these properties we agree that $\mathrm{cat}_X(B, A) = \infty$.

Theorem 3.5 (Multiplicity). *Assume that* (A)–(C) *hold and that* $\nabla(f_n, f, c)$ *holds for every* c *in* $[a, b]$. *Then*

$$\#\{u \in \mathcal{D} \mid u \textit{ is an asymptotically critical point for } ((f_n)_n, f)\ f(u) \in [a, b]\} \geq \limsup_{n \to \infty} \mathrm{cat}_{C_n}(f_n^b, f_n^a).$$

Moreover, when the right hand side above is 1, *there is no need for the local contractibility assumption* (C).

Proof. Suppose that the number of the asymptotically critical points in $f^{-1}([a, b])$ is finite: let $a \leq c_1 < \dots < c_k \leq b$ be the critical values and let $u_{i,1}, \dots, u_{i,h_i}$ be the critical points at level c_i, for $i = 1, \dots, k$.

Using (C) we can find $\rho > 0$ and $\overline{n}$ in $\mathbb{N}$ such that

$$\overline{B(u_{i,j}, 2\rho)} \cap f_n^b \text{ is contractible in } C_n \text{ for all } n \geq \overline{n},\ i = 1, \dots, k,\ j = 1, \dots, h_i.$$

Let $\overline{\varepsilon} := \min\{c_i - c_{i-1} \mid i = 2, \dots, k\}$ $c_i' := (c_i - \overline{\varepsilon}) \vee a$, $c_i'' := (c_i + \overline{\varepsilon}) \wedge b$. In virtue of the nabla property, given $i = 1, \dots, k$, up to taking a bigger $\overline{n}$ we have

$$\sigma_i := \inf_{\substack{n \geq \overline{n} \\ j=1,\dots,h_i}} \{\|\alpha\| \mid \alpha \in \partial^- f_n(u), f(u) \in [c_i', c_i''], \|u - u_{i,j}\| \geq \rho\} > 0$$

Let

$$\varepsilon_i' := \frac{\rho\sigma_i}{4} \vee (c_i - c_i'), \qquad \varepsilon_i'' := \frac{\rho\sigma_i}{4} \vee (c_i'' - c_i),$$

and let

$$F_1 := H \setminus \bigcup_{j=1,\dots,h_i} B(u_{i,j}, 2\rho), \qquad F_2 := H \setminus \bigcup_{j=1,\dots,h_i} B(u_{i,j}, \rho).$$

By Lemma 8.7 we get that $f_n^{c_i - \varepsilon_i'}$ is a strong deformation retract of $f_n^{c_i + \varepsilon_i''} \cap F_1 \cup f_n^{c_i - \varepsilon_i'}$, for $n \geq \overline{n}$. By Lemma 8.6 (using again the nabla properties and possibly enlarging $\overline{n}$), $f_n^{c_i+\varepsilon''_{i-1}}$ is a strong deformation retract of $f_n^{c_i-\varepsilon'_i}$. It follows by the properties of the category that

$$\operatorname{cat}_{C_n}(f_n^{c_i+\varepsilon''_i}, f_n^a) \leq \operatorname{cat}_{C_n}(f_n^{c_{i-1}+\varepsilon''_{i-1}}, f_n^a) + h_i$$

and finally

$$\operatorname{cat}_{C_n}(f_n^b, f_n^a) \leq \sum_{i=1}^{k} h_i. \qquad \square$$

Remark 3.6. Using the same arguments of the proof of Theorem 3.5 one can easily obtain the following version of the multiplicity theorem, which fits better to our needs.

Assume that (A) and (C) hold and that $\nabla(f_n, f, c)$ is verified for all c in $[a,b]$. Then there exist $\varepsilon > 0$ and $\overline{n}$ in $\mathbb{N}$ such that for all $n \geq \overline{n}$ and all a' and b' such that $a - \varepsilon \leq a' \leq b' \leq b + \varepsilon$ and

$$\overline{f_n^{-1}([a', b'])} \subset W_n, \quad f_n^{b'} \subset C_n,$$

one has

$$\#\{u \in \mathcal{D} \mid u \text{ is an asymptotically critical point for } ((f_n)_n, f),\ f(u) \in [a,b]\} \geq \operatorname{cat}_{C_n}(f_n^{b'}, f_n^{a'}) \quad \text{for all } n \geq \overline{n}.$$

This implies that (actually it is equivalent to) if $(a_n)_n$ and $(b_n)_n$ are two sequences in $\mathbb{R}$ such that $a_n \leq b_n$ and

$$\liminf_{n\to\infty} a_n \geq a, \quad \limsup_{n\to\infty} b_n \leq b$$

and if $(k_n)_n$ is a sequence in $\mathbb{N}$ such that $k_n \to \infty$ and

$$\overline{f_{k_n}^{-1}([a_n, b_n])} \subset W_{k_n}, \quad f_{k_n}^{b_n} \subset C_{k_n}$$

then

$$\#\{u \in \mathcal{D} \mid u \text{ is an asymptotically critical point for } ((f_n)_n, f),\ f(u) \in [a,b]\} \geq \limsup_{n\to\infty} \operatorname{cat}_{C_{k_n}}(f_{k_n}^{b_n}, f_{k_n}^{a_n}).$$

Also the following remark can be proved with the arguments used so far.

Remark 3.7. Suppose that $C_n \subset \mathcal{D}(f_n)$ for all n and denote by C_n^* the space C_n endowed with the graph metric:

$$d_n^*(u,v) := \|v - u\| + |f_n(u) - f_n(v)|.$$

Assume that (A) and (B) hold, and that (C) is replaced by

(C*) for every u_0 in $\mathcal{D}$ such that $f(u_0) \in [a,b]$ and u_0 is an asymptotically critical point for $((f_n)_n, f)$, there exist $\rho > 0$ and $\overline{n}$ in $\mathbb{N}$ such that $\overline{B(u_0,\rho)} \cap f_n^b$ is contractible in C_n^* for all $n \geq \overline{n}$.

(Notice that $B(u_0,\rho)$ still denotes the ball in the metric of H.)

If $\nabla(f_n,f,c)$ holds for every c in $[a,b]$ then

$$\#\{u\in\mathcal{D}\mid u \text{ is an asymptotically critical point for } ((f_n)_n,f),\ f(u)\in[0,T]\}\\ \geq \limsup_{n\to\infty}\operatorname{cat}_{C_n^*}(f_n^b,f_n^a).$$

As a simple consequence of Theorem 3.5 and Remark 3.6 we prove now an asymptotic version of the Linking Theorem, in a little more general version which we will use later (see the proof of (b) of Theorem 2.13, in Section 7. We start by introducing some sets and notation.

Let X_1 and X_2 two closed subspaces of H such that $H=X_1\oplus X_2$ and $\dim(X_1)<\infty$. Let $e\in X_2\setminus\{0\}$, $\rho>0$, and let

$$P:=\{x_1+te\mid x_1\in X_1,t\geq 0\},\quad S:=\{x_2\in X_2\mid \|x_2\|=\rho\}.$$

Moreover, let Δ be a bounded subset of P such that Δ is open in $X_1\oplus\operatorname{span}\{e\}$ and $S\cap P\subset\Delta$, and denote by Σ the boundary of Δ in $X_1\oplus\operatorname{span}\{e\}$.

Theorem 3.8. *Assume that* (A) *holds, that for n large $\overline{\Delta}\subset\mathcal{D}(f_n)$, and*

$$\sup f_n(\Sigma)<\inf f_n(S\cap W_n). \tag{3.1}$$

Moreover, if

$$a:=\liminf_{n\to\infty}\inf f_n(S\cap W_n),\quad b:=\limsup_{n\to\infty}\sup f_n(\overline{\Delta}),$$

suppose that $a,b\in\mathbb{R}$ and that there exists a sequence $(a_n)_n$ in $\mathbb{R}$ such that

$$a_n<\inf f_n(S\cap W_n),\quad \overline{f_n^{-1}([a_n,b_n])}\subset W_n \tag{3.2}$$

where $b_n:=\sup f_n(\overline{\Delta})$. Finally, let $\nabla(f_n,f,c)$ hold for all c in $[a,b]$. Then there exists an asymptotically critical point u with $f(u)\in[a,b]$.

Proof. We may assume that $\sup f_n(\Sigma)<a_n$ and that $\liminf_{n\to\infty}a_n=a$. By Remark 3.6 it suffices to prove that $f_n^{a_n}$ is not a retract of $f_n^{b_n}$ for n large. This is proved in the following lemma. □

Lemma 3.9. *Let A and B be two subsets of H such that*

$$\Sigma\subset A\subset B,\quad \overline{\Delta}\subset B,\quad A\cap S=\emptyset.$$

Then A is not a retract of B.

Proof. By contradiction suppose that there exists a retraction r from B into A. It is not difficult to see that there exists a retraction π from H into P such that $\pi^{-1}(S\cap P)\subset S$. Since $\overline{\Delta}\subset B$ we can define $\Psi:\overline{\Delta}\to P$ by $\Psi(u)=\pi(r(u))$. Such a Ψ is continuous, $\Psi(u)=u$ whenever $u\in\Sigma$ (because $\Sigma\subset A\cap P$), and $\Psi(\overline{\Delta})\cap(S\cap P)=\emptyset$ (because $S\cap A=\emptyset$). So Ψ is a continuous map from $\overline{\Delta}$ into P, which keeps the boundary of Δ fixed, but whose image does not cover Δ. This is impossible so the lemma is true. □

4. A variational setting for the bounce problem with fixed end points

As announced in the introduction we now present a variational *asymptotic* setting for the elastic bounce problem with fixed end points. We will introduce a sequence of functionals and after verifying some of their differential properties, in the nonsmooth sense, we will show that the asymptotically critical points for such a sequence are elastic bounce trajectories, and that the ∇-property holds.

We remind that "the billiard table" Ω is a bounded subset of $\mathbb{R}^N$ with $\mathcal{C}^2$ boundary and A, B are two given points in Ω. We also consider a time dependant potential $V\colon[0,T]\times\overline{\Omega}\to\mathbb{R}$ such that $t\mapsto V(t,x)$ is measurable for every x, $x\mapsto V(t,x)$ is of class $\mathcal{C}^2$ for almost every t in $[0,T]$, and there exists a functor C in $L^2(0,T)$ such that for all x in $\overline{\Omega}$ and all t in $[0,T]$

$$|V(t,x)|+\sum_i\left|\frac{\partial}{\partial x_i}V(t,x)\right|+\sum_{i,j}\left|\frac{\partial^2}{\partial x_i x_j}V(t,x)\right|\le C(t). \tag{4.1}$$

For what follows is convenient to extend V as a map $V\colon[0,T]\times\mathbb{R}^N\to\mathbb{R}$ in such a way that V is $\mathcal{C}^2$ in x and (4.1) holds for all x in $\mathbb{R}^N$.

We can introduce a $\mathcal{C}^2$ function $G:\mathbb{R}^N\to\mathbb{R}$ such that

$$\Omega=\{x\mid G(x)<0\}\quad\text{and}\quad|\nabla G(x)|\ge\varepsilon_0>0\quad\text{for all } x \text{ in } (\partial\Omega)_{\eta_0} \tag{4.2}$$

where $(\partial\Omega)_{\eta_0}$ denotes a metric neighbourhood of $\partial\Omega$ with radius $\eta_0>0$. In this way the inward normal ν (introduced in Section 2) verifies

$$\nu(x)=-(\nabla G(x))/(|\nabla G(x)|)\quad\text{for } x \text{ in } \partial\Omega.$$

We can also suppose that

$$\nu(x):=-(\nabla G(x))/(|\nabla G(x)|)\quad\text{for } x \text{ in } (\partial\Omega)_{\eta_0}$$

(we remind that ν is defined everywhere and $\nu(A)=\nu(B)=0$). We can also assume that $\liminf_{|x|\to\infty}(G(x))/(|x|)>0$. Moreover, for a given $p>1$ we set $U(x):=(G(x)^+)^p$. We set

$$\begin{aligned}\mathbb{X}(A,B):=&\{\gamma\in\mathrm{W}^{1,2}(0,1;\mathbb{R}^N)\mid\gamma(0)=A,\ \gamma(1)=B\},\\ \overline{\mathbb{X}}(A,B):=&\{\gamma\in\mathbb{X}(A,B)\mid\gamma([0,T])\subset\overline{\Omega}\},\end{aligned}$$

and for $\omega>0$, we define $g,f_\omega\colon\mathrm{L}^2(0,1;\mathbb{R}^N)\to\mathbb{R}\cup\{\infty\}$ and $f_\infty\colon\overline{X}(A,B)\to\mathbb{R}$ by

$$g(\gamma):=\begin{cases}\displaystyle\int_0^T\left(\frac{1}{2}|\dot\gamma|^2-V(t,\gamma)\right)dt & \text{if } \gamma\in\mathbb{X}(A,B),\\ \infty & \text{otherwise,}\end{cases}$$

$$
\begin{aligned}
f_\omega(\gamma) &:= g(\gamma) - \omega \int_0^T U(\gamma)\,dt \quad \text{for } \gamma \text{ in } L^2(0,T;\mathbb{R}^N),\\
f_\infty(\gamma) &:= g(\gamma) \qquad\qquad\qquad\quad\ \text{for } \gamma \text{ in } \overline{\mathbb{X}}(A,B).
\end{aligned}
$$

For technical reasons we also need another constraint: let $R \in \mathbb{R}$; we set

$$
\begin{aligned}
\mathbb{X}_R(A,B) &:= \{\gamma \in \mathbb{X}(A,B) \mid g(\gamma) \le R\},\\
\overline{\mathbb{X}}_R(A,B) &:= \{\gamma \in \overline{\mathbb{X}}(A,B) \mid g(\gamma) \le R\},
\end{aligned}
$$

and define $f_{R,\omega}\colon \mathrm{L}^2(0,1;\mathbb{R}^N) \to \mathbb{R} \cup \{\infty\}$, $f_{R,\infty}\colon \overline{\mathbb{X}}_R(A,B) \to \mathbb{R}$ by

$$
\begin{aligned}
f_{R,\omega}(\gamma) &:= \begin{cases} f_\omega(\gamma) & \text{if } \gamma \in \mathbb{X}_R(A,B),\\ \infty & \text{otherwise,}\end{cases}\\
f_{R,\infty}(\gamma) &:= f_\infty(\gamma) \quad (\text{for } \gamma \text{ in } \overline{\mathbb{X}}_R(A,B)).
\end{aligned}
$$

The main fact we are going to show now is that bounce trajectories are asymptotically critical points for $((f_{R,\omega})_\omega, f_{R,\infty})$. We emphasize again that the choice of the L^2 metric plays a fundamental role for this property to hold.

The following remark is a simple consequence of the assumption (4.1) on the whole $\mathbb{R}^N$.

Remark 4.1. (a) For every R in $\mathbb{R}$ $\mathbb{X}_R(A,B)$ is bounded in $W^{1,2}(0,T;\mathbb{R}^N)$.

(b) The functional $f_{R,\omega}$ is lower semicontinuous in $L^2(0,T;\mathbb{R}^N)$ for every ω and R. Moreover, $\mathcal{D}(f_{R,\omega}) = \mathbb{X}_R(A,B)$.

Lemma 4.2. *For every γ in $\mathbb{X}_R(A,B)$ and for every δ in $W_0^{1,2}(0,T;\mathbb{R}^N)$:*

$$
\begin{aligned}
g(\gamma+\delta) &\ge g(\gamma) + g'(\gamma)(\delta) - \overline{C}\|\delta\|^2,\\
f_\omega(\gamma+\delta) &\ge f_\omega(\gamma) + f_\omega{}'(\gamma)(\delta) - C_1\|\delta\|^2,
\end{aligned}
$$

where $\overline{C}$ and $C_1 = C_1(\omega,R)$ are suitable constants.

Proof. Both inequalities are simple consequences of the Taylor expansion. For the second one use

$$
C_1 := \overline{C} + N^2\omega \sup_{\substack{i,j=1,\dots,N\\ |x|\le \overline{R}}} \left| \frac{\partial^2}{\partial x_i x_j} U(x) \right|
$$

where $\overline{R} := \sup_{\gamma\in\mathbb{X}_R(A,B)} \|\gamma\|_\infty$. □

Proposition 4.3. *Let $\omega > 0$ and $R \in \mathbb{R}$. Let γ be a curve in $\mathbb{X}_R(A,B)$ such that either $g(\gamma) < R$ or $0 \notin \partial^- g(\gamma)$ and let $\alpha \in L^2(0,T;\mathbb{R}^N)$. Then $\alpha \in \partial^- f_{R,\omega}(\gamma)$ if and only if there exists $\lambda \ge 0$ such that*

$$
(1+\lambda)\int_0^T (\dot\gamma\dot\delta - \nabla V(t,\gamma)\delta)\,dt - \omega \int_0^T \nabla U(\gamma)\delta\,dt = \int_0^T \alpha\delta\,dt \tag{4.3}
$$

for all δ in $W_0^{1,2}(0,T;\mathbb{R}^N)$. Moreover, $\lambda = 0$, if $g(\gamma) < R$.

Proof. Let γ and α be as above. To prove the "only if" part we assume that $\alpha \in \partial^- f_{R,\omega}(\gamma)$. By the definition of the subdifferential we have

$$f_\omega{}'(\gamma)(\delta) \geq \int_0^T \alpha\delta\, dt \tag{4.4}$$

for all δ in $W_0^{1,2}(0,T;\mathbb{R}^N)$ such that $\gamma + t\delta \in \mathbb{X}_R(A,B)$ for $t > 0$ small enough. If $g(\gamma) < R$ all δ's have such a property so (4.4) holds for all δ and (4.3) holds with $\lambda = 0$. Suppose $g(\gamma) = R$; in this case (4.4) holds for all δ in $W_0^{1,2}(0,T;\mathbb{R}^N)$ such that $g'(\gamma)(\delta) < 0$. Since we have $0 \notin \partial^- g(\gamma)$, then we can find δ_0 in $W_0^{1,2}(0,T;\mathbb{R}^N)$ such that $g'(\gamma)(\delta_0) \neq 0$ (if there were no such δ_0, then γ would be critical for g by the first row of Lemma 4.2). Using a simple linearity argument it follows that there exists $\lambda \geq 0$ such that

$$f_\omega{}'(\gamma)(\delta) - \int_0^T \alpha\delta\, dt + \lambda g'(\gamma)(\delta) = 0$$

for all δ in $W_0^{1,2}(0,T;\mathbb{R}^N)$. This is equivalent to saying that (4.3) holds.

Conversely, assume that (4.3) holds for some $\lambda \geq 0$ such that $\lambda = 0$, if $g(\gamma) < 0$. Let δ be a curve in $W_0^{1,2}(0,T;\mathbb{R}^N)$ such that $\gamma + \delta \in \mathbb{X}_R(A,B)$. We have:

$$\begin{aligned} f_{R,\omega}(\gamma+\delta) - f_{R,\omega}(\gamma) - \int_0^T \alpha\delta\, dt &= f_\omega(\gamma+\delta) - f_\omega(\gamma) - \int_0^T \alpha\delta\, dt \\ &\geq f_\omega{}'(\gamma)(\delta) - C_1\|\delta\|^2 - f_\omega{}'(\gamma)(\delta) - \lambda g'(\gamma)\delta) \\ &= -C_1\|\delta\|^2 + \lambda g'(\gamma)(\delta) \geq -C_1\|\delta\|^2 \\ &\quad - \lambda(g(\gamma) - g(\gamma+\delta) - \overline{C}\|\delta\|^2) = (*). \end{aligned}$$

If $g(\gamma) < R$, then $\lambda = 0$ so $(*) \geq -C_1\|\delta\|^2$, otherwise $(*) \geq -(C_1 + \lambda\overline{C})\|\delta\|^2$. In any case we conclude that $\alpha \in \partial^- f_{R,\omega}(\gamma)$. □

Lemma 4.4. *Assume that for all γ in $\overline{\mathbb{X}}(A,B)$ with $g(\gamma) = R$ one has $0 \notin \partial^- g(\gamma)$. Then there exists $\eta > 0$ such that*

$$\sigma := \inf\{\|\alpha\| \mid \alpha \in \partial^- g(\gamma), g(\gamma) = R, \operatorname{dist}_{L^2}(\gamma, \overline{\mathbb{X}}_R(A,B)) \leq \eta\} > 0. \tag{4.5}$$

Proof. By contradiction let $(\gamma_n)_n$ be a sequence in $\mathbb{X}_R(A,B)$ such that $g(\gamma_n) = R$ and $\operatorname{dist}_{L^2}(\gamma_n, \overline{\mathbb{X}}_R(A,B) \to 0)$. Let (α_n) be a sequence in $L^2(0,T;\mathbb{R}^N)$ such that $\alpha_n \in \partial^- g(\gamma_n)$ for all n and $\alpha_n \to 0$. By Remark 4.1 $(\gamma_n)_n$ is bounded in $W^{1,2}(0,T;\mathbb{R}^N)$ hence we may suppose that $\gamma_n \to \gamma$ weakly in $W^{1,2}(0,T;\mathbb{R}^N)$ for a suitable curve γ. This implies that $\gamma_n \to \gamma$ uniforlmly, (and in L^2) so we get $\gamma \in \overline{\mathbb{X}}_R(A,B)$. By the first inequality in Lemma 4.2 we obtain:

$$g(\gamma+\delta) \geq g(\gamma_n) + \langle \alpha_n, \gamma + \delta - \gamma_n \rangle - \overline{C}\|\gamma + \delta - \gamma_n\|^2 \tag{4.6}$$

for all δ in $W_0^{1,2}(0,T;\mathbb{R}^N)$. Using (4.6) with $\delta = 0$ gives $g(\gamma) = R$ and going to the limit as $n \to \infty$:

$$g(\gamma+\delta) \geq g(\gamma) - \overline{C}\|\delta\|^2 \quad \text{for all } \delta \text{ in } W_0^{1,2}(0,T;\mathbb{R}^N),$$

hence $0 \in \partial^- g(\gamma)$ and we have a contradiction. □

Proposition 4.5. *Let R be a real number. Assume that for all γ in $\overline{\mathbb{X}}(A,B)$ with $g(\gamma) = R$ one has $0 \notin \partial^- g(\gamma)$. Then there exists $\eta > 0$ such that for all $\omega > 0$ $f_{R,\omega}$ is of class $C(p,q)$ in W, where W is the η neighbourhood (with respect to the L^2 metric) of $\overline{\mathbb{X}}_R(A,B)$, and $p = p(\omega,R)$ and $q = q(\omega,R)$ are suitable constants.*

Proof. Let η be the number provided by Lemma 4.4 and σ as in (4.5). Let $\gamma \in W := \overline{\mathbb{X}}_R(A,B)_\eta$, $\delta \in W_0^{1,2}(0,T;\mathbb{R}^N)$ and suppose $\gamma, \gamma+\delta \in \mathbb{X}_R(A,B)$; let $\alpha \in \partial^- f_{R,\omega}(\gamma)$. By Lemma 4.4 and Proposition 4.3 there exists $\lambda \geq 0$ such that (4.3) holds. Moreover, $\lambda = 0$ if $g(\gamma) < R$. We have:

$$\begin{aligned} f_{R,\omega}(\gamma+\delta) - {}& f_{R,\omega}(\gamma) - \langle\alpha,\delta\rangle \\ &= f_{R,\omega}(\gamma+\delta) - f_{R,\omega}(\gamma) - (1+\lambda)g'(\gamma)(\delta) + \omega\int_0^T \nabla U(\gamma)\delta\,dt \\ &= f_\omega(\gamma+\delta) - f_\omega(\gamma) - f_\omega{}'(\gamma)(\delta) - \lambda g'(\gamma)(\delta) \\ &\geq -C_1\|\delta\|^2 - \lambda(g(\gamma+\delta) - g(\gamma) + \overline{C}\|\delta\|^2) \geq -C_1\|\delta\|^2 - \lambda\overline{C}\|\delta\|^2 \end{aligned}$$

because either $g(\gamma) < R$ and $\lambda = 0$ or $g(\gamma) = R$ and in that case $g(\gamma+\delta) \leq g(\gamma)$ (remind that $\lambda \geq 0$). Now we want to estimate λ (in the case $g(\gamma) = R$). From (4.3) we deduce that

$$\alpha_0 := \frac{\omega\nabla U(\gamma) + \alpha}{1+\lambda} \in \partial^- g(\gamma);$$

hence $\|\alpha_0\| \geq \sigma$, by Lemma 4.4. This gives

$$1+\lambda = \frac{\|\omega\nabla U(\gamma)+\alpha\|}{\|\alpha_0\|} \leq \frac{\omega M + \|\alpha\|}{\sigma}$$

where $M = \sqrt{T}\max_{x\in B(0,\overline{R})}|\nabla U(x)|$ and $\overline{R} := \sup_{\gamma\in\overline{\mathbb{X}}(A,B)}\|\gamma\|_\infty$. So we have:

$$f_{R,\omega}(\gamma+\delta) \geq f_{R,\omega}(\gamma) + \langle\alpha,\delta\rangle - \left(C_1 + \frac{\overline{C}\omega M}{\sigma} + \frac{\overline{C}}{\sigma}\|\alpha\|\right)\|\delta\|^2 \tag{4.7}$$

for all γ in $W \cap \mathbb{X}_R(A,B)$, all δ in $W_0^{1,2}(0,T;\mathbb{R}^N)$, and all α in $\partial^- f_{R,\omega}(\gamma)$. □

Remark 4.6. The previous proposition shows that, given R in $\mathbb{R}$, the functionals $f_{R,\omega}$ verify (A) of Section 3 on a fixed $W_\omega = W$.

Moreover, given a in $\mathbb{R}$, it is simple to check that there exists $\overline{\omega}$ such that

$$\overline{\{\gamma \in \mathbb{X}_R(A,B) : f_{R,\omega}(\gamma) \geq a\}} \subset W \quad \text{for all } \omega \geq \overline{\omega}$$

so (B) of Section 3 holds for ω large.

Now we study the asymptotically critical points of $((f_{R,\omega})_\omega, f_{R,\infty})$.

Lemma 4.7. *Let $(\mu_n)_n$ be a sequence of nonnegative real numbers, let $(\gamma_n)_n$ be a sequence in $\mathbb{X}(A,B)$ such that $(\gamma_n)_n$ is bounded in $W^{1,2}(0,T;\mathbb{R}^N)$ and $\gamma_n([0,T]) \subset \Omega_{\eta_0}$ (η_0 was given at the beginning of this section), and let $(\beta_n)_n$ be a bounded sequence in $L^1(0,T;\mathbb{R}^N)$ such that:*

$$\int_0^T \beta_n\delta\,dt = \int_0^T (\dot\gamma_n\dot\delta - \nabla V(t,\gamma_n)\delta)\,dt - \mu_n\int_0^T \nabla U(\gamma_n)\delta\,dt \tag{4.8}$$

for all δ in $W_0^{1,2}(0,T;\mathbb{R}^N)$. Then $(\ddot\gamma_n)_n$ and $(\mu_n\nabla U(\gamma_n))_n$ are bounded in $L^1(0,T;\mathbb{R}^N)$ (and (4.8) holds for every δ in $\mathrm{W}_0^{1,1}(0,T;\mathbb{R}^N)$).

Proof. Since $\nu(A)=\nu(B)=0$ we can take $\delta=\nu(\gamma_n)$ in (4.8) and get

$$\int_0^T \beta_n\nu(\gamma_n)\,dt = \int_0^T (\mathrm{d}\nu(\gamma_n)(\dot\gamma_n)\dot\gamma_n - \nabla V(t,\gamma_n)\nu(\gamma_n))\,dt - \mu_n\int_0^T |\nabla U(\gamma_n)|\,dt$$

Since $\mathrm{d}\nu$ is bounded in Ω_{η_0} (due to the regularity of $\partial\Omega$), then $(\mu_n|\nabla U(\gamma_n)|)_n$ is bounded in $L^1(0,T;\mathbb{R}^N)$. Since

$$\ddot\gamma_n = -\nabla V(t,\gamma_n) - \mu_n\nabla U(\gamma_n)$$

we get that $(\ddot\gamma_n)_n$ is bounded in $L^1(0,T;\mathbb{R}^N)$ too. □

The following lemma is strictly related to Remark 2.4.

Lemma 4.8. *Let $(\mu_n)_n$ be a sequence of nonnegative real numbers. Let (γ_n) be a sequence in $\mathbb{X}(A,B)$ which converges in $W^{1,2}(0,T;\mathbb{R}^N)$ to a curve γ in $\overline{\mathbb{X}}(A,B)$. Let $(\beta_n)_n$ be a sequence in $L^1(0,T;\mathbb{R}^N)$ such that $\beta_n\to 0$ in $L^1(0,T;\mathbb{R}^N)$ and (4.8) holds. Then γ is an elastic bounce trajectory and*

$$\lim_{n\to\infty}\mu_n\int_0^T U(\gamma_n)\,dt = 0. \tag{4.9}$$

Proof. *Step 1.* We first prove (4.9). By Lemma 4.7 $\mu_n\int_0^T|\nabla U(\gamma_n)|\,dt$ are bounded. Moreover, since $\gamma_n\to\gamma$ uniformly we have that $\gamma_n([0,T])\subset\Omega_{\eta_0}$ for n large (η_0 was given in (4.2)). Then, by (4.2):

$$\begin{aligned}\mu_n\int_0^T U(\gamma_n)\,dt &= \mu_n\int_0^T (G(\gamma_n)^+)^p\,dt \le \|G^+(\gamma_n)\|_\infty\mu_n\int_0^T (G(\gamma_n)^+)^{p-1}\,dt\\ &\le \|G^+(\gamma_n)\|_\infty\frac{\mu_n}{p\varepsilon_0}\int_0^T|\nabla U(\gamma_n)|\,dt \le \text{const}\ \|G^+(\gamma_n)\|_\infty\to 0.\end{aligned}$$

Step 2. We prove that (2.2) holds. We first take δ in $W_0^{1,2}(0,T;\mathbb{R}^N)$ such that $\nu(\gamma(t))\cdot\delta(t)>0$ for all t in $C(\gamma)$. Since $\gamma_n([0,T])\subset\Omega_{\eta_0}$ for n large, there exist $\varepsilon>0$ and $\overline{n}$ in $\mathbb{N}$ such that

$$\nu(\gamma_n(t))\cdot\delta(t)\ge\varepsilon \quad \text{for all } n\ge\overline{n} \text{ and all } t \text{ in } C(\gamma)_\varepsilon.$$

Up to shrinking ε, $\gamma(C(\gamma)_\varepsilon) \subset (\partial\Omega)_{\eta_0}$, so for n large $\gamma_n(C(\gamma)_\varepsilon) \subset (\partial\Omega)_{\eta_0}$ and

$$t \in C(\gamma)_\varepsilon \Rightarrow -\frac{\nabla G(\gamma_n(t))}{|\nabla G(\gamma_n(t))|} = \nu(\gamma_n(t)),$$
$$t \notin C(\gamma)_\varepsilon \Rightarrow \gamma_n(t) \in \Omega.$$

Then $\nabla U(\gamma_n) \cdot \delta \leq 0$ in $[0, T]$ for all $n \geq \overline{n}$. By (4.8) this implies

$$\int_0^T \dot{\gamma}_n \dot{\delta}\, dt - \int_0^T \nabla V(t, \gamma_n)\delta\, dt \leq \int_0^T \beta_n \delta\, dt \Rightarrow \int_0^T \dot{\gamma}\dot{\delta}\, dt - \int_0^T \nabla V(t, \gamma)\delta\, dt \leq 0.$$

Finally, if δ is such that $\nu(\gamma(t)) \cdot \delta(t) \geq 0$, we can get the same conclusion by means of an approximation argument.

Step 3. We prove the energy conservation law. If $\varphi \in \mathcal{C}_0^\infty(0, T; \mathbb{R})$ let $\delta = \dot{\gamma}_n\varphi$. We have $\dot{\delta} = \ddot{\gamma}_n\varphi + \dot{\gamma}_n\dot{\varphi}$. Then $\delta \in \mathrm{W}_0^{1,1}(0, T; \mathbb{R}^N)$ by Lemma 4.7, because $\gamma_n \in \mathrm{W}^{2,1}$, and δ is an admissible test in (4.8). We obtain

$$\begin{aligned}
\int_0^T (\beta_n \cdot \gamma_n)\varphi
&= \int_0^T \dot{\gamma}_n(\ddot{\gamma}_n\varphi + \dot{\gamma}_n\dot{\varphi})\, dt - \int_0^T ((\nabla V(t, \gamma_n) + \mu_n \nabla U(\gamma_n)) \cdot \dot{\gamma}_n)\varphi\, dt \\
&= \int_0^T \left(\frac{1}{2}\frac{d}{dt}|\dot{\gamma}_n|^2\varphi + |\dot{\gamma}_n|^2\dot{\varphi}\right) dt \\
&\quad + \int_0^T \nabla V(t, \gamma_n)\dot{\gamma}_n\varphi\, dt - \int_0^T \mu_n\left(\frac{d}{dt}U(\gamma_n)\right)\varphi\, dt \\
&= \int_0^T \left(\frac{1}{2}|\dot{\gamma}_n|^2 + \mu_n U(\gamma_n)\right)\dot{\varphi}\, dt + \int_0^T \nabla V(t, \gamma_n)\dot{\gamma}_n\varphi\, dt.
\end{aligned}$$

Letting $n \to \infty$ we obtain, by (4.9)

$$\int_0^T \left(\frac{1}{2}|\dot{\gamma}|^2\dot{\varphi} + \nabla V(t, \gamma)\dot{\gamma}\varphi\right) dt = 0 \quad \text{for all } \varphi \text{ in } \varphi \in \mathcal{C}_0^\infty(0, T; \mathbb{R}^N),$$

that is (b) of Definition 2.1 holds. □

Remark 4.9. Notice that, in the previous statement, if $(\mu_n)_n$ is bounded, then $\mu_n \int_0^T |\nabla U(\gamma_n)|\, dt \to 0$ so γ is a solution of

$$\ddot{\gamma} + \nabla V(t, \gamma) = 0. \tag{4.10}$$

The following statements represent an "asymptotic" Hamilton principle for the elastic bounce problem.

Theorem 4.10. *Let γ in $\overline{\mathbb{X}}_R(A, B)$ be an asymptotically critical point for $((f_{R,\omega})_\omega, f_{R,\infty})$. Then γ is an elastic bounce trajectory in Ω joining A to B.*

Proof. We can suppose $0 \notin \partial^- g(\gamma)$, otherwise the claim is true, because γ solves (4.10). Let $(\omega_n)_n$ be a sequence in $\mathbb{R}$ such that $\omega_n \to \infty$, let $(\gamma_n)_n$

be a sequence in $\mathbb{X}_R(A,B)$ such that $\gamma_n \to \gamma$ in $L^2(0,T;\mathbb{R}^N)$ and $f_{R,\omega_n}(\gamma_n) \to f_{R,\infty}(\gamma)$, and let $(\alpha_n)_n$ be a sequence in $L^2(0,T;\mathbb{R}^N)$ such that $\alpha_n \in \partial^- f_{\omega_n}(\gamma_n)$ for all n, and $\alpha_n \to 0$ in $L^2(0,T;\mathbb{R}^N)$. It is clear that for n large $0 \notin \partial^- g(\gamma_n)$. Since $\gamma_n \in \mathbb{X}_R(A,B)$ for all n, then $(\gamma_n)_n$ is bounded in $W^{1,2}(0,T;\mathbb{R}^N)$, by Remark 4.1. This implies that $\gamma_n \to \gamma$ uniformly, so eventually $\gamma_n \in \Omega_{\eta_0}$. By Proposition 4.3 for every n there exists $\lambda_n \geq 0$ such that

$$(1+\lambda_n)\int_0^T (\dot\gamma_n \dot\delta - \nabla V(t,\gamma_n)\delta)\,dt - \omega_n \int_0^T \nabla U(\gamma_n)\delta\,dt = \int_0^T \alpha_n \delta\,dt \tag{4.11}$$

for all $\delta \in W_0^{1,2}(0,T;\mathbb{R}^N)$. Applying Lemma 4.7 with $\mu_n = 1/(1+\lambda_n)$ and $\beta_n = \alpha_n/(1+\lambda_n)$ gives that $(\ddot\gamma_n)_n$ is bounded in $L^1(0,T;\mathbb{R}^N)$. This implies that $\gamma_n \to \gamma$ in $\mathrm{W}^{1,p}$, for all $p>1$. The conclusion now follows by Lemma 4.8.□

Lemma 4.11. *Let R be a real number such that:*

for all γ in $\overline{\mathbb{X}}(A,B)$ with $g(\gamma)=R$ one has $0 \notin \partial^- g(\gamma)$.

Let c be a real number. Let $(\omega_n)_n$ be a sequence in $\mathbb{R}$ such that $\omega_n \to \infty$, let $(\gamma_n)_n$ be a sequence in $\mathbb{X}_R(A,B)$ such that $f_{R,\omega_n}(\gamma_n) \to c$, and let $(\alpha_n)_n$ be a sequence in $L^2(0,T;\mathbb{R}^N)$ such that $\alpha_n \in \partial^- f_{\omega_n}(\gamma_n)$ for all n, and $\alpha_n \to 0$ in $L^2(0,T;\mathbb{R}^N)$. Then there exist a strictly increasing sequence $(k_n)_n$ in $\mathbb{N}$ and a curve γ in $\overline{\mathbb{X}}_R(A,B)$ such that $\gamma_{k_n} \to \gamma$ in $W^{1,2}(0,T;\mathbb{R}^N)$. Moreover, either $g(\gamma)=R$ or $f_{R,\infty}(\gamma)=c$.

Proof. *Step 1.* By Proposition 4.3, for all n there exists $\lambda_n \geq 0$ such that (4.11) holds. By Lemma 4.7 with $\mu_n = 1/(1+\lambda_n)$ and $\beta_n = \alpha_n/(1+\lambda_n)$ we have that $(\ddot\gamma_n)_n$ is bounded in $L^1(0,T;\mathbb{R}^N)$ hence $(\dot\gamma_n)_n$ is relatively compact in L^p for every $p \geq 1$. So we can find $(k_n)_n$ and γ such that $\gamma_{k_n} \to \gamma$ in $W^{1,2}(0,T;\mathbb{R}^N)$.

Step 2. Since $f_{R,\omega_n}(\gamma_n)$ is bounded we get that $\omega_n \int_0^T U(\gamma_n)\,dt$ is bounded; then $\int_0^T U(\gamma_n)\,dt \to 0$ which in turn gives $\gamma \in \overline{\mathbb{X}}_R(A,B)$, because $\gamma_{k_n} \to \gamma$ uniformly.

Step 3. By Lemma 4.8 we get that γ is an elastic bounce trajectory and $\omega_n/(1+\lambda_n)\int_0^T U(\gamma_n)\,dt \to 0$.

Step 4. Now we conclude by distinguishing two cases. If $g(\gamma_{k_n}) = R$ for infinitely many n, then $g(\gamma)=R$. If this is not the case we can suppose $g(\gamma_{k_n}) < R$ for all n. Then $\lambda_{k_n} = 0$ for all n and $\omega_{k_n}\int_0^T U(\gamma_{k_n})\,dt \to 0$. This implies $f_{R,\omega_{k_n}}(\gamma_{k_n}) \to g(\gamma) = f_{R,\infty}(\gamma)$. □

The following result follows immediately from the previous lemma.

Proposition 4.12. *Let R be a real number such that:*

there are no elastic bounce trajectories γ in Ω such that $g(\gamma)=R$.

Then the condition $\nabla(f_{R,\omega}, f_{R,\infty}, c)$ *holds for all real numbers* c.

5. Asymptotic ∇-theorems

In this section we present an asymptotic version of the ∇-theorems introduced in [14] and [15]. As we already said, we will use these theorems in the proof of Theorem 2.13.

As in Section 3 let us consider a Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Throughout this section we also assign a closed subspace X of H and a continuous linear projection Q having X as its kernel.

We introduce the map $\Phi: H \setminus X \to H$ defined by:

$$\Phi(z) = z - \frac{Q(z)}{\|Qz\|}$$

and the set C given by $C = \{z \in H \mid \|Q(z)\| \geq 1\}$.

The following notations will also turn useful: if $z \in H \setminus X$ we let

$$Q_z w := Qw - \left\langle Qw, \frac{Qz}{\|Qz\|} \right\rangle \frac{Qz}{\|Qz\|} \quad \text{for all } w \text{ in } H$$

and $X_z := \mathrm{Ker}(Q_z) = X \oplus \mathrm{span}(z)$.

We first point out some properties of Φ whose proof can be accomplished in a standard way.

Remark 5.1. The following facts are true.

(a) Φ is of class $\mathcal{C}^\infty(H \setminus X)$ and, if $z \in H \setminus X$,

$$d\Phi(z)(w) = w - \frac{Q_z w}{\|Qz\|} \quad \text{for all } w \text{ in } H.$$

(b) Φ is a diffeomorphism from $\mathrm{int}(C)$ into $H \setminus X$ and for all u in $H \setminus X$:

$$d(\Phi^{-1})(u)(v) = v + \frac{Q_u v}{\|Qu\|} \quad \text{for all } v \text{ in } H.$$

For the notations used in the following lemma we refer to the Appendix.

Lemma 5.2. *Let* W *be an open subset of* Ω *and* $f : W \to \mathbb{R} \cup \{\infty\}$ *be a function of class* $\mathcal{C}(p, q)$. *Assume that:*

- X *has a finite codimension;*
- $\mathcal{D}(f)$ *and* X *are not tangent at any* u *in* $\mathcal{D}(f) \cap X$.

Then the function $g := f \circ \Phi + I_C$, *which is defined in* $\Phi^{-1}(W)$, *is of class* $\mathcal{C}(p', q')$ *for suitable* p' *and* q'. *Moreover, for every* z *in* $\mathcal{D}(g) = \Phi^{-1}(\mathcal{D}(f)) \cap C$

$$(5.1) \quad \partial^- g(z) = \begin{cases} \{d\Phi(z)^*(\alpha) \mid \alpha \in \partial^- f(\Phi(z))\} & \text{if } z \in \mathrm{int}(C), \\ \{d\Phi(z)^*(\alpha) - \lambda Q^* Qz \mid \alpha \in \partial^- f(\Phi(z)), \lambda \geq 0\} & \text{if } z \in \partial C. \end{cases}$$

Proof. We first consider the function $g_1 : \Phi^{-1}(W) \to \mathbb{R} \cup \{\infty\}$ defined by $g_1 := f \circ \Phi$ (so that $g = g_1 + I_C$). Let $z \in \mathcal{D}(g_1)$ and let $u := \Phi(z)$.

Step 1. We remark that (as one can easily see from the definitions)

$$\{d\Phi(z)^*\alpha \mid \alpha \in \partial^- f(u)\} \subset \partial^- g_1(z), \tag{5.2}$$

$$\text{if } z \notin \partial C, \text{ then } \{d\Phi(z)^*\alpha \mid \alpha \in \partial^- f(u)\} = \partial^- g_1(z). \tag{5.3}$$

Notice that (5.3) holds since Φ is a local diffeomorphism outside ∂C.

Step 2. We claim that

$$\begin{aligned} &\text{if } z \in \partial C \text{ and } \beta \in \partial^- g_1(z) \\ &\qquad \text{then } \beta \in \partial^-(f + I_{X_z})(u), \quad \langle \beta, Q_z w\rangle = 0 \quad \text{for all } w \text{ in } H. \end{aligned} \tag{5.4}$$

Indeed for the first claim notice that $\Phi|_{X_z} : X_z \to X_z$ is a translation in a neighbourhood of z and its differential is the identity, so

$$\beta \in \partial^- g_1(z) \Rightarrow \beta \in \partial^-(g_1 + I_{X_z})(z) \Leftrightarrow \beta \in \partial^-(f + I_{X_z})(u).$$

The second claim follows since Φ is constant over $S := \{u + Qw \mid w \in H,\ \|Qw\| = 1\}$ and the tangent plane to S at z is $\{Q_z w \mid w \in H\}$.

Step 3. Now we want to prove that, wherever z lies in $\mathcal{D}(g_1)$

$$\partial^- g_1(z) = \{d\Phi(z)^*\alpha \mid \alpha \in \partial^- f(u)\}. \tag{5.5}$$

If $z \notin \partial C$ this was already proved in (5.3). So let $z \in \partial C$ and let $\beta \in \partial^- g_1(z)$. In view of (5.2) it suffices to show that there exists α in $\partial^- f(u)$ such that $\beta = d\Phi(z)^*\alpha$. By (5.4) $\beta \in \partial^-(f + I_{X_z})(u)$. Using the nontangency between $\mathcal{D}(f)$ and X we get that $\mathcal{D}(f)$ and X_z are not tangent too. By Theorem 8.9, we obtain that $\beta = \alpha + \nu$ for suitable α in $\partial^- f(u)$ and ν in $N_u(X_z)$. Using this decomposition and the second condition in (5.4) we have

$$\begin{aligned} \langle \beta, w\rangle &= \langle \beta, d\Phi(z)w + Q_z w\rangle = \langle \beta, d\Phi(z)w\rangle = \langle \alpha, d\Phi(z)w\rangle + \langle \nu, d\Phi(z)w\rangle \\ &= \langle \alpha, d\Phi(z)w\rangle + \langle \nu, w - Q_z w\rangle = \langle \alpha, d\Phi(z)w\rangle, \end{aligned}$$

for all w in H, since $w - Q_z w \in X_z$. This concludes the proof of (5.5).

From (5.5), with easy computations, it follows that g_1 is of class $C(p_1, q_1)$ for suitable functions p_1, q_1.

Step 4. We claim now that C and $\mathcal{D}(g_1)$ are not tangent at any point of their intersection. Indeed, by Theorem 8.10, we derive that $I_{\mathcal{D}(f)}$ is of class $C(p, 0)$. Therefore, noticing that $I_{\mathcal{D}(g_1)} = I_{\mathcal{D}(f)} \circ \Phi$ and using (5.5) with f replaced by $I_{\mathcal{D}(f)}$, we have

$$\begin{aligned} N_z(\mathcal{D}(g_1)) &= \partial^- I_{\mathcal{D}(g_1)}(z) = \{d\Phi(z)^*\alpha \mid \alpha \in \partial^- I_{\mathcal{D}(f)}(u)\} \\ &= \{d\Phi(z)^*\nu \mid \nu \in N_u(\mathcal{D}(f))\}. \end{aligned}$$

Now, assume that $\nu_1 \in N_z(\mathcal{D}(g_1))$ and $-\nu_1 \in N_z(C)$; clearly we may suppose that $z \in \partial C$ (otherwise the conclusion is trivial). In particular ν_1 is orthogonal to X. Moreover, $\nu_1 = d\Phi(z)^*\nu$ for a suitable ν in $N_u(\mathcal{D}(f))$. It follows that, for any v in X

$$0 = \langle \nu_1, v\rangle = \langle d\Phi(z)^*\nu, v\rangle = \langle \nu, d\Phi(z)v\rangle = \langle \nu, v\rangle$$

($d\Phi(z) = \mathrm{id} - Q_z$ is the identity on X), so $-\nu \in N_u(X)$. Since $\mathcal{D}(f)$ and X are not tangent, it follows $\nu = 0$, hence $\nu_1 = 0$ and the proof of the claim is over.

Step 5. Using the previous step and Theorem 8.9 again we get that $g = g_1 + I_C$ is of class $C(p', q')$ for suitable p', q' and that $\partial^- g(z) = \partial^- g_1(z) + N_z(C)$. To prove the formula (5.1) and conclude, it suffices to notice that, if $z \in \partial C$, then $N_z(C) = \{-\lambda Q^* Q z \mid \lambda \geq 0\}$. □

From now on we consider a sequence $(f_n)_n$ of functions, such that $f_n: W_n \to \mathbb{R} \cup \{\infty\}$, where W_n are open subsets of H, and a function $f: \mathcal{D} \to \mathbb{R}$, $\mathcal{D}$ being a subset of H.

We also use the following notation: given a closed supspace Y of H we denote by Π_Y the orthogonal projection ont Y.

Definition 5.3. Let $c \in \mathbb{R}$.

(a) We say that a sequence $(u_n)_n$ in H is a $\nabla(f_n, X, c)$ sequence if there exist (k_n) in $\mathbb{N}$ strictly increasing and $(\alpha_n)_n$ in H such that:
- for all n $u_n \in \mathcal{D}(f_{k_n})$, $\mathrm{dist}(u_n, X) \to 0$, $f_{k_n}(u_n) \to c$,
- for all n $\alpha_n \in \partial^- f_{k_n}(u_n)$, $\Pi_{X \oplus \mathrm{span}(u_n)} \alpha_n \to 0$.

(b) We say that the $\nabla(f_n, X, c)$-condition holds if any $\nabla(f_n, f, X, c)$-sequence admits a subsequence converging to some point u in $\mathcal{D}$ such that $f(u) = c$.

(c) We say that a point u in $\mathcal{D} \cap X$ is an X-constrained asimptotically critical point for $((f_n), f)$, if there exists a $\nabla(f_n, X, f(u))$ sequence which converges to u.

Lemma 5.4. *Assume that for all n f_n is of class $C(p_n, q_n)$, and $\mathcal{D}(f_n)$ and X are non tangent. Let $\widetilde{W}_n := \Phi^{-1}(W_n)$ and define $g_n: \widetilde{W}_n \to \mathbb{R} \cup \{\infty\}$ by $g_n := f_n \circ \Phi + I_C$. Moreover, let $\widetilde{\mathcal{D}} := \Phi^{-1}(\mathcal{D}) \cap C$ and $g : \widetilde{\mathcal{D}} \to \mathbb{R}$ defined by $g := f \circ \Phi$. Then the following facts are true.*

(a) *Let z in $\widetilde{\mathcal{D}}$ be an asymptotically critical point for $((g_n)_n, g)$. Then*
  (a1) *if $z \in \mathrm{int}(C)$, then $u := \Phi(z)$ is an asymptotically critical point for $((f_n)_n, f)$;*
  (a2) *if $z \in \partial C$, then $u := \Phi(z)$ is an X-constrained asymptotically critical point for $((f_n)_n, f)$.*

(b) *Let $c \in \mathbb{R}$. If $\nabla(f_n, f, c)$ and $\nabla(f_n, f, X, c)$ hold, then $\nabla(g_n, g, c)$ holds.*

Proof. Let $(z_n)_n$ be a sequence in C, $(k_n)_n$ a strictly increasing sequence in $\mathbb{N}$ and $(\beta_n)_n$ a sequence in H such that

$$z_n \in \mathcal{D}(g_n) \quad \text{for all } n, \qquad \beta_n \in \partial^- g_{k_n}(z_n) \quad \text{for all } n, \qquad \beta_n \to 0.$$

We claim that there exists $(\alpha_n)_n$ in H such that $\alpha_n \in \partial^- f_{k_n}(\Phi(z_n))$ for all n and

if $\liminf_{n\to\infty} \operatorname{dist}(z_n, \partial C) > 0$ then $\alpha_n \to 0$,
if $\liminf_{n\to\infty} \operatorname{dist}(z_n, \partial C) = 0$ then $\Pi_{X\oplus\operatorname{span}(\Phi(z_n))}\alpha_n \to 0$.

Case 1. Suppose that $\inf \operatorname{dist}(z_n, \partial C) > 0$. From Lemma 5.2 it follows that $\alpha_n := (d\Phi(z_n)^*)^{-1}\beta_n$ belongs to $\partial^- f_{k_n}(\Phi(z_n))$ and from (b) of Remark 5.1 it turns out that $\alpha_n \to 0$, because

$$\|(d\Phi(z_n)^*)^{-1}\| = \|d\Phi(z_n)^{-1}\| \le 1 + \frac{\|Q_{\Phi(z_n)}\|}{\|Q\Phi(z_n)\|}$$

and the last term is bounded due to the fact that $\operatorname{dist}(z_n, \partial C)$ is far away from zero.

Case 2. We can suppose that $\operatorname{dist}(z_n, \partial C) \to 0$. Let $u_n := \Phi(z_n)$. From Lemma 5.2 we have that for any n there exist α_n in $\partial^- f_{k_n}(u_n)$ and $\lambda_n \ge 0$ such that

$$\beta_n = d\Phi(z_n)^*\alpha_n - \lambda_n Q^* Q z_n.$$

Now, we distinguish the terms z_n with $z_n \in \operatorname{int}(C)$ and the terms with $z_n \in \partial C$. In the first case $\lambda_n = 0$ so

$$\Pi_{X\oplus\operatorname{span}(z_n)}\beta_n = (d\Phi(z_n)\Pi_{X\oplus\operatorname{span}\{z_n\}})^*\alpha_n = \Pi_{X\oplus\operatorname{span}(u_n)}\alpha_n$$

(because $Q_{z_n}\Pi_{X\oplus\operatorname{span}(z_n)} = 0$) and $X \oplus \operatorname{span}(z_n) = X \oplus \operatorname{span}(u_n)$, since $u_n \notin X$. In the second case $\Pi_X Q^* Q = (Q\Pi_X)^* Q = 0$, because $X = \operatorname{Ker}(Q)$, so we deduce that $\Pi_{X\oplus\operatorname{span}(u_n)}\beta_n = \Pi_{X\oplus\operatorname{span}(u_n)}\alpha_n$. In both cases we get $\Pi_{X\oplus\operatorname{span}(u_n)}\alpha_n \to 0$.

From the claim it is easy to derive (a). To prove (b) just notice that, if $(\Phi(z_n))_n$ converges, then $(Qz_n/(\|Qz_n\|))_n$ is relatively compact, hence $(z_n)_n$ is relatively compact. □

For the next theorem we consider three closed subspaces X_1, X_2, X_3 of H such that $H = X_1 \oplus X_2 \oplus X_3$, and $\dim(X_1 \oplus X_2) < \infty$. We also suppose that:

S is a sphere in $X_2 \oplus X_3$ centered at 0,
Δ is a compact subset of $X_1 \oplus X_2$ such that $S \cap (X_1 \oplus X_2) \subset \operatorname{int}_{X_1\oplus X_2}(\Delta)$,
$\Sigma := (\partial_{X_1\oplus X_2}\Delta) \cup (\Delta \cap X_1)$.

Theorem 5.5 (∇-Asymptotic Theorem). *Assume that*

(a) *for all n f_n is lower semicontinuous and is of class $C(p_n, q_n)$ on W_n;*
(b) *for all n $\Delta \subset \mathcal{D}(f_n)$ and*

$$\limsup_{n\to\infty} \sup f_n(\Sigma) < \liminf_{n\to\infty} \inf f_n(S \cap W_n);$$

(c) *letting*

$$a := \liminf_{n\to\infty} \inf f_n(S \cap W_n), \quad b_n := \sup f_n(\Delta), \quad b := \limsup_{n\to\infty} b_n,$$

then $a \in \mathbb{R}$, $b \in \mathbb{R}$ *and there exists a sequence* $(a_n)_n$ *such that for all* n

$$a_n < \inf f_n(S \cap W_n) \quad \textit{and} \quad \overline{f_n^{-1}([a_n, b_n])} \subset W_n;$$

(d) *for all* n $\mathcal{D}(f_n)$ *and* $X_1 \oplus X_3$ *are not tangent;*

(e) $\nabla(f_n, f, c)$ *and* $\nabla(f_n, f, X_1 \oplus X_3, c)$ *hold for all* c *in* $[a, b]$.

Then

$$\#\{\textit{asymptotically critical points } u \textit{ for } ((f_n)_n, f) \mid a \le f(u) \le b\}$$
$$+ \#\{(X_1 \oplus X_3)\textit{-constrained a. c. p.'s } u \textit{ for } ((f_n)_n, f) \mid a \le f(u) \le b\} \ge 2.$$

Proof. Let us denote by P_1, P_2, P_3 the projections associated with the decomposition $H = X_1 \oplus X_2 \oplus X_3$. From now on we set $X := X_1 \oplus X_3$, $Q := P_1 + P_3$, and consider Φ and C defined as in the beginnig of this section, with this choice of X and Q.

Moreover, we set $\widetilde{W}_n := \Phi^{-1}(W_n)$, $\widetilde{\mathcal{D}} := \Phi^{-1}(\mathcal{D})$, and consider $g_n \colon \widetilde{W}_n \to \mathbb{R} \cup \{\infty\}$ and $g \colon \widetilde{\mathcal{D}}$ defined as before.

Step 1. We first show that $((g_n)_n, g)$ fulfill the assumptions of the multiplicity Theorem 3.5, more precisely of Remark 3.6, where $C_N = C$ for all n. By (a) and (d), using Lemma 5.2, we get that g_n is of class $C(p_n, q_n)$ and lower semicontinuous in $\widetilde{W}_n$. From (c) we deduce $\overline{g_n^{-1}([a_n, b_n])} \subset \widetilde{W}_n$. It is also clear that assumption (C) of Section 3, holds, since C is locally contractible. Finally, from (a), (d), (e), using (b) of Lemma 5.4, we obtain that $\nabla(g_n, g, c)$ holds for any c in $[a, b]$.

Step 2. At this point, in view of (a) of Lemma 5.4, it suffices to prove that there exist two asimptotically critical points for $((g_n)_n, g)$ in $g^{-1}([a, b])$. We shall prove this fact by showing that for n large $\operatorname{cat}_C(g_n^{b_n}, g_n^{a_n}) \ge 2$ and by applying Remark 3.6. It is clear that, up to getting closer to $\inf f_n(S)$, we can suppose

$$a_n > \sup f_n(\Sigma), \qquad \liminf_{n\to\infty} a_n = a.$$

For $r_1, r_2 > 0$ we set

$$D := \{u \in X_1 \oplus X_2 \mid \|P_1 u\| \le r_1, \ \|P_2 u\| \le r_2\}$$
$$T := (\partial_{X_1 \oplus X_2} D) \cup (D \cap X_1).$$

We can choose $r_1 > 0$ and $r_2 > 0$ such that $\Delta \subset \operatorname{int}_{X_1 \oplus X_2}(D)$. We also define

$$\mathbf{S} := \Phi^{-1}(S) \cap C, \qquad \mathbf{D} := \Phi^{-1}(D) \cap C,$$
$$\boldsymbol{\Delta} := \Phi^{-1}(\Delta) \cap C, \qquad \mathbf{T} := \Phi^{-1}(T) \cap C,$$
$$\boldsymbol{\Sigma} := \Phi^{-1}(\Sigma) \cap C, \qquad \boldsymbol{\Gamma} := \mathbf{S} \cap \mathbf{D}.$$

It is clear that $\mathbf{\Delta} \subset X_1 \oplus X_2$, $\mathbf{\Sigma} = \partial_{X_1 \oplus X_2} \mathbf{\Delta}$, $\mathbf{T} = \partial_{X_1 \oplus X_2} \mathbf{D}$ and

$$\mathbf{D} = \{z \in H \mid \|P_1 z\| \leq r_1,\ 1 \leq \|P_2 z\| \leq r_2 + 1,\ P_3 z = 0\} \subset X_1 \oplus X_2.$$

We show now that for n large the assumptions of Lemma 5.6 are satisfied, with the sets introduced above and with $A := g_n^{a_n}$, $B := g_n^{b_n}$.

We first show that there exists a retraction $\pi: C \to \mathbf{D}$ such that $\Pi^{-1}(\mathbf{\Gamma}) \subset \mathbf{S}$. Actually we can first define $\pi_1: C \to (X_1 \oplus X_2) \cap C$ by

$$\pi_1(z) := P_1 z + \left(1 + \left\| z - P_1 z - \frac{P_2 z}{\|P_2 z\|} \right\| \right) \frac{P_2 z}{\|P_2 z\|}.$$

It is clear that π_1 is a retraction of C into $(X_1 \oplus X_2) \cap C$, such that $\pi^{-1}(\mathbf{\Gamma}) \subset \mathbf{S}$. Now we can define π by composing π_1 and π_2, where

$$\pi_2(z) := \left(1 \wedge \frac{r_1}{\|P_1 z\|}\right) P_1 z + \left(1 \wedge \frac{(r_2 + 1)}{\|P_2 z\|}\right) P_2 z$$

(the first term being zero if $P_1 z = 0$).

It is clear that $\mathbf{T} \subset \mathbf{D} \setminus \mathbf{\Gamma}$, because $S \cap (X_1 \oplus X_2) \subset \operatorname{int}_{X_1 \oplus X_2}(\Delta)$, and that $\mathbf{T}$ is a deformation retract of $\mathbf{D} \setminus \mathbf{\Gamma}$ in $\mathbf{D}$. Then (a) and (b) of Lemma 5.6 are verified. It is also evident that (c) holds too, and that, finally, for n large

$$\sup g_n(\mathbf{\Sigma}) < a_n < \inf g_n(\mathbf{S}), \qquad \sup g_n(\mathbf{\Delta}) = b_n,$$

hence $g_n^{a_n} \cap \mathbf{S} = \emptyset$ and $\mathbf{\Delta} \subset g_n^{b_n}$, $\mathbf{\Sigma} \subset g_n^{a_n}$, $\pi(g_n^{a_n}) \cap \mathbf{\Gamma} = \emptyset$.

Applying Lemma 5.6 we get $\operatorname{cat}_C(g_n^{b_n}, g_n^{a_n}) \geq \operatorname{cat}_{\mathbf{D}}(\mathbf{D}, \mathbf{T})$.

It is well known (see for instance in [14]) that $\operatorname{cat}_{\mathbf{D}}(\mathbf{D}, \mathbf{T}) = 2$, so the conclusion follows. □

Lemma 5.6. *Let C be a topological space and let $\mathbf{D}$ be a closed subspace of C. We assume that*

(a) *there exists a continuous retraction $\pi : C \to \mathbf{D}$;*
(b) *there exist two subset $\mathbf{T}$ and $\mathbf{\Gamma}$ of $\mathbf{D}$ such that $\mathbf{T}$ is closed, $\mathbf{T} \subset \mathbf{D} \setminus \mathbf{\Gamma}$, and $\mathbf{T}$ is a strong deformation retract in the space $\mathbf{D}$ of $\mathbf{D} \setminus \mathbf{\Gamma}$;*
(c) *there exist two other closed sets $\mathbf{\Delta}$ and $\mathbf{\Sigma}$ such that $\mathbf{\Sigma} \subset \mathbf{\Delta} \subset \mathbf{D}$ and $\partial_{\mathbf{D}} \mathbf{\Delta} \subset \mathbf{\Sigma}$, $\mathbf{T} \cap (\mathbf{\Delta} \setminus \mathbf{\Sigma}) = \emptyset$, $\mathbf{\Gamma} \subset \mathbf{\Delta}$.*

Then, for any pair $(B; A)$ of closed sets in C such that $\mathbf{\Delta} \subset B$, $\mathbf{\Sigma} \subset A$, $\pi(A) \cap \mathbf{\Gamma} = \emptyset$ we have

$$\operatorname{cat}_C(B, A) \geq \operatorname{cat}_{\mathbf{D}}(\mathbf{D}, \mathbf{T}).$$

Proof. Let $U_0, \dots U_k$ closed subsets of C, which we may suppose contained in B, such that

$$B = \bigcup_{i=0}^{k} U_i, \qquad U_1, \dots, U_k \text{ are contractible in } C,$$
$$A \subset U_0, \quad A \text{ is a strong deformation retract in } C \text{ of } U_0.$$

Let $V_i := U_i \cap \mathbf{\Delta}$, if $i = 1, \dots, k$ and $V_0 := (U_0 \cap \mathbf{\Delta}) \cup (\mathbf{D} \setminus \mathbf{\Delta})$. It is trivial that $\mathbf{D} = \bigcup_{i=0}^k V_i$, since $\mathbf{\Delta} \subset B$, and that $V_1, \dots, V_k$ are closed. It is also easy to check that they are contractible in $\mathbf{D}$, by using the retraction π.

Now we notice that, since $\mathbf{\Sigma} \subset A \cap \mathbf{\Delta} \Rightarrow \mathbf{\Sigma} \subset U_0 \cap \mathbf{\Delta}$, then

$$V_0 = (U_0 \cap \mathbf{\Delta}) \cup \mathbf{\Sigma} \cup (\mathbf{D} \setminus \mathbf{\Delta}) = (U_0 \cap \mathbf{\Delta}) \cup (\mathbf{D} \setminus (\mathbf{\Delta} \setminus \mathbf{\Sigma})).$$

Furthermore $\mathbf{\Delta} \setminus \mathbf{\Sigma}$ is open in $\mathbf{D}$, since $\partial_{\mathbf{D}} \mathbf{\Delta} \subset \mathbf{\Sigma}$, hence $\mathbf{D} \setminus (\mathbf{\Delta} \setminus \mathbf{\Sigma})$ is closed, so V_0 is closed. It is also clear that $\mathbf{T} \subset V_0$. We want to show that $\mathbf{T}$ is a strong deformation retract of V_0.

Composing the strong deformation of U_0 with π we can find a deformation $\eta : U_0 \cap \mathbf{\Delta} \times [0,1] \to \mathbf{D}$ such that $E := \eta(U_0 \cap \mathbf{\Delta}, 1) \subset \pi(A)$ and $\eta(x,t) = x$ whenever $x \in A \cap \mathbf{\Delta}$ (in particular if $x \in \mathbf{\Sigma}$). Since

$$(U_0 \cap \mathbf{\Delta}) \cap (\mathbf{D} \setminus (\mathbf{\Delta} \setminus \mathbf{\Sigma})) = (U_0 \cap \mathbf{\Delta}) \cap ((\mathbf{D} \setminus \mathbf{\Delta}) \cup \mathbf{\Sigma}) = (U_0 \cap \mathbf{\Delta}) \cap \mathbf{\Sigma} = \mathbf{\Sigma},$$

we can extend η to $V_0 \times [0,1]$ in such a way that $\eta(x,t) = x$ whenever $x \in (\mathbf{D} \setminus (\mathbf{\Delta} \setminus \mathbf{\Sigma}))$ and in particular $\eta(x,t) = x$ if $x \in \mathbf{T} \subset (\mathbf{D} \setminus (\mathbf{\Delta} \setminus \mathbf{\Sigma}))$. In this way $\eta(V_0, 1) = E \cup (\mathbf{D} \setminus \mathbf{\Delta})$.

Since $E \subset \pi(A)$ we have $E \cap \mathbf{\Gamma} = \emptyset$. Moreover, $(\mathbf{D} \setminus \mathbf{\Delta}) \cap \mathbf{\Gamma} = \emptyset$, so by (b) one can deform $E \cup (\mathbf{D} \setminus \mathbf{\Delta})$ in $\mathbf{D}$ to $\mathbf{T}$ keeping $\mathbf{T}$ fixed. Glueing the two deformations one can finally see that $\mathbf{T}$ is a strong deformation retract in $\mathbf{D}$ of V_0 and the conclusion follows. □

6. Constrained bounce trajectories in a star-shaped domain

Let Ω, ν, V, A, and B be as in Section 4. Let $\gamma_0 \in \mathbb{X}(A,B)$ be such that $\gamma_0([0,T]) \subset \Omega$ and let us assume that Ω is uniformly star-shaped with respect to Ω. Let X be a closed subspace of $L^2(0,T;\mathbb{R}^N)$ with finite codimension.

We remind that Π_Y denotes the orthogonal L^2 projection on a closed subspace Y of L^2.

Lemma 6.1. *Let $(\gamma_n)_n$ be a sequence of curves in $\mathbb{X}(A,B)$ such that $(\gamma_n)_n$ is bounded in $W^{1,2}(0,T;\mathbb{R}^N)$ and*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \operatorname{dist}(\gamma_n(t), \Omega) = 0.$$

Suppose that $(\mu_n)_n$ is a sequence of positive numbers and $(\beta_n)_n$ is a sequence in $L^2(0,T;\mathbb{R}^N)$ such that

$$\ddot{\gamma}_n + \nabla V(t,\gamma_n) + \mu_n(U(\gamma_n)) = \beta_n \tag{6.1}$$

and $\Pi_{X\oplus \operatorname{span}(\gamma_n-\gamma_0)}\beta_n$ are bounded in $L^1(0,T;\mathbb{R}^N)$. Then

(a) *$(\ddot{\gamma}_n)_n$ and $(\mu_n \nabla U(\gamma_n))_n$ are bounded in $L^1(0,T;\mathbb{R}^N)$;*

(b) *if* $\Pi_{X\oplus\mathrm{span}(\gamma_n-\gamma_0)}\beta_n \to 0$ *in* $L^1(0,T;\mathbb{R}^N)$, *then there exists a subsequence* $(\gamma_{n_k})_k$ *such that* $\gamma_{n_k}\to\gamma$ *in* $W^{1,2}(0,T;\mathbb{R}^N)$, *for a suitable curve* γ *in* $\mathbb{X}(A,B)$ *with the properties*

$$
\begin{cases}
\textit{there exists a nonnegative measure } \mu \textit{ such that} \\
\int_0^T(\dot\gamma\dot\delta-\nabla V(t,\gamma)\delta)\,dt+\int_0^T\nu(\gamma)\delta\,d\mu=0 \\
\textit{for all } \delta \textit{ in } X\cap W_0^{1,2}(0,T;\mathbb{R}^N), \\
\mathrm{spt}(\mu)\subset\{t\in[0,T]\mid\gamma(t)\in\partial\Omega\}.
\end{cases}
\tag{6.2}
$$

Proof. *Step 1.* Since Ω is uniformly star-shaped we have

$$\nabla U(\gamma_n(t))(\gamma_n(t)-\gamma_0(t))\geq\frac{\varepsilon}{2}|\nabla U(\gamma_n(t))|\quad\text{for all } t \text{ in } [0,T],$$

for n large enough. Multiplying (6.2) by $\gamma_n-\gamma_0$, we get

$$
\begin{aligned}
\int_0^T\dot\gamma_n(\dot\gamma_n-\dot\gamma_0)\,dt-\int_0^T\nabla V(t,\gamma_n)(\gamma_n-\gamma_0)\,dt+\int_0^T\beta_n(\gamma_n-\gamma_0)\,dt \\
=\mu_n\int_0^T\nabla U(\gamma_n)(\gamma_n-\gamma_0)\,dt\geq\mu_n\frac{\varepsilon}{2}\int_0^T|\nabla U(\gamma_n)|\,dt.
\end{aligned}
$$

Since

$$\int_0^T\beta_n(\gamma_n-\gamma_0)\,dt=\int_0^T\Pi_{X\oplus\mathrm{span}(\gamma_n-\gamma_0)}\beta_n(\gamma_n-\gamma_0)\,dt$$

we get that $\mu_n\nabla U(\gamma_n)$ is bounded in $L^1(\Omega)$.

Step 2. Let Y be a finite dimensional subspace of $W_0^{1,2}(0,T;\mathbb{R}^N)$ such that $L^2(0,T;\mathbb{R}^N)=X\oplus Y$ (such a subspace exists since $W_0^{1,2}(0,T;\mathbb{R}^N)$ is dense in $L^2(0,T;\mathbb{R}^N)$). If $\delta\in X\cap W_0^{1,2}(0,T;\mathbb{R}^N)$, multiplying (6.1) by δ yields

$$\int_0^T\dot\gamma_n\dot\delta\,dt=\int_0^T\nabla V(t,\gamma_n)\delta\,dt+\mu_n\int_0^T\nabla U(\gamma_n)\delta\,dt-\int_0^T\Pi_X\beta_n\delta\,dt.$$

Then

$$\left|\int_0^T\dot\gamma_n\dot\delta\,dt\right|\leq K_1\|\delta\|_\infty\quad\text{for all } \delta \text{ in } X\cap W_0^{1,2}(0,T;\mathbb{R}^N)$$

for a suitable constant K_1. On the other hand if $\delta\in Y$

$$\int_0^T\dot\gamma_n\dot\delta\,dt\leq\|\gamma_n\|_W\|\delta\|_W\leq K_2\|\delta\|_\infty\quad\text{for all } \delta \text{ in } Y$$

for another constant K_2 (since Y is finite dimensional).

Step 3. Denote by P and Q the projections of $L^2(0,T;\mathbb{R}^N)$ onto X and Y respectively. It is clear that the restriction of Q to $W_0^{1,2}(0,T;\mathbb{R}^N)$ is continuous as a map from $W_0^{1,2}(0,T;\mathbb{R}^N)$ into Y with respect to the norm $\|\cdot\|_\infty$ (since Y is finite dimensional). By difference also the restriction of P to $W_0^{1,2}(0,T;\mathbb{R}^N)$, as a map from $W_0^{1,2}(0,T;\mathbb{R}^N)$ to $X \cap W_0^{1,2}(0,T;\mathbb{R}^N)$, is continuous with respect to $\|\cdot\|_\infty$. Then for any δ in $W_0^{1,2}(0,T;\mathbb{R}^N)$

$$\int_0^T \dot\gamma_n \dot\delta\, dt = \int_0^T \dot\gamma_n (P\delta + Q\delta)'\, dt \le K_1 \|P\delta\|_\infty + K_2\|Q\delta\|_\infty \le K\|\delta\|_\infty$$

for a suitable constant K. This concludes the proof of the first claim.

Step 4. To prove the second claim we first notice that, since $(\mu_n \nabla U(\gamma_n))_n$ and $(\ddot\gamma_n)_n$ are bounded in $L^1(0,T;\mathbb{R}^N)$, then there exists $(n_k)_k$ such that $\gamma_{n_k} \to \gamma$ in $W^{1,2}(0,T;\mathbb{R}^N)$, for a suitable γ in $\overline{\mathbb{X}}(A,B)$ and $\mu_{n_k}\nabla U(\gamma_{n_k})$ converge weakly to a nonnegative measure μ (in the dual of $\mathcal{C}_0^0(]0,T[)$). Since $\gamma_{n_k} \to \gamma$ uniformly it is clear that the support of μ is contained in $\{t \mid \gamma(t) \in \partial\Omega\}$. If we multiply by δ in (6.1) and pass to the limit we get the conclusion. □

Now let $R > g(\gamma_0)$ and consider the functionals $\widetilde f_{R,\omega}\colon L^2(0,T;\mathbb{R}^N) \to \mathbb{R}\cup\{\infty\}$ defined by $\widetilde f_{R,\omega}(\delta) := f_{R,\omega}(\gamma_0+\delta) - f_{R,\omega}(\gamma_0)$ and the functional $\widetilde f_{R,\infty}\colon \mathcal{D}_{\infty,R} \to \mathbb{R}$ defined by $\widetilde f_{R,\infty}(\delta) := f_{R,\infty}(\gamma_0+\delta) - f_{R,\infty}(\gamma_0)$, where $\mathcal{D}_{\infty,R} := \overline{\mathbb{X}}_R(A,B) - \gamma_0$.

Proposition 6.2. *Suppose that there are no γ's in $\overline{\mathbb{X}}_R(A,B) \cap (\gamma_0 + X)$ such that $g(\gamma) = R$ and $0 \in \partial^- g(\gamma)$.*

(a) *If $\delta \in \mathcal{D}_{\infty,R} \cap X$, δ is an X-constrained asymptotical critical point for $((\widetilde f_{R,\omega})_\omega, f_{R,\infty})$, then $\gamma := \gamma_0 + \delta$ belongs to $\overline{\mathbb{X}}_R(A,B) \cap (X + \gamma_0)$ and verifies* (6.2).

(b) *If in addition there exist no γ's in $\overline{\mathbb{X}}(A,B) \cap (X+\gamma_0)$ with $g(\gamma) = R$ such that* (6.2) *holds, then $\nabla(\widetilde f_{R,\omega}, \widetilde f_{R,\infty}, X, c)$ holds for every c in $\mathbb{R}$.*

Proof. We prove the first claim. Let δ be an X-constrained asymptotical critical point for $((\widetilde f_{R,\omega})_\omega, f_{R,\infty})$; then there exist a sequence $(\omega_n)_n$ such that $\omega_n \to \infty$ and a $\nabla(\widetilde f_{R,\omega_n}, \widetilde f_{R,\infty}, X, c)$-sequence $(\delta_n)_n$ such that $\delta_n \to \delta$ in $L^2(0,T;\mathbb{R}^N)$. Let $\gamma_n := \gamma_0 + \delta_n$; we claim that $(\gamma_n)_n$ verifies the assumptions of Lemma 6.1. We have indeed:

Step 1. $(\gamma_n)_n$ is bounded in $W^{1,2}(0,T;\mathbb{R}^N)$, since $\gamma_n \in \mathbb{X}_R(A,B)$ for all n and $\mathbb{X}_R(A,B)$ is bounded in $W^{1,2}(0,T;\mathbb{R}^N)$;

Step 2. $\omega_n \int_0^T U(\gamma_n)\, dt$ are bounded, since $f_{R,\omega_n}(\gamma_n)$ are bounded below; hence

$$\sup_{t\in[0,T]} \operatorname{dist}(\gamma_n(t), \Omega) \to 0;$$

it follows, by the assumption, that there exists $\overline{n}$ in $\mathbb{N}$ such that for all $n \ge \overline{n}$ it cannot happen that $g(\gamma_n) = R$ and $0 \in \partial^- g(\gamma_n)$;

Step 3. Let $(\alpha_n)_n$ be a sequence in $L^2(0,T;\mathbb{R}^N)$ such that $\alpha_n \in \partial^- \widetilde f_{R,\omega_n}(\delta_n) = \partial^- f_{R,\omega_n}(\gamma_n)$ for all n and $\Pi_{X\oplus \operatorname{span}(\delta_n)}\alpha_n \to 0$; by Proposition 4.3 there exists a sequence $(\lambda_n)_n$ in $\mathbb{R}$ such that

$$(6.3)\qquad \begin{cases} \ddot{\gamma}_n + \nabla V(t,\gamma_n) + \dfrac{\omega_n}{1+\lambda_n}\nabla U(\gamma_n) + \dfrac{1}{1+\lambda_n}\alpha_n = 0, \\ \lambda_n \geq 0, \quad \lambda_n = 0 \quad \text{if } g(\gamma_n) = R. \end{cases}$$

By Lemma 6.1, up to a subsequence, we have that $\gamma_n \to \gamma$ in $W^{1,2}(0,T;\mathbb{R}^N)$ for a curve γ verifying (6.2). It is clear that $\gamma \in X + \gamma_0$ and that $\gamma = \gamma_0 + \delta$ so the first conclusion is true.

To prove the second claim let $(\omega_n)_n$ and $(\delta_n)_n$ be as before. Arguing as above we can find $(\lambda_n)_n$ such that (6.3) holds and a curve γ such that $\gamma_0 + \delta_n \to \gamma$ in $W^{1,2}(0,T;\mathbb{R}^N)$, up to passing to a subsequence. It is also clear that $\gamma \in \overline{\mathbb{X}}(A,B) \cap (\gamma_0 + X)$; to get the conclusion we just need to show that $\widetilde{f}_{R,\infty}(\gamma - \gamma_0) = c$. We claim that $(\lambda_n)_n$ is bounded; if not we would have $g(\gamma_n) = R$ for n large, hence $g(\gamma) = R$ which is not allowed by the assumptions. Since $(\lambda_n)_n$ bounded we have $\omega_n \int_0^T U(\gamma_n)\,dt \to 0$ because

$$\omega_n \int_0^T U(\gamma_n)\,dt \leq \frac{\omega_n}{p}\|G(\gamma_n)\|_\infty \int_0^T \nabla U(\gamma_n)\,dt \leq \text{const}\|G(\gamma_n)\|_\infty$$

by Lemma 6.1. Then $f_{R,\omega_n}(\gamma_n) \to f_{R,\infty}(\gamma)$ that is $\widetilde{f}_{R,\omega_n}(\delta_n) \to \widetilde{f}_{R,\infty}(\gamma - \gamma_0)$. □

7. Proofs of the main results

Throughout this section we assume that (V) and (Λ_0) of Section 2 hold, for suitable β and x_0. For the sake of convenience we assume that $V(\lambda,t,x,) = (\lambda/2)\beta(t)(x)x + x_0(t)x$ for all λ in Λ_0, t in $[0,T]$ and x in a neighbourhood Ω_1 of Ω. We also use all the auxiliary definitions and notations introduced in Section 4. Moreover, we denote by $\|\cdot\|$ the L^2-norm while, when needed, we denote by $\|\cdot\|_W$ the norm in $W^{1,2}(0,T;\mathbb{R}^N)$.

For $\omega > 0$, λ in Λ_0, and $R \geq g(\gamma_{0,\lambda})$ we consider again the functionals $\widetilde{f}_{R,\omega}\colon L^2(0,T;\mathbb{R}^N) \to \mathbb{R} \cup \{\infty\}$ defined by

$$\widetilde{f}_{R,\omega}(\delta) := f_{R,\omega}(\gamma_{0,\lambda} + \delta) - f_{R,\omega}(\gamma_{0,\lambda}),$$

and

$$\begin{aligned} \mathcal{D}_\infty &:= \{\delta \in W_0^{1,2}(0,T;\mathbb{R}^N) \mid \gamma_{0,\lambda} + \delta \in \overline{\mathbb{X}}(A,B)\}, \\ \mathcal{D}_{R,\infty} &:= \{\delta \in W_0^{1,2}(0,T;\mathbb{R}^N) \mid \gamma_{0,\lambda} + \delta \in \overline{\mathbb{X}}_R(A,B)\}. \end{aligned}$$

It is easy to check that, if $\gamma_{0,\lambda} + \delta \in \mathbb{X}_R(A,B), (\gamma_{0,\lambda} + \delta)([0,T]) \subset \Omega_1$, then

$$\widetilde{f}_{R,\omega}(\delta) = Q_\lambda(\delta) - \omega \int_0^T U(\gamma_{0,\lambda} + \delta)\,dt.$$

where

$$Q_\lambda(\delta) := \frac{1}{2}\int_0^T |\dot{\delta}|^2\,dt - \frac{\lambda}{2}\int_0^T \beta(t)(\delta)\delta\,dt \quad \text{for } \delta \text{ in } W_0^{1,2}(0,T;\mathbb{R}^N).$$

Finally we define $\widetilde{f}_{R,\infty}\colon \mathcal{D}_{R,\infty} \to \mathbb{R}$ by

$$\widetilde{f}_{R,\infty}(\delta) := Q_\lambda(\delta) \quad \text{for all } \delta \text{ in } \mathcal{D}_{R,\infty}.$$

Notice that all these definitions depend on λ, which we do not write explicitly to keep the notation simpler.

Given λ_i eigenvalue of (2.5) we set

$$\begin{aligned}\mathbb{X}^-_{\lambda_i} &:= \operatorname{span}(e_j \mid 0 \wedge \lambda_i \le \lambda_j \le 0 \vee \lambda_i),\\ \mathbb{X}^+_{\lambda_i} &:= \bigg\{\delta \in W^{1,2}_0(0,T;\mathbb{R}^N) \;\bigg|\; \int_0^T \dot\delta \dot e\, dt = 0 \quad \text{for all } e \in \mathbb{X}^-_{\lambda_i}\bigg\}.\end{aligned}$$

Remark 7.1. Let λ_i be an eigenvalue of (2.5). Then (remind (2.6)):

(a) $\mathbb{X}^+_{\lambda_i}$ is the $W^{1,2}(0,T;\mathbb{R}^N)$-closure of $\operatorname{span}(e_j \mid \lambda_j \notin [\lambda_i \wedge 0, \lambda_i \vee 0]) \oplus E_0$;

(b) $Q_\lambda(\delta) \le 0$ for δ in $\mathbb{X}^-_{\lambda_i}$, whenever either $\lambda \ge \lambda_i > 0$ or $\lambda \le \lambda_i < 0$;

(c) if $\lambda \in \mathbb{R}$, then

$$Q_\lambda(\delta) \ge c_\lambda \int_0^T |\dot\delta|^2\, dt \ge c_\lambda S \|\delta\|^2 \quad \text{for all } \delta \text{ in } \mathbb{X}^+_{\lambda_i}$$

where

$$c_\lambda := \min\bigg\{\frac{1}{2}\bigg(1 - \frac{\lambda}{\lambda_j}\bigg) \;\bigg|\; e_j \in \mathbb{X}^+_{\lambda_i}\bigg\},\ S := \inf\bigg\{\int_0^T |\dot\delta|^2\, dt \;\bigg|\; \int_0^T |\delta|^2\, dt = 1\bigg\};$$

moreover, if either $\lambda_{-1} < \lambda < \lambda_{i+1}$ and $\lambda_{i+1} > \lambda_i > 0$, or $\lambda_0 > \lambda > \lambda_{i+1}$ and $\lambda_{i-1} < \lambda_i < 0$, then $c_\lambda > 0$;

(d) we have

$$L^2(0,T;\mathbb{R}^N) = \mathbb{X}^-_{\lambda_i} \oplus \overline{\mathbb{X}^+_{\lambda_i}}$$

(of course the closure is taken in $L^2(0,T;\mathbb{R}^N)$); equivalently, the projections onto $\mathbb{X}^-_{\lambda_i}$ and $\mathbb{X}^+_{\lambda_i}$ with respect to $W^{1,2}_0(0,T;\mathbb{R}^N)$, which we may denote by $P^-_{\lambda_i}$ and $P^+_{\lambda_i}$, are well defined and continuous with respect to the $L^2(0,T;\mathbb{R}^N)$-norm.

Proof. We prove (d). Let $\mathcal{B}(\delta) := (1/2)\int_0^T \beta(t)(\delta)\delta\, dt$; it is easy to see that

$$\mathcal{B}(\delta) > 0 \text{ (resp. } \mathcal{B}(\delta) < 0) \quad \text{for } \delta \text{ in } \mathbb{X}^-_{\lambda_i} \setminus \{0\}, \quad \text{if } \lambda_i > 0 \text{ (resp. if } \lambda_i < 0).$$

Furthermore, $\mathcal{B}$ is L^2-continuous and $\mathcal{B}'(\delta_1)(\delta_2) = 0$, whenever $\delta_1 \in \mathbb{X}^-_{\lambda_i}$ and $\delta_2 \in \mathbb{X}^+_{\lambda_i}$. It follows that, if $\delta \in \mathbb{X}^-_{\lambda_i} \cap \overline{\mathbb{X}^+_{\lambda_i}}$, we have $\mathcal{B}(\delta) = 2\mathcal{B}'(\delta)(\delta) = 0$. Moreover, $P^-_{\lambda_i}(\delta) = \sum_{j \mid e_j \in X^-_{\lambda_i}} \lambda_j \mathcal{B}'(e_j)(\delta) e_j$ which is continuous with respect to the L^2 norm (since $\beta \in \mathrm{L}^2(0,T;\mathbb{R}^{N^2})$). By difference, the same is true for the complementary projection $P^+_{\lambda_i}$. □

Let λ_i be an eigenvalue of (2.5) and let $\overline{e} = \overline{e}(\lambda_i)$ be an element of $(\mathbb{X}^+_{\lambda_i} \setminus \{0\}) \cap \mathbb{X}^-_{\lambda_k}$, where $\lambda_i = \lambda_{k-1} < \lambda_k$, if $0 < \lambda_i$, or $\lambda_i = \lambda_{k+1} > \lambda_k$, if $0 > \lambda_i$. Given $\rho > 0$ and $\sigma > 0$ we set:

$$\begin{aligned}
P(\lambda_i) &:= \{\delta + t\overline{e} \mid \delta \in \mathbb{X}^-_{\lambda_i},\ t \geq 0\},\\
\Delta(\lambda_i) &:= P(\lambda_i) \cap \mathcal{D}_\infty, \qquad \Delta_\sigma(\lambda_i) := \{\delta \in P(\lambda_i) \setminus \mathbb{X}_{\lambda_i} \mid \operatorname{dist}_{L^2}(\delta, \Delta(\lambda_i)) < \sigma\},\\
\Sigma_\sigma(\lambda_i) &:= \partial_{\mathbb{X}^-_{\lambda_i} \oplus \operatorname{span}(\overline{e})} \Delta_\sigma(\lambda_i), \qquad S_\rho(\lambda_i) := \{\delta \in \overline{\mathbb{X}^+_{\lambda_i}} \mid \|\delta\| = \rho\}.
\end{aligned}$$

LEMMA 7.2. *Let λ_i be an eigenvalue of* (2.5)*. Then $\Delta(\lambda_i)$ is bounded and the following facts hold.*

(a) *Given λ in Λ_0 we have:*

(a1) *for any $R > \sup g(\gamma_{0,\lambda} + \Delta(\lambda_i))$ there exists $\sigma(\lambda, R) > 0$ such that for every σ in $]0, \sigma(\lambda, R)]$*

$$\sup g(\gamma_{0,\lambda} + \Delta_\sigma(\lambda_i)) < R$$

and therefore

$$\begin{aligned}
\lim_{\omega\to\infty} \sup \widetilde{f}_{R,\omega}(\Delta_\sigma(\lambda_i)) &= \sup Q_\lambda(\Delta(\lambda_i)),\\
\lim_{\omega\to\infty} \sup \widetilde{f}_{R,\omega}(\Sigma_\sigma(\lambda_i)) &= \sup Q_\lambda(\mathcal{D}_\infty \cap \mathbb{X}^-_{\lambda_i});
\end{aligned}$$

if $(\lambda/\lambda_i) \geq 1$ we can be more explicit in the last equality and say

$$\sup \widetilde{f}_{R,\omega}(\Sigma_\sigma(\lambda_i)) = 0 \quad \textit{for } \omega \textit{ large enough;} \tag{7.1}$$

(a2) *for any R in $\mathbb{R}$ there exists $\rho = \rho(\lambda, R) > 0$ such that $\emptyset \neq S_\rho(\lambda_i) \cap P(\lambda_i) \subset \Delta(\lambda_i)$ and such that every curve γ in $(\gamma_{0,\lambda} + S_\rho(\lambda_i)) \cap \mathbb{X}_R(A,B)$ verifies $\gamma([0,T]) \subset \Omega$; it follows that*

$$\inf Q_\lambda(S_\rho(\lambda_i)) \leq \inf \widetilde{f}_{R,\omega}(S_\rho(\lambda_i)) \quad \textit{for all } \omega. \tag{7.2}$$

(b) *Let $\lambda \in \Lambda_0$ and $R > \sup g(\gamma_{0,\lambda} + \Delta(\lambda_i))$. If either $0 < \lambda_i \leq \lambda < \lambda_{i+1}$ or $\lambda_{i-1} < \lambda \leq \lambda_i < 0$, then for $0 < \sigma \leq \sigma(\lambda, R)$ and $\rho = \rho(\lambda, R)$ we have*

$$\sup \widetilde{f}_{R,\omega}\,(\Sigma_\sigma(\lambda_i)) = 0 < \inf Q_\lambda(S_\rho(\lambda_i)) \leq \inf \widetilde{f}_{R,\omega}(S_\rho(\lambda_i)) \quad \textit{for } \omega \textit{ large.} \tag{7.3}$$

PROOF. (a1) The existence of σ such that the first property holds is trivial since $\overline{\Delta_\sigma(\lambda_i)}$ is compact. Concerning the two limits notice that, if $\gamma \in \mathbb{X}_R(A,B)$ and $\gamma - \gamma_{0,\lambda} \in P(\lambda_i) \setminus \Delta(\lambda_i)$, then

$$\lim_{\omega\to\infty} \widetilde{f}_{R,\omega}(\gamma - \gamma_{0,\lambda}) = -\infty;$$

moreover, if $(\lambda/\lambda_i) \geq 1$, the last conclusion follows from

$$\widetilde{f}_{R,\omega}(\gamma - \gamma_{0,\lambda}) \leq 0 \quad \text{for all } \gamma \text{ in } \mathbb{X}_R(A,B) \text{ such that } \gamma - \gamma_{0,\lambda} \in \mathbb{X}^-_{\lambda_i}\ .$$

(a2) The existence of $\rho(\lambda, R)$ follows from the interpolation

$$\|\delta\|_{L^\infty} \leq \text{const}\|\dot{\delta}\|^{1/2}\|\delta\|^{1/2} \leq \text{const}_1(R)\|\delta\|^{1/2} \qquad \text{if } \gamma_{0,\lambda} + \delta \in \mathbb{X}_R(A,B)$$

(using Remark 4.1 and the fact that V is bounded). From this (7.2) follows immediately.

(b) To get the conclusion it suffices to combine (7.1) and (7.2), noticing that $(\lambda/\lambda_i) \geq 1$, that the constant c_λ in (c) of Remark 7.1 is positive, and that $\inf Q_\lambda S_\rho(\lambda_i)) \geq S(c_\lambda/2)\rho^2$. □

Lemma 7.3. *Let λ_i be an eigenvalue of* (2.5) *with λ_i in Λ_0. Then there exists $\varepsilon > 0$ such that $[\lambda_i - \varepsilon, \lambda_i + \varepsilon] \subset \Lambda_0$ and for every λ in $[\lambda_i - \varepsilon, \lambda_i + \varepsilon]$ one has*

$$\sup Q_\lambda(\mathcal{D}_\infty \cap \mathbb{X}^-_{\lambda_i}) < \inf_{|R| \leq R(\lambda)+1} Q_\lambda(S_{\rho(\lambda,R)}(\lambda_i)) \tag{7.4}$$

where $R(\lambda) := \sup g(\gamma_{0,\lambda} + \Delta(\lambda_i))$. Then for such λ it turns out that for all R in $]R(\lambda), R(\lambda)+1]$ and all σ in $]0, \sigma(\lambda, R)]$

$$\sup \widetilde{f}_{R,\omega}(\Sigma_\sigma(\lambda_i)) < \inf \widetilde{f}_{R,\omega}(S_{\rho(\lambda,R}(\lambda_i)) \quad \textit{for } \omega \textit{ large.} \tag{7.5}$$

Proof. By continuity we have

$$\liminf_{\lambda \to \lambda_i} \inf\{Q_\lambda(S_{\rho(\lambda,R)}(\lambda_i)) \mid |R| \leq R(\lambda)+1\} > 0$$

since

$$\liminf_{\lambda \to \lambda_i} \inf\{\rho(\lambda,R) \mid |R| \leq R(\lambda)+1\} > 0.$$

Moreover,

$$\lim_{\lambda \to \lambda_i} \sup Q_\lambda(\mathcal{D}_\infty \cap \mathbb{X}^-_{\lambda_i}) = 0.$$

Then for λ close to λ_i (7.4) holds; finally if we fix R in $]R(\lambda), R(\lambda)+1]$ we derive (7.5) from (a1) of Lemma 7.2 and from (7.2). □

From now on we denote by $\varepsilon(\lambda_i)$ the number ε provided by Lemma 7.3.

Lemma 7.4. *Let λ_i be an eigenvalue of* (2.5) *with λ_i in Λ_0. We can suppose $\lambda_{i+1} > \lambda_i > 0$ (or $\lambda_{i-1} < \lambda_i < 0$). Then, for every λ in $[\lambda_i - \varepsilon(\lambda_i), \lambda_{i+1}[\cap \Lambda_0$ (resp. in $]\lambda_{i-1}, \lambda_i + \varepsilon(\lambda_i)] \cap \Lambda_0$), there exist $\rho > 0$ and a true elastic bounce trajectory $\gamma_{\lambda,\lambda_i}$ such that*

(a) *if $\lambda_i = \lambda_{j+1} > \lambda_j > 0$ (resp. $\lambda_i = \lambda_{j-1} < \lambda_j < 0$) we have*

$$\begin{aligned} 0 \leq \sup Q_\lambda(\Delta(\lambda_j)) &\leq \sup Q_\lambda(\mathcal{D}_\infty \cap \mathbb{X}^-_{\lambda_i}) \\ &< Q_\lambda(\gamma_{\lambda,\lambda_i} - \gamma_{0,\lambda}) \leq \sup Q_\lambda(\Delta(\lambda_i)); \end{aligned} \tag{7.6}$$

(b) *if $\lambda_i = \lambda_0$ (resp. $\lambda_i = \lambda_{-1}$) then* (7.6) *holds with $\Delta(\lambda_j)$ replaced by*

$$\Delta^* := \{te^* \mid t \geq 0, te^* \in \mathcal{D}_\infty\}$$

where e^* is any nontrivial eigenvector with eigenvalue λ_i.

Proof. To prove the first claim we consider for example $\lambda_{i+1} > \lambda_i > 0$. Let λ be in $[\lambda_i - \varepsilon(\lambda_i), \lambda_{i+1}[\cap \Lambda_0$. We remind that $R(\lambda) = \sup g(\gamma_{0,\lambda} + \Delta(\lambda_i))$.

Step 1. Suppose that for every R in $]R(\lambda), R(\lambda)+1]$ there exists an elastic bounce trajectory γ_R such that $g(\gamma_R) = R$. Then, by Remark 2.4, there exists an elastic bounce trajectory $\overline{\gamma}$ such that $g(\overline{\gamma}) = R(\lambda)$, that is

$$Q_\lambda(\overline{\gamma} - \gamma_{0,\lambda}) = \sup Q_\lambda(\Delta(\lambda_i)).$$

Now, let $\rho := \rho(\lambda, R(\lambda))$, it follows $S_\rho \cap \Delta(\lambda_i) \neq \emptyset$, hence

$$\sup Q_\lambda(\Delta(\lambda_i)) = Q_\lambda(\overline{\gamma} - \gamma_{0,\lambda}) \geq \inf Q_\lambda(S_\rho(\lambda_i)) > \sup Q_\lambda(\mathcal{D}_\infty \cap \mathbb{X}^-_{\lambda_i})$$

by (7.4) in the case $\lambda \in [\lambda - \varepsilon(\lambda_i), \lambda_i]$, or by (b), (c) of Remark 7.1, if $\lambda_i \leq \lambda < \lambda_{i+1}$. If $\lambda \in [\lambda_i, \lambda_{i+1}[$. On the other hand, since

$$\Delta(\lambda_j) \subset \mathbb{X}^-_{\lambda_i} \cap \mathcal{D}_\infty, \text{ if } \lambda_i = \lambda_{j+1} > \lambda_j > 0 \quad (\Delta^* \subset \mathbb{X}^-_{\lambda_i} \cap \mathcal{D}_\infty, \text{ if } \lambda_i = \lambda_0),$$

we have

$$\begin{aligned} \sup Q_\lambda(\Delta(\lambda_j)) &\leq \sup Q_\lambda(\mathcal{D}_\infty \cap \mathbb{X}^-_{\lambda_i}), \\ (\sup Q_\lambda(\Delta^*) &\leq \sup Q_\lambda(\mathcal{D}_\infty \cap \mathbb{X}^-_{\lambda_i})) \end{aligned} \tag{7.7}$$

and the proof is over, in this case.

Step 2. From now on we can take R in $]R(\lambda), R(\lambda)+1]$ such that there are no elastic bounce trajectories γ with $g(\gamma) = R$. By Proposition 4.5 we can find an L^2 metric neighbourhood W of $\mathcal{D}_\infty$ such that for all ω $\widetilde{f}_{R,\omega}$ is lower semicontinuous and of class $C(p(\omega), q(\omega))$. Now we want to employ Theorem 3.8 in the case where $f_{R,\omega}$ play the role of f_n, f is $\widetilde{f}_{R,\infty}$, W_n are all equal to W, $\mathcal{D}$ is $\mathcal{D}_{R,\infty}$, and where $\Delta = \Delta_\sigma(\lambda_i)$, $\Sigma = \Sigma_\sigma(\lambda_i)$ with σ in $]0, \sigma(\lambda, R)]$, $S = S_{\rho(\lambda,R)}(\lambda_i)$ ($\sigma(\lambda, R)$ and $\rho(\lambda, R)$ were defined in Lemma 7.2). We have just proved that Assumption (A) of Section 3 is fulfilled.

Step 3. Up to shrinking σ we can suppose that

$$\overline{\Delta_\sigma(\lambda_i)} \subset W \cap (\mathbb{X}_R(A, B) - \gamma_{0,\lambda}) = \mathcal{D}(\widetilde{f}_{R,\omega}).$$

We claim that the linking assumpion (3.1) is verified. If $\lambda \in [\lambda_i, \lambda_{i+1}[$ this follows from (7.3); if $\lambda \in [\lambda_i - \varepsilon(\lambda_i), \lambda_i[$ this follows from (7.5).

Step 4. As in Theorem 3.8 we set

$$\begin{aligned} a &:= \liminf_{\omega\to\infty} \inf \widetilde{f}_{R,\omega}(S_\rho(\lambda_i) \cap W), \\ b &:= \limsup_{\omega\to\infty} b_\omega \quad \text{where } b_\omega := \sup \widetilde{f}_{R,\omega}(\Delta_\sigma(\lambda_i)). \end{aligned}$$

By Lemma 7.2 it turns out that

$$b = \sup Q_\lambda(\Delta(\lambda_i)), \qquad a \geq \inf Q_\lambda(S_\rho(\lambda_i)) \tag{7.8}$$

and setting (for instance) $a_\omega = 0$ for all ω, we have

$$a_\omega < \inf Q_\lambda(S_\rho(\lambda_i)) \le \inf \widetilde{f}_{R,\omega}(S_\rho(\lambda_i) \cap W) \quad \text{for all } \omega$$

by (c) of Remark 7.1 and (7.2). Using Remark 4.6

$$\overline{\widetilde{f}_{R,\omega}^{-1}([a_\omega, b_\omega])} \subset W \quad \text{for } \omega \text{ large enough.}$$

So also Assumption (3.2) of Theorem 3.8 holds.

Step 5. By Proposition 4.12 and the way R has been choosen, we derive that $\nabla((\widetilde{f}_{R,\omega})_\omega, \widetilde{f}_{R,\infty}, c)$ holds for every c in $\mathbb{R}$. Then we can apply Theorem 3.8 to obtain that there exists an asymptotically critical point δ such that $a \le \widetilde{f}_{R,\infty}(\delta) \le b$. By Theorem 4.10 $\gamma_{\lambda,\lambda_i} := \gamma_{0,\lambda} + \delta$ is an elastic bounce trajectory. By (7.4) we have $\sup Q_\lambda(\mathcal{D}_\infty \cap \mathbb{X}^-_{\lambda_i}) < a$, therefore by (7.7) we get (7.6).

Step 6. Finally we notice that $\gamma_{\lambda,\lambda_i}$ is a true bounce trajectory since $Q_\lambda(\gamma_{\lambda,\lambda_i} - \gamma_{0,\lambda}) > 0$. It is indeed trivial to see that any solution γ of the "free equation":

$$\ddot{\delta} + \lambda\beta(t)\delta = 0, \qquad \delta \in W_0^{1,2}(0,T;\mathbb{R}^N)$$

satisfies $Q_\lambda(\delta) = 0$. □

With the same arguments one can also prove the following result.

Lemma 7.5. *If $\lambda \in]\lambda_{-1}, \lambda_0[\cap \Lambda_0$ and $e^* \in W^{1,2}(0,T;\mathbb{R}^N)$, there exists an elastic bounce trajectory γ^*_λ such that*

$$0 < Q_\lambda(\gamma^*_\lambda - \gamma_{0,\lambda}) \le \sup Q_\lambda(\Delta^*) \tag{7.9}$$

where $\Delta^* = \{te^* \mid t \ge 0, te^* \in \mathcal{D}_\infty\}$.

Now we are in position to prove the first two statements of the main theorem.

Proof of (a) and (b) of Theorem 2.13. (a) Let $\lambda \in \Lambda_0$. If $\lambda \notin]\lambda_{-1}, \lambda_0]$, we can take λ_i such that either $0 < \lambda_i \le \lambda < \lambda_{i+1}$ or $\lambda_{i-1} < \lambda \le \lambda_i < 0$. Then the curve $\gamma_{\lambda,\lambda_i}$ found in Lemma 7.4 is a true elastic bounce trajectory. If $\lambda_{-1} < \lambda < \lambda_0$ the desired trajectory can be found using Lemma 7.5.

(b) We consider $\varepsilon = \varepsilon(\lambda_i)$ the positive number ε found in Lemma 7.4. Assume for instance that $\lambda_i > 0$. If $\lambda_i > \lambda_0$ let j be such that $\lambda_i = \lambda_{j+1} > \lambda_j > 0$: if we take $\lambda \in [\lambda_i - \varepsilon, \lambda_i[$ we can set $\gamma_\lambda := \gamma_{\lambda,\lambda_j}$ and $\eta_\lambda := \gamma_{\lambda,\lambda_j}$. Using (7.6), since $\lambda \in [\lambda_j, \lambda_{j+1}[$, we get

$$Q_\lambda(\gamma_\lambda - \gamma_{0,\lambda}) \le \sup Q_\lambda(\Delta(\lambda_j)) < Q_\lambda(\eta_\lambda - \gamma_{0,\lambda}),$$

so $\gamma_\lambda \ne \eta_\lambda$. In the case $\lambda_i = \lambda_0$ we set $\gamma_\lambda := \gamma^*_\lambda$ (as in Lemma 7.5, with e^* choosen to be any eigenfunction with eigenvalue λ_0) and $\eta_\lambda := \gamma_{\lambda,\lambda_0}$. Then by (7.6) and (7.9)

$$Q_\lambda(\gamma_\lambda - \gamma_{0,\lambda}) \le \sup Q_\lambda(\Delta^*) < Q_\lambda(\eta_\lambda - \gamma_{0,\lambda}). \qquad \square$$

Lemma 7.6. *Let λ_i be an eigenvalue of* (2.5) *and $\lambda_i \in \Lambda_0$. Assume that Ω is uniformly star-shaped with respect to γ_{0,λ_i}. Then there exists $\sigma = \sigma(\lambda_i) > 0$ such that for every λ in $]\lambda_i - \sigma, \lambda_i[\,\cap \Lambda_0$ if $\lambda_i > 0$ (for every λ in $]\lambda_i, \lambda_i + \sigma[\,\cap \Lambda_0$ if $\lambda_i < 0$) the following alternative holds:*

- *either there exists $\varepsilon > 0$ such that for every c with*

$$0 \le c - \sup Q_\lambda(\mathbb{X}^-_{\lambda_i} \cap \mathcal{D}_\infty \le \varepsilon$$

there is a true elastic bounce trajectory γ with $Q_\lambda(\gamma - \gamma_{0,\lambda}) = c$,

- *or there exist two distinct true bounce trajectories $\gamma_{1,\lambda,\lambda_i}$, $\gamma_{2,\lambda,\lambda_i}$ such that*

$$0 < Q_\lambda(\gamma_{h,\lambda,\lambda_i} - \gamma_{0,\lambda}) \le \sup Q_\lambda(\mathbb{X}^-_{\lambda_i} \cap \mathcal{D}_\infty) \le \sup Q_\lambda(\Delta(\lambda_i)) \tag{7.10}$$

for $h = 1, 2$ and for $\rho > 0$ small enough.

Proof. We consider, for instance, the case $\lambda_i > 0$ and we take j such that $\lambda_j < \lambda_{j+1} = \lambda_i$; we can also suppose $\lambda_i < \lambda_{i+1}$.

Step 1. Let $\varepsilon_0 > 0$ be the number provided by (b) of Lemma 7.7, relative to $\sigma_0 = (1/2)(\lambda_i - \lambda_j) \wedge (\lambda_{i+1} - \lambda_i)$. Since

$$\lim_{\lambda \to \lambda_i^-} \sup Q_\lambda(\mathbb{X}^-_{\lambda_i} \cap \mathcal{D}_\infty) = 0,$$

then there exists $\sigma = \sigma(\lambda_i)$ such that $\sigma_0 > \sigma > 0$ and for every λ in $[\lambda_i - \sigma, \lambda_i[$ one has

$$\sup Q_\lambda(\mathbb{X}^-_{\lambda_i} \cap \mathcal{D}_\infty) < \varepsilon_0.$$

We can also suppose that $\sigma(\lambda_i) \le \varepsilon(\lambda_i)$ ($\varepsilon(\lambda_i)$ was defined in the previous Lemma 7.3) and that Ω is uniformly star-shaped with respect to $\gamma_{0,\lambda}$, for all λ's in $[\lambda_i - \sigma, \lambda_i]$. From now on let λ be fixed in $]\lambda_i - \sigma, \lambda_i[$.

Step 2. If the first altenative doesn't hold we can find $\overline{c}$ in $[\sup Q_\lambda(\mathbb{X}^-_{\lambda_i} \cap \mathcal{D}_\infty), \varepsilon_0]$ such that there are no true elastic bounce trajectories γ with $g(\gamma) = g(\gamma_{0,\lambda}) + \overline{c}$. Since $\overline{c} > 0$ then there are also no free solutions γ with $g(\gamma) = g(\gamma_{0,\lambda}) + \overline{c}$, so there are no elastic bounce trajectories with such a property. Moreover, using Remark 2.4, we can suppose that $\overline{c} > \sup Q_\lambda(\mathbb{X}^-_{\lambda_i} \cap \mathcal{D}_\infty)$.

Step 3. Let

$$X_1 := \begin{cases} \mathbb{X}^-_{\lambda_j} & \text{if } \lambda_i > \lambda_0, \\ \{0\} & \text{if } \lambda_i = \lambda_0, \end{cases} \qquad X_2 := \operatorname{span}(e_h \mid \lambda_h = \lambda_i), \qquad X_3 := \overline{\mathbb{X}^+_{\lambda_i}}$$

(the closure being in $L^2(0,T;\mathbb{R}^N)$ as usual). We have that $L^2(0,T;\mathbb{R}^N) = X_1 \oplus X_2 \oplus X_3$, by (d) of Remark 7.1. Furthermore we set $R := g(\gamma_{0,\lambda}) + \overline{c}$

and consider the functionals $\widetilde{f}_{R,\omega}$ and $\widetilde{f}_{R,\infty}$. Given $\rho, \sigma > 0$ we set

$$\begin{aligned} \Delta :=& \{\delta \in \mathbb{X}^-_{\lambda_i} \mid \operatorname{dist}(\delta, \mathcal{D}_\infty) \le \sigma\}, \\ \Sigma :=& (\partial_{X^-_{\lambda_i}} \Delta) \cup (X_1 \cap \Delta), \\ S :=& \begin{cases} S_\rho(\lambda_j) & \text{if } \lambda_i > \lambda_0, \\ \{\delta \in L^2(0,T;\mathbb{R}^N) \mid \|\delta\| = \rho\} & \text{if } \lambda_i = \lambda_0. \end{cases} \end{aligned}$$

Step 4. We verify the assumptions of Theorem 5.5.

(a) By Proposition 4.5 every functional $\widetilde{f}_{R,\omega}$ is lower semicontinuous and of class $C(p(\omega), q(\omega))$ in a fixed L^2 metric neighbourhood W of $\mathcal{D}_\infty$.

(b) If σ is small enough we have $\sup g(\gamma_{0,\lambda} + \Delta) < R$, hence $\Delta \subset \mathcal{D}(\widetilde{f}_{R,\omega})$. Moreover, for ω large enough, $\sup \widetilde{f}_{R,\omega}(\Sigma) = 0$, because $\sup \widetilde{f}_{R,\omega}(\partial_{X^-_{\lambda_i}} \Delta) \to -\infty$ as $\omega \to \infty$ and $f_{R,\omega}(\delta) \le 0$ for δ in $X_1 \cap \Delta$. Furthermore, for $\rho > 0$ sufficently small we have $S \cap (X_1 \oplus X_2) \subset \operatorname{int}_{X_1 \oplus X_2}(\Delta)$ and $S \cap (\mathbb{X}_R(A,B) - \gamma_{0,\lambda}) \subset \mathcal{D}_\infty$. It follows that, for all ω,

$$\inf \widetilde{f}_{R,\omega}(S \cap W) = \inf Q_\lambda(S \cap (\mathbb{X}_R(A,B) - \gamma_{0,\lambda})) =: a \ge \inf Q_\lambda(S) > 0.$$

Then

$$\limsup_{\omega \to \infty} \sup \widetilde{f}_{R,\omega}(\Sigma) < \liminf_{\omega \to \infty} \inf \widetilde{f}_{R,\omega}(S \cap W) = a.$$

(c) We set $b_\omega := \sup \widetilde{f}_{R,\omega}(\Delta)$. It is clear that

$$\lim_{\omega \to \infty} b_\omega = \sup Q_\lambda(\mathcal{D}_\infty \cap \mathbb{X}^-_{\lambda_i}) =: b \in \mathbb{R}.$$

Moreover, setting $a_\omega := a/2$, it is straightforward that (see Remark 4.6)

$$\overline{\widetilde{f}^{-1}_{R,\omega}([a_\omega, b_\omega])} \subset W \quad \text{for } \omega \text{ large enough.}$$

(d) We show that $\mathcal{D}(\widetilde{f}_{R,\omega})$ and $X_1 \oplus X_3$ are not tangent. Notice that

$$\mathcal{D}(\widetilde{f}_{R,\omega}) = \{\delta \in W^{1,2}_0(0,T;\mathbb{R}^N) \mid Q_\lambda(\delta) \le \overline{c}\}$$

and that $\delta \in \mathcal{D}(f_{R,\omega})$, $w \in N_\delta(\mathcal{D}(\widetilde{f}_{R,\omega}))$ if and only if

$$\begin{cases} \text{there exists } \theta \ge 0 \text{ such that} & \\ \qquad \langle w, \delta_1 \rangle = \theta Q'(\delta)(\delta_1) \text{ for all } \delta_1 \text{ in } W^{1,2}_0(0,T;\mathbb{R}^N) & \text{if } Q_\lambda(\delta) = \overline{c}, \\ w = 0 & \text{if } Q_\lambda(\delta) < \overline{c}. \end{cases}$$

By contradiction, assume $\mathcal{D}(\widetilde{f}_{R,\omega})$ and $X_1 \oplus X_3$ to be tangent at some point δ in $\mathcal{D}(\widetilde{f}_{R,\omega}) \cap (X_1 \oplus X_3)$, that is there exists $w \ne 0$ such that $w \in N_\delta(\mathcal{D}(\widetilde{f}_{R,\omega}))$, $-w \in N_\delta(X_1 \oplus X_3)$. Then for a positive θ

$$0 = \langle -w, \delta \rangle = -\theta Q'_\lambda(\delta)(\delta) = -2\theta Q_\lambda(\delta) < 0$$

leading to a contradiction.

(e) By Proposition 4.12 $\nabla(\widetilde{f}_{R,\omega}, \widetilde{f}_{R,\infty}, c)$ holds for any real number c, since there are no elastic bounce trajectories γ with $g(\gamma) = R$. For what follows we show that there are no curves γ such that (6.2) holds, where $\gamma_0 = \gamma_{0,\lambda}$, $X = X_1 \oplus X_3$, with the condition $g(\gamma_{0,\lambda}) < g(\gamma) \leq R$. Equivalently we show that there exist no δ in $X_1 \oplus X_3$ such that (7.11) holds, for a suitable nonnegative Radon measure μ on $]0, T[$, and $0 < Q_\lambda(\delta) \leq \overline{c}$. Indeed for any such δ we would get $\delta = 0$, by (b) of Lemma 7.7, because $Q_\lambda(\delta) \leq \varepsilon_0$ since we chose $\overline{c} \leq \varepsilon_0$ in Step 2. But $Q_\lambda(\delta) > 0$, so we have a contradiction.

The property above implies that $\nabla(\widetilde{f}_{R,\omega}, \widetilde{f}_{R,\infty}, X_1 \oplus X_3, c)$ holds for any real number c, as a consequence of Proposition 6.2.

Step 5. Using again the arguments in (e) of the previous step we also derive that there are no $(X_1 \oplus X_3)$-constrained asymptotically critical points δ for $((\widetilde{f}_{R,\omega})_\omega, \widetilde{f}_{R,\infty})$ such that $0 < f_{R,\infty}(\delta) \leq \varepsilon_0$, by (a) of Proposition 6.2, since $0 < a < b \leq \overline{c}$.

Using Theorem 5.5 we find two distinct asymptotically critical points δ_1 and δ_2 such that $a \leq \widetilde{f}_{R,\infty}(\delta_i) \leq b$, for $i = 1, 2$. Letting $\gamma_{1,\lambda,\lambda_i} = \gamma_{0,\lambda} + \delta_1$ and $\gamma_{2,\lambda} = \gamma_{0,\lambda,\lambda_i} + \delta_2$ we obtain two elastic bounce trajectories verifying (7.6). □

Lemma 7.7. *Let λ_i be an eigenvalue of* (2.5) *and assume that $\lambda_{-1} \leq \lambda_j < \lambda_{j+1} \leq \lambda_i < \lambda_{i+1}$ in the case $\lambda_i > 0$ (resp. $\lambda_{i-1} < \lambda_i \leq \lambda_{j-1} < \lambda_j \leq \lambda_0$ in the case $\lambda_i < 0$). We set*

$$X := \begin{cases} \mathbb{X}^-_{\lambda_j} \oplus \mathbb{X}^+_{\lambda_i} & \text{if } \lambda_i\lambda_j > 0, \\ \mathbb{X}^+_{\lambda_i} & \text{if } \lambda_i\lambda_j < 0. \end{cases}$$

Moreover, let $\gamma_0 \in \mathbb{X}(A, B)$ and suppose

$$\gamma_0([0, T]) \subset \Omega, \quad \Omega \text{ uniformly star-shaped with respect to } \gamma_0.$$

Then for every $\sigma_0 > 0$ the following facts hold:

(a) *There exist $C_1, C_2 \geq 0$ such that for every λ in $[\lambda_j + \sigma_0, \lambda_{i+1} - \sigma_0]$, for every δ in X and for every Radon measure μ on $]0, T[$ such that*

$$(7.11) \quad \begin{cases} (\gamma_0 + \delta)([0, T]) \subset \overline{\Omega}, \quad \mu \geq 0, \\ \operatorname{spt}(\mu) \subset \{t \in]0, T[\mid (\gamma_0 + \delta)(t) \in \partial\Omega\}, \\ \displaystyle\int_0^T \dot{\delta}\dot{\eta}\, dt - \lambda \int_0^T \beta(t)\delta\eta\, dt + \int_{]0,T[} \nu(\gamma_0 + \delta)\eta\, d\mu = 0 \quad \text{for all } \eta \text{ in } X \end{cases}$$

the following inequalities hold:

$$(7.12) \quad \|\delta\|_{W^{1,2}} \leq C_1 \mu(]0, T[) \leq C_2 Q_\lambda(\delta);$$

(b) *There exists $\varepsilon_0 > 0$ such that for every λ in $[\lambda_j + \sigma_0, \lambda_{i+1} - \sigma_0]$ if $\lambda_i > 0$ (for every λ in $[\lambda_{i-1} + \sigma_0, \lambda_j - \sigma_0]$if $\lambda_i < 0$), for every δ in X such that* *there exists a Radon measure* μ *on* $]0,T[$ *verifying* (7.11) *one has*

$$Q_\lambda(\delta) \le \varepsilon_0 \Rightarrow \delta = 0.$$

Proof. We consider for instance $\lambda_i > 0$. Let

$$L_\lambda: W_0^{1,2}(0,T;\mathbb{R}^N) \to W_0^{1,2}(0,T;\mathbb{R}^N)$$

be the linear operator defined by

$$\langle L_\lambda \delta, \eta\rangle_W = \int_0^T \dot\delta\dot\eta\,dt - \lambda\int_0^T \beta(t)\delta\eta\,dt.$$

Clearly L_λ maps X into itself. Since λ is far away from λ_j and from λ_{i+1}, there exists a constant $K_1 > 0$ such that

$$\|L_\lambda\delta\|_W \ge K_1\|\delta\|_W \quad \text{for all } \delta \text{ in } X.$$

Conversely, it is clear that there exists another constant K_2 such that

$$\|L_\lambda\delta\|_W \le K_2\mu(]0,T[) \quad \text{for every } (\delta,\mu) \text{ in } X \text{ verifying (7.11)}.$$

Taking $\eta = \delta$ in (7.11) and using the fact that Ω is uniformly star-shaped with respect to γ_0 yields

$$2Q_\lambda(\delta) = -\int_{]0,T[} \nu(\gamma_0+\delta)\delta\,d\mu \ge \varepsilon\mu(]0,T[)$$

for a suitable $\varepsilon > 0$. Therefore (7.12) holds. To prove (b) just notice that, if $Q_\lambda(\delta) < (K_2\sqrt{T})^{-1}\operatorname{dist}(\gamma_0([0,T]),\partial\Omega)$ then $\|\delta\|_\infty < \operatorname{dist}(\gamma_0([0,T]),\partial\Omega)$, which in turn gives $(\gamma_0+\delta)([0,T]) \subset \Omega$, hence $\mu = 0$ and finally $\delta = 0$. □

Proof of (c) of Theorem 2.13. Let, for instance, $\lambda_i > 0$ and let $\lambda \in [\lambda_i - \sigma(\lambda_i), \lambda_i[$. By Lemma 7.4 we can find an elastic bounce trajectory $\eta_\lambda := \gamma_{\lambda,\lambda_i}$ such that, by (7.6),

$$\sup Q_\lambda(\mathbb{X}^-_{\lambda_i} \cap \mathcal{D}_\infty) < Q_\lambda(\eta_\lambda - \gamma_{0,\lambda}).$$

By Lemma 7.6 in both alternatives of its conclusion there exist two distinct elastic bounce trajectories, $\gamma_{1,\lambda,\lambda_i}$ and $\gamma_{2,\lambda,\lambda_i}$ such that

$$Q_\lambda(\gamma_{h,\lambda,\lambda_i} - \gamma_{0,\lambda}) < Q_\lambda(\eta_\lambda - \gamma_{0,\lambda}) \qquad h = 1,2$$

(if the first alternative occurs this is trivial, otherwise we use (7.10)). By Remark 2.12 λ_i is a transition value. The conclusion is thus proved. □

8. Appendix

In this section we recall briefly the properties of the Φ-convex functions which we used throughout the paper. For more details and for the proofs we refer the reader to [8], [5], [6], [17] and [13].

Let H be a Hilbert space with inner product $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|$. Let W be an open subset of H and let $f\colon W \to \mathbb{R} \cup \{\infty\}$ be a function. We define the *domain* of f as the set $\mathcal{D}(f) := \{u \in W \mid f(u) \in \mathbb{R}\}$. Moreover, for any real number c we use the standard notation $f^c := \{u \in \mathcal{D}(f) \mid f(u \leq c\}$.

Definition 8.1. Let $u \in \mathcal{D}(f)$. We introduce the *Frechét subdifferential* of f at u, denoted by $\partial^- f(u)$, as the set of all α's in H such that

$$\liminf_{v \to u} \frac{f(v) - f(u) - \langle \alpha, v - u \rangle}{\|v - u\|} \geq 0.$$

It is easy to see that $\partial^- f(u)$ is a closed convex subset of H (possibly empty). If $\partial^- f(u) \neq \emptyset$, we can define the *subgradient* of f at u, denoted by $\operatorname{grad}^- f(u)$, as the the element α_0 in $\partial^- f(u)$ such that $\|\alpha_0\| \leq \|\alpha\|$ for all α's in $\partial^- f(u)$.

We say that u in $\mathcal{D}(f)$ is a (lower) *critical point* for f, if $0 \in \partial^- f(u)$.

Definition 8.2. Let $\phi\colon \mathcal{D}(f)^2 \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function.

(a) We say that f is ϕ-convex if

$$f(v) \geq f(u) + \langle \alpha, v - u \rangle - \phi(u, v, f(u), f(v), \|\alpha\|)\|v - u\|^2 \tag{8.1}$$

for all u, v in $\mathcal{D}(f)$, for all α in $\partial^- f(u)$ (notice that the previous property holds true whenever $\partial^- f(u) = \emptyset$).

(b) Let r be a nonnegative number. We say that f is ϕ-convex of order r, if it is ϕ-convex and there exists a continuous function $\phi_0\colon \mathcal{D}(f)^2 \times \mathbb{R}^2 \to \mathbb{R}$ such that

$$\phi(u, v, f(u), f(v), \|\alpha\|) \leq \phi_0(u, v, f(u), f(v))(1 + \|\alpha\|^r)$$

for all u, v in $\mathcal{D}(f)$, for all α in $\partial^- f(u)$.

(c) Let $p, q\colon \mathcal{D}(f) \to \mathbb{R}$ be two continuous functions. We say that f is of class $C(p, q)$ if f is ϕ convex and

$$\phi(u, v, f(u), f(v), \|\alpha\|) \leq p(u)\|\alpha\| + q(u)$$

for all u, v in $\mathcal{D}(f)$, for all α in $\partial^- f(u)$.

Definition 8.3. Let E be a subset of H. We define the indicator function of E, $I_E\colon H \to \mathbb{R} \cup \{\infty\}$, by

$$I_E(u) := \begin{cases} 0 & \text{if } u \in E, \\ \infty & \text{if } u \notin E. \end{cases}$$

If $u \in E$ we define the *normal cone* to E at u, denoted by $N_u(E)$, by $N_u(E) := \partial^- I_E(u)$. An element ν in $N_u(E)$ will be called a *normal* to E at u.

With the above definitions we study a function f on a constraint E by studying the *constrained function* $f + I_E$. In particular the critical points of $f + I_E$ will be called critical points for f on E.

As an example it is not difficult to see that, if $h: \mathbb{R}^N \to \mathbb{R}$ is a convex function, $g: \mathbb{R}^N \to \mathbb{R}$ is a $\mathcal{C}^2$ function, and M is a $\mathcal{C}^2$ submanifold of $\mathbb{R}^N$, then $f := h + g + I_M$ is of class $C(p, q)$, for suitable p and q.

Now we give an account of two fundamental theorems concerning ϕ-convex functions which are quite relevant in our paper.

Theorem 8.4 (curves of maximal slope). *Let $f: W \to \mathbb{R} \cup \{\infty\}$ be lower semicontinuous and ϕ-convex of order two.*

(a) *For every u in $\mathcal{D}(f)$ there exist $T > 0$ and a unique curve $\mathcal{U}: [0, T[\to \mathcal{D}(f)$ such that $\mathcal{U}(0) = u$ and*

$$\text{(8.2)} \quad \begin{cases} \mathcal{U} \text{ and } f \circ \mathcal{U} \text{ are absolutely continuous in } [0,T[\text{ and locally Lipschitz} \\ \text{continuous in }]0,T[\text{, moreover, if } t \in]0,T[\ \partial^- f(\mathcal{U}(t)) \neq \emptyset \text{ and} \\ \mathcal{U}'_+(t) = -\operatorname{grad}^- f(\mathcal{U}(t)), \ (f \circ \mathcal{U})'_+(t) = -\|\operatorname{grad}^- f(\mathcal{U}(t))\|^2. \end{cases}$$

We call $\mathcal{U}$ a curve of maximal slope for f starting from u.

(b) *Given u_0 in $\mathcal{D}(f)$ and $c \geq f(u_0)$, there exist $\rho > 0$ and $T > 0$ such that for every u in $f^c \cap B(u_0, \rho)$ the curve of maximal slope $\mathcal{U}$ starting from u is defined on $[0, T]$. If we denote by $\Phi(u)$ such a curve, then, letting $u \to \overline{u}$ in $f^c \cap B(u_0, \rho)$, we have that $\Phi(u)$ converges to $\Phi(\overline{u})$ uniformly on $[0, T]$, while $f \circ \Phi(u)$ converges to $f \circ \Phi(\overline{u})$ uniformly on any compact subinterval of $]0, T]$.*

The following remark related to the maximal interval of existence is easy to prove.

Remark 8.5. Let $\mathcal{U}: [0, T[\to \mathcal{D}(f)$ be a curve of maximal slope for f (i.e. let (8.2) be verified for $\mathcal{U}$). If $T < \infty$ and $\inf_{0 \leq t < T} f \circ \mathcal{U}(t) > -\infty$, then there exists $\lim_{t \to T^-} \mathcal{U}(t)$.

The following *Deformation Lemmas* were used in the proof of the multiplicity Theorem 3.5. They can be easily obtained from Lemma 8.4 and Remark 8.5, using standard arguments along with the assumption on W.

Lemma 8.6(First Deformation Lemma). *Let $f: W \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuos ϕ-convex function of order two and let a, b be two real numbers such that $a < b$,*

$$\text{(8.3)} \qquad \overline{f^{-1}([a, b])} \subset W$$

and $\inf\{\|\alpha\| \mid \alpha \in \partial^- f(u),\ a \le f(u) \le b\} > 0$. *Then* f^a *is a strong deformation retract of* f^b *in* f^b.

Lemma 8.7 (Second Deformation Lemma). *Let* $f: W \to \mathbb{R} \cup \{\infty\}$ *be a lower semicontinuos* ϕ*-convex function of order two. Let* c', c'', *and* c *be real numbers such that* $c' \le c \le c''$ *and*

$$\overline{f^{-1}([c', c''])} \subset W. \tag{8.4}$$

Moreover, let F_1 *and* F_2 *be two closed subsets of* H *such that*

$$\begin{aligned} &\sigma := \inf\{\|\alpha\| \mid \alpha \in \partial^- f(u), u \in f^{-1}([c', c'']) \cap F_2\} > 0, \\ &\rho := \operatorname{dist}(F_1, H \setminus F_2) > 0. \end{aligned} \tag{8.5}$$

Then, for every $\varepsilon', \varepsilon''$ *such that* $0 \le \varepsilon' < (\rho\sigma/2)$, $0 \le \varepsilon'' < (\rho\sigma/2)$ *and* $c' \le c - \varepsilon' \le c + \varepsilon'' \le c''$, *the set* $f^{c-\varepsilon'}$ *is a strong deformation retract of* $(f^{c+\varepsilon''} \cap F_1) \cup f^{c-\varepsilon'}$ *in* $f^{c+\varepsilon''}$.

In view of the proof of Lemma 5.2, we recall now a result about contrained functions. For the proof we refer the reader to [6] and to [13]. We first need a definition.

Definition 8.8. Let V_1 and V_2 be two subsets of H and let $u \in V_1 \cap V_2$. We say that V_1 and V_2 are (*externally*) *tangent* at u if

$$N_u(V_1) \cap (-N_u(V_2)) \ne \{0\}.$$

Theorem 8.9. *Let* $f: W \to \mathbb{R} \cup \{\infty\}$ *be a lower semicontinuous function of class* $C(p, q)$, *for two suitable functions* p *and* q. *Let* M *be a* $\mathcal{C}^2$ *submanifold of* H *with finite codimension (possibly with boundary). If* $\mathcal{D}(f)$ *and* M *are not tangent at any* u *in* $\mathcal{D}(f) \cap M$, *then*

$$\partial^-(f + I_M)(u) = \partial^- f(u) + N_u(M) \quad \text{for all } u \text{ in } \mathcal{D}(f) \cap M.$$

Moreover, $f + I_M$ *is of class* $C(\overline{p}, \overline{q})$ *for suitable* $\overline{p}, \overline{q}: \mathcal{D}(f) \to \mathbb{R}$.

While proving Lemma 5.2 we used the following result.

Theorem 8.10. *Let* $f: W \to \mathbb{R} \cup \{\infty\}$ *be a lower semicontinuous function of class* $C(p, q)$. *Then* $I_{\mathcal{D}(f)}$ *is of class* $C(p, 0)$, *that is*

$$\langle \nu, v - u \rangle \le p(u) \|\nu\| \|v - u\|^2 \tag{8.6}$$

for all u, v *in* $\mathcal{D}(f)$ *and all* ν *in* $N_u(\mathcal{D}(f))$.

Proof. Let $u \in \mathcal{D}(f)$ and $\nu \in N_u(\mathcal{D}(f))$.

Step 1. We claim that there exist a strictly increasing sequence $(n_k)_k$ in $\mathbb{N}$ and sequences $(u_k)_k$ in $\mathcal{D}(f)$, $(\alpha_k)_k$ in H such that

$$u_k \to u, \quad \alpha_k \to \nu, \quad k\alpha_k \in \partial^- f(u_k) \quad \text{for all } k.$$

If not, defining $g_n\colon W \to \mathbb{R}\cup\{\infty\}$, by $g_n(v) := (1/n)f(v) - \langle \nu, v-u\rangle$, there would exist $\overline{n}$ in $\mathbb{N}$, $R, \sigma > 0$ such that $\inf f(B(u,R)) > -\infty$ and

$$n \geq \overline{n}, \quad v \in B(u,R), \quad \alpha \in \partial^- g_n(v) \quad \Rightarrow \quad \|\alpha\| \geq \sigma.$$

It follows that for all $n \geq \overline{n}$ and for all $\rho < R$ there exists $u_{n,\rho}$ such that

$$\|u_{n,\rho} - u\| = \rho, \qquad g_n(u_{n,\rho}) \leq g_n(u) - \sigma\rho.$$

Indeed let us denote by $\mathcal{U}_n$ the curve of maximal slope for g_n starting from u. If t is such that $\mathcal{U}(\tau) \in B(u,R)$ for all τ in $[0,t]$, then

$$g_n(\mathcal{U}_n(t)) - g_n(u) \leq -\int_0^t \|\mathcal{U}_n'(\tau)\|^2\,d\tau \leq \begin{cases} -\sigma^2 t, \\ -\sigma\|\mathcal{U}(t) - u\|. \end{cases}$$

Since g_n is bounded below in $B(u,R)$, it follows that there exists t_n such that $\|\mathcal{U}_n(t_n) - u\| = \rho$. As a consequence $g_n(\mathcal{U}_n(t_n)) \leq g_n(u) - \rho\sigma$. Then $u_{n,\rho} := \mathcal{U}(t_n)$ is the desired point. In particular

$$\frac{f(u_{\rho,n}) - f(u)}{n} + \rho\sigma \leq \langle \nu, u_{n,\rho} - u\rangle.$$

So for n large

$$\frac{\sigma}{2}\|u - u_{n,\rho}\| = \frac{\rho\sigma}{2} \leq \langle \nu, u_{n,\rho} - u\rangle.$$

This contradicts the fact that $\nu \in N_u(\mathcal{D}(f))$, i.e. $\langle \nu, v-u\rangle \leq o(\|v-u\|)$ for v in $\mathcal{D}(f)$.

Step 2. Since f is of class $C(p,q)$ we get that for all v in $\mathcal{D}(f)$:

$$f(v) \geq f(u_k) + \langle n_k\alpha_k, v - u_k\rangle - (n_k\|\alpha_k\|p(u_k) + q(u_k))\,\|v - u_k\|^2.$$

Dividing by n_k and letting $k \to \infty$ gives (8.6). □

References

[1] T. Bartsch and M. Clapp, *Critical point theory for indefinite functionals with symmetries*, J. Funct. Anal. **138** (1996), 107–136.

[2] V. Benci and F. Giannoni, *Periodic bounce trajectories with a low number of bounce points*, Ann. Inst. H. Poincaré **6** (1989), 73–93.

[3] G. Buttazzo and D. Percivale, *On the approximation of the elastic bounc problem on riemanian manifolds*, J. Differential Equations **47** (1983), 227–245.

[4] M. Carriero, A. Leaci and E. Pascali, *Convergenza per l'equazione degli integrali primi associati al problema del rimbalzo unidimensionale*, Ann. Mat. Pura Appl. **133** (1983), 227–256.

[5] G. Chobanov, A. Marino and D. Scolozzi, *Evolution equation for the eigenvalue problem for the Laplace operator with respect to an obstacle*, Rend. Accad. Naz. Sci. XL Mem. Mat. **14** (1990), no. 5, 139–162.

[6] ———, *Multiplicity of eigenvalues for the Laplace operator with respect to an obstacle, and nontangency conditions*, Nonlinear Anal. **15** (1990), no. 3, 199–215.

[7] M. DEGIOVANNI, *Multiplicity of solutions for the bounce problem*, J. Differential Equations **54** (1984), 414–428.

[8] M. DEGIOVANNI, A. MARINO AND M. TOSQUES, *Evolution Equations with Lack of Convexity*, Nonlinear Anal. **9** (1985), 1401–1443.

[9] G. FOURNIER, D. LUPO, M. RAMOS AND M. WILLEM, *Limit relative category and critical point theory*, Dynamics Reported, Springer, Berlin, 1994, pp. 1–24.

[10] F. GIANNONI, *Bounce trajectories with one bounce point*, Ann. Mat. Pura Appl. **CLIX** (1991), 101–115.

[11] A. MARINO AND D. MUGNAI, *Asymptotical multiplicity and some reversed variational inequalities*, Topol. Methods Nonlinear Anal. **17** (2002), 43–62.

[12] ———, *Asymptotically critical points and their multiplicity*, Topol. Methods Nonlinear Anal. **20** (2002), 29–38.

[13] A. MARINO AND C. SACCON, *Some elements of subdifferential analysis and some eigenvalue problems for eigenvalue problems*, Autumn School: Variational Inequalities, S. Charles University of Prague, 1993, pp. 1–26.

[14] ———, *Some variational theorems of mixed type and elliptic problems with jumping nonlinearities*, Ann. Scuola Norm. Sup. Pisa **XXV** (1997), 631–665.

[15] ———, *Nabla theorems and multiple solutions for some noncooperative elliptic systems*, Topol. Methods Nonlinear Anal. **17** (2001), 213–237.

[16] ———, *Asymptotically critical points and multiple solutions in the elastic bounce problem*, Variational Analysis and Applications (F.Giannessi and A. Maugeri, eds.), vol. 79, Springer, 2005, pp. 651–663.

[17] A. MARINO AND M. TOSQUES, *Some variational problems with lack of convexity and some partial differential inequalities*, Method of Nonconvex Analysis, Lecture Notes in Math., vol. 1446, Springer, Berlin, 1990, pp. 58–83.

[18] L. PENROSE AND R. PENROSE, *Puzzles for Christmas*, New Scientist, 1958, pp. 1580–1581, 1597.

[19] D. PERCIVALE, *Uniqueness in elastic bounce problems*, J. Differential Equations **54** (1985), 1984.

[20] J. RAUCH, *Illumination of Bounded Domains*, Amer. Math. Monthly (1978), 359–361.

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