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CRITICAL POINTS OF NON-C^2 FUNCTIONALS

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Abstract. We establish flows on normed spaces. Applying it we extend the results of Gromoll, Meyer, Morse and Palais for non-C^2 functionals.

1. Introduction

In [16] Palais proved the Morse–Palais lemma for C^3 functions. This result was extended for C^2 functions by Kuiper in [11] (see also [15]). Recently Li, Li and Liu [14] obtained a version of Morse–Palais lemma without the C^2-smoothness. But the functions studied in [14] are of class C^2 in some sense (see [14, p. 440]). In the present paper we get the Morse–Palais lemma, which does not request the C^2-smoothness of functions nor the completeness of the spaces (see Theorem 1.1). Our Morse–Palais lemmas are applicable to the following function

$$J(x,y)=\begin{cases} x^2-y^2+\dfrac{1}{40}(x^2+y^2)^5\sin\dfrac{1}{(x^2+y^2)^2} & \text{for all } (x,y)\in\mathbb{R}^2\setminus(0,0),\\ 0 & (x,y)=(0,0).\end{cases}$$

This example illustrates our idea: the shapes of the graphs of $g(x,y)=x^2-y^2$ and its perturbed function J are similar near $(0,0)$ even if the perturbed part $J-g$ is not in C^2 but its second derivatives in any direction are controlled in some sense (see condition (e) in Theorem 1.1). We note that the results in [11],

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[14] can not be applied to this case (see also Remark 1.4 for infinite-dimensional cases).

Since the considered functionals are not C^2-smooth, we can not use the Implicit Mapping Theorem to define the mapping φ in Theorem 1.1, and we apply the method in [5], [11], [15]: using flows corresponding to considered functionals to define φ.

In this paper we establish flows on normed spaces corresponding to non-smooth functionals (see Theorem 2.1). We obtain the flows without the completeness of the spaces, which is essential in [5], [11], [15]. Our ideas are as follows: we reduce the problems on deformations into finite-dimensional problems and we can relax the completeness of the spaces and we only need very weak smoothness of the functional (see Definition 3.1 and Theorem 3.5). This reduction is very successful in studying Lagrange multipliers (see [2]).

Using the deformation results, we relax some conditions on compactness and smoothness of Mountain-Pass Theorem and Gromoll–Meyer theorem (see Theorems 3.10, 1.3 and the example in Remark 1.4. Applications of these results to resonance problems will be appeared elsewhere. Our version of Morse–Palais lemma is as follows.

Theorem 1.1 (Morse–Palais Lemma). *Let H be a vector space with a norm $||\cdot||_H$ defined by an inner product $\langle\,\cdot\,,\,\cdot\,\rangle$, O be an open subset of H, J be a twice H-differentiable real function on O (see Definition 3.1). Let a in O be an isolated critical point of J. Assume that there exist a Hermite bounded linear operator A on H and positive real numbers α, Γ, δ and θ such that:*

(a) $D^2J(a)(u,v) = \langle Au, v\rangle$ *for all* $u, v \in H$,
(b) $\Gamma||x|| \geq ||A(x)|| \geq \alpha||x||$ *for all* $x \in H$,
(c) $\theta < \min\{\alpha/2, \alpha^2/\Gamma\}$,
(d) *for any h in H, the map $x \mapsto DJ(x)h$ is continuous on O,*
(e) $||D^2J(z)(z-a,h) - D^2J(a)(z-a,h)|| < \theta||z-a||||h||$ *for any* $h \in H\setminus\{0\}$, $z \in B_H(a,\delta)$.

Then there exist two closed vector subspaces E and F of H, two open neighbourhoods U and W of a and 0 in H respectively and an isomorphism φ from W onto U such that $H = E \oplus F$, $\varphi(0) = a$ and

$$J(\varphi(y+z)) = \langle y, y\rangle - \langle z, z\rangle + J(a) \quad \text{for all } y \in E,\ z \in F,\ y+z \in W.$$

Remark 1.2. The conditions (a)–(e) hold when J is of class C^2, H is a Hilbert space and A is invertible.

Theorem 1.3 (Gromoll–Meyer Splitting Theorem). *Let H be a Hilbert space, and J be a twice H-differentiable real function on H (see Definition 3.1. Let a in H be an isolated critical point of J. Assume that $D^2J(a)$ is a Hermite bounded* *linear operator A on H; A and DJ are of class S_+ on H. Then there are positive real numbers C, α and Γ, a closed vector subspace H^+, two finite-dimensional vector subspaces H^0 and H^- of H such that $H^- \oplus H^0 \oplus H^+$ is a orthogonally direct decomposition of H, $H^0 = \ker A$,*

$$\begin{aligned} \langle Ax, x\rangle &\geq C||x||^2 && \text{for all } x \in H^+, \\ \langle Ax, x\rangle &\leq -C||x||^2 && \text{for all } x \in H^-, \\ \Gamma||y|| \geq ||A(y)|| &\geq \alpha||y|| && \text{for all } y \in Y \equiv H^+ \oplus H^-. \end{aligned}$$

Assume further that there exists positive real numbers δ and θ such that:

(a) $\theta < \min\{\alpha/2, \alpha^2/\Gamma\}$,
(b) *the map $x \mapsto DJ(x)h$ is continuous on O,*
(c) $||D^2J(x)(x-a+z,h) - D^2J(a)(x-a+z,h)|| < \theta||x-a+z||||h||$ *for all $x \in B(a,\delta)$, $z \in Z \equiv H^0$, $h \in H \setminus \{0\}$,*
(d) $\langle DJ(z+x_1+y_1) - DJ(z+x_2+y_2), (x_1-x_2)-(y_1-y_2)\rangle > 0$ *for all $x_1, x_2 \in H^+$, $y_1, y_2 \in H^-$ and $x_1+y_1 \neq x_2+y_2 \in B'_Y(0,\delta/2)$, $z \in B'_Z(0,\delta/2)$.*

Then there exist a continuous mapping ψ from $B_Z(0,\delta/2)$ into $B_Y(0,\delta/2)$ and two open neighbourhoods U and W of 0 in H, and an isomorphism φ from W onto U such that $\varphi(0)=0$, $DJ(z+\psi(z))|_Y = 0$ and

$$\begin{aligned} J(z+\psi(z)) &= \min\{J(z+Q\psi(z)+x) : x \in H^+,\ Q\psi(z)+x \in B_Y(0,\delta_1)\}, \\ J(z+\psi(z)) &= \max\{J(z+P\psi(z)+t) : t \in H^-,\ P\psi(z)+t \in B_Y(0,\delta_1)\}, \\ J(\varphi(y+z)) &= \frac{1}{2}\langle Ay, y\rangle + J(z+\psi(z)), \end{aligned}$$

for any $y \in H^+ \oplus H^-$, $z \in H^0$, $y+z \in U$, where P and Q are defined in Definition 5.7.

REMARK 1.4. This theorem is proved in [5] if J is C^2 and A is a compact vector field. In the following example we relax this smoothness. Let $\Omega = B(0,1)$ be the open unit ball in $\mathbb{R}^N$, $N \geq 3$, and H be the Sobolev space $W_0^{1,2}(\Omega)$ with the norm

$$||u|| = \left\{\int_\Omega |\nabla u|^2\right\}^{1/2}.$$

Put

$$\rho(x) = -(1-||x||)^{-2} \quad \text{for all } x \in \Omega.$$

By Poincaré inequality, there is a positive real number C such that

$$\left|\int_\Omega \rho w^2\,dx\right| \leq C||w||^2 \quad \text{for all } w \in H. \tag{1.1}$$

Let γ be a real number, ε be a positive real number and k be a function of class $C^2(\mathbb{R},\mathbb{R})$ such that:

$$k(t) = \begin{cases} \dfrac{1}{6}t^3 & \text{for all } t \in (-1,1),\\ \dfrac{1}{2}t|t| & \text{for all } t \in (-\infty,-2)\cup(2,\infty). \end{cases}$$

Note that $k''(t) = t$ for any t in $[-1,1]$ and there is a real positive real number M such that $|k''(s)| \le M$ for any s in $\mathbb{R}$. Put

$$J(u) = \int_\Omega \left[\frac{1}{2}|\nabla u(x)|^2 + \gamma u(x)^2 + \varepsilon\rho(x)k(u(x))\right] dx \quad \text{for all } u \in H.$$

By (1.1) we see that J is H-differentiable on H and for any u, v and w in H,

$$\begin{aligned} DJ(u)(v) &= \int_\Omega [\nabla u(x)\nabla v(x) + \gamma u(x)v(x) + \varepsilon\rho(x)k'(u(x))v(x)]\,dx,\\ D^2J(u)(v,w) &= \int_\Omega [\nabla v(x)\nabla w(x) + \gamma v(x)w(x) + \varepsilon\rho(x)k''(u(x))v(x)w(x)]\,dx,\\ |D^2J(u)(v,w) - D^2J(0)(v,w)| &= \varepsilon\left|\int_\Omega [\rho(x)k''(u(x))v(x)w(x)]\,dx\right|\\ &\le \varepsilon M||v||\,||w||. \end{aligned}$$

Hence J satisfies the conditions (a)–(c) of Theorem 1.3 for $a = 0$ if ε is sufficiently small. Now let H^0, H^- and H^+ be as in Theorem 1.3, u be in H^0, v_1 and v_2 be in H^+, and w_1 and w_2 be in H^-. We have

$$\begin{aligned} &[DJ(u+v_1+w_1) - DJ(u+v_2+w_2)][(v_1-v_2)-(w_1-w_2)]\\ &\quad\ge \int_\Omega [|\nabla(v_1-v_2)|^2 + \gamma(v_1-v_2)^2]\,dx - 2\varepsilon M\int_\Omega \rho(v_1-v_2)^2\,dx\\ &\qquad - \int_\Omega [|\nabla(w_1-w_2)|^2 + \gamma(w_1-w_2)^2]\,dx - 2\varepsilon M\int_\Omega \rho(w_1-w_2)^2\,dx. \end{aligned}$$

It implies that J satisfies the condition (d) of Theorem 1.3 for $a = 0$ if ε is sufficiently small. We shall prove that D^2J is not continuous at 0. Put $a_i = (1-2^{-i},0,\ldots,0)$ and $r_i = 2^{-i-2}$ for any positive integer i. Choose ψ in $C_c^\infty(\mathbb{R}^N)$ such that

$$\psi(x) = \begin{cases} 1 & \text{if } |x| < 1,\\ \in [0,1] & \text{if } 1 \le |x| \le 3/2,\\ 0 & \text{if } |x| > 3/2. \end{cases}$$

Note that $\{\int_\Omega |\psi|^2\,dx\}^{1/2} = \gamma_0 > 0$ and $\{\int_{\mathbb{R}^N} |\nabla\psi|^2\,dx\}^{1/2} = \gamma_1 > 0$. We define

$$\begin{aligned} \phi_i(x) &= r_i^{1-N/2}\psi\left(\frac{x-a_i}{r_i}\right),\\ \psi_i(x) &= \psi\left(\frac{x-a_i}{r_i}\right) \qquad \text{for all } x \in \mathbb{R}^N,\ i \in \mathbb{N}. \end{aligned}$$

We see that ϕ_i and ψ_i are in $W_0^{1,2}(\Omega)$, $||\psi_i|| = r_i^{N/2-1}\gamma_1$ and $||\phi_i|| = \gamma_1$ for any integer i. Therefore $\{\psi_i\}$ converges to 0 in $W_0^{1,2}(\Omega)$. On the other hand we have

$$\begin{aligned}|D^2J(\psi_i)(\phi_i,\phi_i) - D^2J(0)(\phi_i,\phi_i)| &= \varepsilon\int_\Omega (1-||x||)^{-2}\psi_i\phi_i^2\,dx\\ \geq \varepsilon\int_{B(a_i,r_i)}(1-||x||)^{-2}\phi_i^2\,dx &\geq \varepsilon\frac{1}{25}\int_{B(a_i,r_i)} r_i^{-2}\phi_i^2\,dx = \varepsilon\bigg(\frac{\gamma_0}{5}\bigg)^2 > 0.\end{aligned}$$

Therefore $\{D^2J(\psi_i)\}$ does not converge to $D^2J(0)$ and the functional J is not of class C^2 on $W_0^{1,2}(\Omega)$.

This remark show that our results can be applied to very strongly singular elliptic equations without compactness. In [7] we can study the buckling of thick shells with such relaxation of compactness.

2. Flows

In this section we study flows in normed spaces. Our results do not require the completeness of the spaces as follows.

Theorem 2.1. *Let $\widetilde{E}$ be the completion of a normed space $(E, ||\cdot||)$ and W be an open set of $\widetilde{E}$ with boundary ∂W. For any positive real number s, we put*

$$W_s = \{x \in W : \delta(x,\partial W) \equiv \inf\{||x-y|| : y \in \partial W\} > s\}.$$

Let g be a mapping from W into E such that $g(W_s)$ is bounded for every s. Assume for any v in W, there exist a finite-dimensional vector subspace E_v of E and two positive real numbers r_v and L_v such that

$$\begin{aligned}||g(x)-g(y)|| &\leq L_v||x-y|| \quad \text{for all } x,y \in B'_{\widetilde{E}}(v,r_v),\\ g(x) &\in E_v \quad\quad\quad\quad \text{for all } x \in B'_{\widetilde{E}}(v,r_v).\end{aligned} \tag{2.1}$$

Then, for any b in $W\cap E$, we have

(a) *There exist $t_b \in (0,\infty]$ and a unique continuous map ν_b from $[0,t_b)$ into $W\cap E$ such that*

$$\begin{cases}\nu_b'(t) = g(\nu_b(t)) & \text{for all } t \in (0,t_b),\\ \nu_b(0) = b.\end{cases}$$

Furthermore, if t_b is finite then $\inf_{t\in[0,t_b)}\delta(\nu_b(t),\partial W) = 0.$

(b) *For any $s \in (0,t_b)$, there exists a positive real number β having the following property: for any $x \in B(b,\beta)\cap W\cap E$, there is a unique continuous map ν_x from $[0,s]$ into $W\cap E$ such that*

$$\begin{cases}\nu_x'(t) = g(\nu_x(t)) & \text{for all } t \in (0,s),\\ \nu_x(0) = x.\end{cases}$$

To prove (a) of the theorem we need the following lemmas, whose proofs are standard and omitted.

Lemma 2.2. *Let F be a finite-dimensional vector subspace of a normed space $(E, \|\cdot\|)$, U be an open subset of E and g be a mapping from U into E. Assume there exist a in F and positive real numbers r_a, L_a and M_a such that $B'(a, r_a) \subset U$ and*

(g1) $||g(x) - g(y)|| \leq L_a||x - y||$ *for all* $x, y \in B'(a, r_a)$,
(g2) $||g(x)|| \leq M_a$, $x \in B'(a, r_a)$
(g3) $g(x) \in F$ *for all* $x \in B'(a, r_a)$.

Put $t_a = \min\{1/(2L_a), r_a/M_a\}$. Then there is a unique continuous mapping ν_a from $[0, t_a]$ into $B'_F(a, r_a)$ such that:

$$\begin{cases} \nu_a\,'(t) = g(\nu_a\,(t)) & \textit{for all } t \in (0, t_a), \\ \nu_a\,(0) = a\,. \end{cases}$$

Lemma 2.3. *Let V be an open set in a Banach space H and g be a mapping from V into H. Assume there exist a in V and positive real numbers r_a, L_a and M_a such that $B'(a, r_a) \subset V$ and g satisfies* (g1) *and* (g2). *Put $t_a = \min\{1/(2L_a), r_a/M_a\}$. Then there is a unique continuous mapping ν_a from $[0, t_a]$ into $B'(a, r_a)$ such that*

$$\begin{cases} \nu_a\,'(t) = g(\nu_a\,(t)) & \textit{for all } t \in (0, t_a), \\ \nu_a\,(0) = a\,. \end{cases}$$

Proof of Theorem 2.1. (a) Fix b in $W \cap E$. Put T_b be the set of all positive real number s for which there is a solution ν_s in $C([0, s), W \cap E)$ to the following Cauchy problem:

$$\begin{cases} \nu_s'(t) = g(\nu_b(t)) & \text{for all } t \in (0, s), \\ \nu_s(0) = b. \end{cases}$$

Applying Lemma 2.2 for $U = W \cap E$, $a = b$, $r_a = r_b/2$ and the finite dimensional vector subspace F generated by $E_b \cup \{b\}$ we see that T_b is not empty. Let α and β in T_b such that $\alpha < \beta$, we shall prove

$$\nu_\alpha(t) = \nu_\beta(t) \quad \text{for all } t \in (0, \alpha). \tag{2.2}$$

Assume by contradiction that (2.2) is false. Denote by S the set $\{t \in (0, \alpha) : \nu_\alpha(t) \neq \nu_\beta(t)\}$. Then $S \neq \emptyset$. Put $t_0 \equiv \inf S$. We show

$$\nu_\alpha(t_0) = \nu_\beta(t_0) \tag{2.3}$$

If $t_0 = 0$, we have (2.3) since $\nu_\alpha(t_0) = b = \nu_\beta(t_0)$.

Consider the case $t_0 > 0$. We have $\nu_\alpha(t) = \nu_\beta(t)$ for all $t \in (0, t_0)$. Since ν_α and ν_β are continuous on $(0, t_0]$, we get (2.3).

Put $c = \nu_\alpha(t_0) = \nu_\beta(t_0)$. By the continuity of ν_α and ν_β there are two positive real numbers t_1 and t_2 such that

$$\nu_\alpha(t_0 + t) \in B'_{\widetilde{E}}(c, r_c) \quad \text{for all } t \in (0, t_1), \tag{2.4}$$

$$\nu_\beta(t_0 + t) \in B'_{\widetilde{E}}(c, r_c) \quad \text{for all } t \in (0, t_2). \tag{2.5}$$

Applying Lemma 2.3 for $H = \widetilde{E}$, $V = W$ and $a = c$, we have a positive real number s_c and a unique mapping from $[0, t_c)$ into $B'_{\widetilde{E}}(c, s_c)$ such that

$$\begin{cases} \nu_c'(t) = g(\nu_c(t)) \quad \text{for all } t \in (0, t_c), \\ \nu_c(0) = c. \end{cases} \tag{2.6}$$

Put $t_3 = \min\{t_1, t_2, t_c\}$. Combining (2.4)–(2.6) we get

$$\nu_\alpha(t_0 + t) = \nu_\beta(t_0 + t) \quad \text{for all } t \in (0, t_3).$$

It implies that $[0, t_0 + t_3]$ is contained in $[0, \infty) \setminus S$, which is a contradiction since $\inf S = t_0 < t_0 + t_3$. Thus (2.2) holds. If T_b is unbounded from above, (a) of Theorem 2.1 is obtained with $t_b = \infty$.

Now suppose that $t_b \equiv \sup T_b < \infty$, we shall prove $\inf\{\delta(\nu_b(t), \partial W) : t \in [0, t_b)\} = 0$. Assume by contradiction

$$\inf\{\delta(\nu_b(t), \partial W) : t \in [0, t_b)\} > d > 0. \tag{2.7}$$

In this case $\nu_b([0, t_b)) \subset W_d$. By the hypothesis of the theorem, there is a positive real number M such that $g(W_d)$ is contained in $B'(0, M)$ and

$$||\nu_b(t) - \nu_b(s)|| = \left\| \int_t^s g(\nu_b(\xi))\, d\xi \right\| \le \left| \int_t^s ||g(\nu_b(\xi))||\, d\xi \right| \le M|t - s|$$

for all $t, s \in [0, t_b)$. Thus ν_b is uniformly continuous on $[0, t_b)$ and there exists $c = \lim_{t \to t_b} \nu_b(t)$ in $\widetilde{E}$. By (2.7) we see that c is in W. Applying Lemma 2.3 as above, we get a positive number t_c as in Lemma 2.3. We can find t_0 in $(t_b - t_c/4, t_b)$ such that $||\nu_b(t_0) - c|| < r_c/2$. Thus $B'_{\widetilde{E}}(\nu_b(t_0), r_c/2) \subset B'_{\widetilde{E}}(c, r_c)$ and

$$\begin{aligned} ||g(x) - g(y)|| &\le L_c ||x - y|| && \text{for all } x, y \in B'_{\widetilde{E}}(\nu_b(t_0), r_c/2), \\ ||g(x)|| &\le M_c && \text{for all } x \in B'_{\widetilde{E}}(\nu_b(t_0), r_c/2), \\ g(x) &\in E_c && \text{for all } x \in B'_{\widetilde{E}}(\nu_b(t_0), r_c/2), \end{aligned}$$

where L_c, M_c and E_c are appeared in hypotheses of the theorem.

Put $d = \nu_b(t_0)$, $t_4 = \min\{1/(2L_c), r_c/(2M_c)\}$. Then $t_4 \ge t_c/2$, and by Lemma 2.2 there is a unique mapping ν_d from $[0, t_4)$ into $B'_{E_c}(\nu_b(t_0), r_c/2)$ such that

$$\begin{cases} \nu_d'(t) = g(\nu_d(t)) \quad \text{for all } t \in (0, t_4), \\ \nu_d(0) = \nu_b(t_0). \end{cases}$$

Put

$$h(t) = \begin{cases} \nu_b(t) & \text{for all } t \in (0, t_0), \\ \nu_d(t - t_0) & \text{for all } t \in [t_0, t_0 + t_4). \end{cases}$$

We have

$$\begin{cases} h'(t) = g(h(t)) & \text{for all } t \in (0, t_0 + t_4), \\ h(0) = b. \end{cases}$$

Thus $t_0 + t_4$ is in T_b. Since $t_0 \in (t_b - t_c/4, t_b)$ and $t_4 \geq t_c/2$, we have

$$t_b = \sup T_b \geq t_0 + t_4 > t_b - \frac{t_c}{4} + \frac{t_c}{2} = t_b + \frac{t_c}{4} > t_b,$$

which is a contradiction and completes the proof of (a) of Theorem 2.1.

(b) Since $\nu_b([0, s])$ is a compact subset of $W \cap E$, by (a) of the theorem, there are two positive real numbers L and γ such that $\nu_b([0, s]) + B(0, \gamma) \subset W \cap E$ and

$$||g(x) - g(y)|| \leq L||x - y|| \quad \text{for all } x, y \in \nu_b([0, s]) + B(0, \gamma).$$

Fix z in $\nu_b([0, s]) + B(0, \gamma)$. By (a) of the theorem, there is a unique mapping ν_z from $[0, t_z)$ into $W \cap E$ such that

$$\begin{cases} \nu_z'(t) = g(\nu_z(t)) & \text{for all } t \in (0, t_z), \\ \nu_z(0) = z. \end{cases}$$

Furthermore, by Lemma 2.3, there is a positive real number ε such that

$$t_z > 3\varepsilon \quad \text{for all } z \in \nu_b([0, s]) + B(0, \gamma). \tag{2.8}$$

Put $\beta = \gamma e^{-Ls}/2$. Assume there is x in $B(b, \beta) \cap W \cap E$ such that $t_x < s$. Define

$$S = \{t \in (0, t_x) : \nu_x(\xi) \in \nu_b([0, s]) + B(0, \gamma) \text{ for any } \xi \in [0, t]\}, \quad s_1 = \sup S.$$

We have

$$\begin{aligned} ||\nu_x(\zeta) - \nu_b(\zeta)|| &\leq ||x - b|| + \int_0^\zeta ||g(\nu_x(\zeta)) - g(\nu_b(\zeta))||\, d\xi \\ &\leq ||x - b|| + \int_0^\zeta L||\nu_x(\zeta) - \nu_b(\zeta)||\, d\xi \quad \text{for all } \zeta \in [0, s_1). \end{aligned}$$

Using the Gronwall inequality (see [12]) we have

$$||\nu_x(\zeta) - \nu_b(\zeta)|| \leq ||x - b|| e^{L\zeta} \leq \beta e^{Ls} = \frac{1}{2}\gamma \quad \text{for all } \zeta \in [0, s_1). \tag{2.9}$$

If $s_1 < t_x$, $\nu_x(s_1)$ is defined and belongs to $\nu_b([0, s]) + B'(0, \gamma/2)$, therefore by above argument s_1 is not $\sup S$. Thus $s_1 = t_x$.

Choose a positive real number s_2 in $(t_x - \varepsilon, t_x)$ and put $z = \nu_x(s_2)$. We see that z contained in $\nu_b([0, s]) + B(0, \gamma/2)$. By (2.8), $t_z > 3\varepsilon$. Thus ν_x can be defined on $[0, t_x + 2\varepsilon)$. This is the contradiction and $t_x > s$ for any x in $B(b, \beta) \cap W \cap E$. □

The following lemmas will be used in the next section.

LEMMA 2.4. *Assume all hypotheses of Theorem* 2.1 *hold. Let b be in $W \cap E$, and s be in $(0, t_b)$. Then there is a finite-dimensional vector subspace F_s of E such that $\nu_b([0, s]) \subset F_s$, where ν_b is defined as in* (a) *of Theorem* 2.1.

PROOF. Since ν_b is continuous from $[0, s]$ into $W \cap E$ we have $\nu_b([0, s])$ is compact in E. On the other hand:

$$\nu_b([0, s]) \subset \bigcup_{a \in \nu_b([0,s])} B_E(a, r_a).$$

Thus there is a finite subset A of $\nu_b([0, s])$ such that

$$\nu_b([0, s]) \subset \bigcup_{a \in A} B_E(a, r_a).$$

Let F_s be the vector subspace generated by $\{b\} \cup \{\bigcup_{a \in A} g(B_{\widetilde{E}}(a, r_a))\}$. By (2.1) we have F_s is a finite dimensional subspace of E and

$$\nu_b(t) = b + \int_0^t g(\nu_b(s))ds \in F_s \quad \text{for all } t \in [0, s].$$

This completes the proof. □

LEMMA 2.5. *Assume all hypotheses of Theorem* 2.1 *hold and $g(W)$ is contained in a vector space V of E. Let b be in $W \cap E$, and s be in $(0, t_b)$. Then there is a finite-dimensional vector subspace V_s of V such that $\nu_b([0, s])$ is contained in $b + V_s$, where ν_b is defined as in* (a) *of Theorem* 2.1.

PROOF. By Lemma 2.4, $\nu_b([0, s])$ and $g(\nu_b([0, s]))$ are contained in a finite-dimensional vector subspace F_s of E. On the other hand, since $g(W)$ is contained in V, this implies $g(\nu_b([0, s]))$ is contained in $F_s \cap V$. Moreover, since $F_s \cap V$ is finitely dimensional then

$$\nu_b(t) = b + \int_0^t g(\nu_b(\xi))\, d\xi \in b + V \cap F_s \quad \text{for all } t \in [0, s].$$

Put $V_s = F_s \cap V$, we have $\nu_b[0, s] \subset b + V_s$. □

3. Deformation theorem

In this section we prove a deformation theorem for continuously V-differentiable functionals (see Definition 3.1). We do not need the completeness of the space, which is essential conditions in [1], [5], [6], [17]. First we need the following definitions.

In this section, we assume that J is a continuous mapping.

Definition 3.1. Let V be a non-trivial vector subspace of a normed space $(E, \|\cdot\|_E)$ and J be a mapping from an open subset U of E into a normed space $(F, \|\cdot\|_F)$. We say

(a) J is *V-differentiable* on U if and only if for any x in U there is a linear mapping $DJ(x)$ from V into F such that

$$DJ(x)(v) = \lim_{t \to 0} \frac{J(x+tv) - J(x)}{t} \quad \text{for all } v \in V.$$

(b) J is *continuously V-differentiable* on U if and only if J is V-differentiable on U and

$$\lim_{y \to x} DJ(y)(v) = DJ(x)(v) \quad \text{for all } (x, v) \in E \times V.$$

(c) J is *twice V-differentiable* on U if and only if J is V-differentiable on U, $DJ(x)$ is a continuous operator on E for any x in U and DJ is a V-differentiable mappings from U into $L(E, F)$, where $L(E, F)$ is the usual normed space of bounded linear mapping from E into F.

Now we introduce a notation on $||DJ(x)||$.

Definition 3.2. Let V be a non-trivial vector subspace of a normed space $(E, \|\cdot\|)$, $\widetilde{E}$ be the completion of E and J be a continuously V-differentiable function on E. We put

$$||DJ(x)|| = \sup_{h \in V, ||h||=1} \lim_{r \to 0} \inf\{|DJ(y)h| : y \in B(x, r) \cap E\} \quad \text{for all } x \in \widetilde{E},$$

where $||DJ(x)||$ may be ∞.

If x belongs to E, our notation $||DJ(x)||$ just is the classical one. Now we introduce some other notations for V-differentiable functions.

Definition 3.3. Let $\widetilde{E}$ be the completion of a normed space $(E, \|\cdot\|)$, V be a non-trivial vector subspace of E, J be a continuously V-differentiable mapping from E into $\mathbb{R}$ and c be a real number. Put $U(J) = \{x \in \widetilde{E}$: there is a sequence $\{x_m\}$ in E such that $\{x_m\}$ converges to x and $\{J(x_m)\}$ converges in $\mathbb{R}\}$, $J^*(y) = \{\alpha \in \mathbb{R}$: there is a sequence $\{y_m\}$ in E converges to y and $\{J(y_m)\}$ converges to α in $\mathbb{R}\}$ for any y in $U(J)$ and

$$K_c = \{y \in U(J) : c \in J^*(y) \text{ and } ||DJ(y)|| = 0\}.$$

If $K_c \neq \emptyset$, we say c is a generalized critical value of J, and each vector u in K_c is called *generalized critical point* of J.

If y belongs to E, $J^*(y) = \{J(y)\}$. So our notation $J^*(y)$ is similar to the classical one.

The notation of generalized critical points has been introduced in [6], where a version of mountain-pass theorem has been proved and applied to solve a problem of Yamabe. Now we have the following definition on Palais–Smale condition.

Definition 3.4 (Palais–Smale condition). Let V be a non-trivial vector subspace of a normed space $(E, \|\cdot\|)$, $\widetilde{E}$ be the completion of E, J be a real continuously V-differentiable function on E and $\{x_m\}$ be a sequence in $U(J)$. We say

(a) $\{x_m\}$ is a $(\mathrm{PS})_c$-sequence if $\lim_{m\to\infty} ||DJ(x_m)|| = 0$ and there is a real sequence $\{\alpha_m\}$ such that $\alpha_m \in J^*(x_m)$ for any integer m and $\{\alpha_m\}$ converges to c.
(b) J satisfies $(\mathrm{PS})_c$ condition if and only if every $(\mathrm{PS})_c$-sequence has a subsequence converging in $\widetilde{E}$.

We have the following result.

Theorem 3.5 (Deformation Theorem). *Let V be a non-trivial vector subspace of a normed space $(E, \|\cdot\|)$, $\widetilde{E}$ be the completion of E, J be a real continuously V-differentiable function on E, and a and b be two real numbers such that $a < b$. Assume:*

(a) *J satisfies $(\mathrm{PS})_c$ condition for any c in $[a, b]$,*
(b) *$K_\alpha = \emptyset$ for any α in (a, b),*
(c) *K_a, $K_b \subset E$,*
(d) *every connected subset T of K_a has at most one element.*

Then J_a is a deformation retract of $J_b \setminus K_b$, where $J_\alpha = J^{-1}((-\infty, \alpha])$ for any α in $\mathbb{R}$.

In order to prove this theorem we need the following lemmas.

Lemma 3.6. *Let E, $\widetilde{E}$ and V be as in Theorem 3.5. Let J be a real continuously V-differentiable function on E and x be in $\widetilde{E}$. Then two propositions are equivalent:*

(a) $||DJ(x)|| = 0$.
(b) *For any vector h in V there is a sequence $\{x_n\}$ in E such that $\{x_n\}$ converges to x in $\widetilde{E}$ and $\lim_{n\to\infty} DJ(x_n)(h) = 0$.*

Proof. First, we proof (a) implies (b). Assume $||DJ(x)|| = 0$. Fix a vector h in V, by the definition of $||DJ(x)||$ we have

$$\lim_{r\to 0} \inf\{|DJ(y)h| : y \in B(x, r) \cap E\} = 0.$$

On the other hand, for every positive integer m, there is x_m in the set $B(x, 1/m) \cap E$ such that

$$|DJ(x_m)(h)| \le \inf\left\{|DJ(y)h| : y \in B\left(x, \frac{1}{m}\right) \cap E\right\} + \frac{1}{m} \quad \text{for all } m \in \mathbb{N}.$$

It is clear that the sequence $\{x_m\}$ converges to x in $\widetilde{E}$ and

$$\lim_{m\to\infty} DJ(x_m)(h) = 0.$$

Now assume (b) holds, that is for any vector h in V there is a sequence $\{x_n\}$ in E such that converges to x in $\widetilde{E}$ and $\lim_{n\to\infty} DJ(x_n)(h) = 0$. We have

$$\inf\{|DJ(y)h| : y \in B(x,r) \cap E\} = 0 \quad \text{for all } r > 0,\ h \in V.$$

This implies $||DJ(x)|| = 0$. □

LEMMA 3.7. *Let E, $\widetilde{E}$, V and J be as in Theorem* 3.5 *and $\{x_m\}$ be a sequence converging to x in $\widetilde{E}$. Then*

$$||DJ(x)|| \le \liminf_{m\to\infty} ||DJ(x_m)||.$$

PROOF. Fix a vector h in V such that $||h|| = 1$. By definition we have

$$\lim_{r\to 0}\{\inf\{|DJ(y)h| : y \in B(x_n,r) \cap E\}\} \le ||DJ(x_n)|| \quad \text{for all } n \in \mathbb{N}.$$

It implies that

$$\inf\{|DJ(y)h| : y \in B(x_n,1/n) \cap E\} \le ||DJ(x_n)|| \quad \text{for all } n \in \mathbb{N}.$$

Thus there is z_n in $B(x_n,1/n) \cap E$ such that

$$|DJ(z_n)h| \le ||DJ(x_n)|| + \frac{1}{n}.$$

We have $\{x_n\}$ converges to x and $||x_n - z_n|| \le 1/n$. This implies $\{z_n\}$ converges to x and

$$\begin{aligned}\lim_{r\to 0}\inf\{|DJ(y)h| : y \in B(x,r) \cap E\} &\le \liminf_{n\to\infty} |DJ(z_n)h)| \\ &\le \liminf_{n\to\infty}\left[||DJ(x_n)|| + \frac{1}{n}\right] = \liminf_{m\to\infty} ||DJ(x_n)||.\end{aligned}$$

It implies

$$||DJ(x)|| = \sup_{h\in V,||h||=1} \lim_{r\to 0}\{\inf\{|DJ(y)h| : y \in B(x,r)\cap E\}\} \le \liminf_{m\to\infty} ||DJ(x_n)||.$$

□

LEMMA 3.8. *Let E, $\widetilde{E}$ and V be as in Theorem* 3.5, *y be in $\widetilde{E}$ and J be a real continuously V-differentiable function on E. Assume $||DJ(y)|| \in (0,\infty]$. Then there exist h_y in V and a positive real number r_y such that:*

$$\begin{aligned}\|h_y\|\cdot\|DJ(y)\| &< 2 &&\text{if } ||DJ(y)|| < \infty,\\ ||h_y|| &= 1 &&\text{if } ||DJ(y)|| = \infty,\\ DJ(x)h_y &< -1 &&\text{for all } x \in B_{\widetilde{E}}(y,r_y) \cap E.\end{aligned}$$

PROOF. By definition there exist two positive real numbers r_y, s and a vector h in V with $||h|| = 1$ such that

- If $||DJ(y)|| < \infty$: $||DJ(y)|| < s$ and $|DJ(x)h| > s/2$ for all $x \in B_{\widetilde{E}}(y, r_y) \cap E$.
- If $||DJ(y)|| = \infty$: $s = 2$ and $|DJ(x)h| > s/2$ for all $x \in B_{\widetilde{E}}(y, r_y) \cap E$.

Since the map $x \mapsto DJ(x)h$ is continuous on the connected set $B_{\widetilde{E}}(y, r_y) \cap E$, we can (and shall) suppose $DJ(x)h < -s/2$ for any x in $B_{\widetilde{E}}(y, r_y) \cap E$. Putting $h_y = 2h/s$ we get the lemma. □

LEMMA 3.9. *Let E, $\widetilde{E}$, V and J be as in Theorem 3.5. Assume J satisfies the* (PS)*-condition. The K_c is a compact set for any real number c.*

PROOF. By Definitions 3.3 and 3.4 we get the lemma. □

PROOF OF THEOREM 3.5. Denote by $\overline{J^{-1}([a,b])}$ the closure of $J^{-1}([a,b])$ in $\widetilde{E}$. Put

$$A = \overline{J^{-1}([a,b])} \setminus \{K_a \cup K_b\}.$$

Fix x in A. By Lemma 3.9 we can choose h_x and r_x as in Lemma 3.8 such that $B_{\widetilde{E}}(x, r_x) \cap \{K_a \cup K_b\} = \emptyset$. We have $A \subset \bigcup_{x \in A} B_{\widetilde{E}}(x, r_x)$. By the Dugunji theorem there is a subset B of A such that

$$A \subset W \equiv \bigcup_{x \in B} B_{\widetilde{E}}(x, r_x).$$

So $W \cap \{K_a \cup K_b\} = \emptyset$ and for any a in W there is a positive real number s having the following property: the set $\{x \in B : B_{\widetilde{E}}(x, r_x) \cap B_{\widetilde{E}}(a, s) \neq \emptyset\}$ is finite. Put

$$g(y) = \sum_{x \in B} \Bigl(\sum_{x \in B} \varphi_x(y)\Bigr)^{-1} \varphi_x(y) h_x \quad \text{for all } y \in W,$$

where $\varphi_x(y) = \delta(y, \widetilde{E} \setminus B_{\widetilde{E}}(x, r_x)) \equiv \inf\{||y - z|| : z \in \widetilde{E} \setminus B_{\widetilde{E}}\}$, and h_x is as in Lemma 3.8. By the choosing of B the mapping g is well defined. We prove that g satisfies the assumptions of Theorem 2.1. First we consider the boundedness of g on $W_\alpha \equiv \{y \in W : \delta(y, \partial W) > \alpha\}$ for any positive real number α. Put

$$T_\alpha = \{x \in B : B_{\widetilde{E}}(x, r_x) \cap W_\alpha \neq \emptyset\}.$$

We have

$$g(y) = \sum_{x \in B} \Bigl(\sum_{x \in B} \varphi_x(y)\Bigr)^{-1} \varphi_x(y) h_x = \sum_{x \in T_\alpha} \Bigl(\sum_{x \in T_\alpha} \varphi_x(y)\Bigr)^{-1} \varphi_x(y) h_x$$

for all $y \in W_\alpha$. We shall show that $\delta(x, \partial W) > \alpha/2$ for any x in T_α. Assume by contradiction that there is x in T_α and z in ∂W such that $||x - z|| \leq \alpha/2$. Hence

$r_x \leq \alpha/2$ since $B_{\widetilde{E}}(x, r_x)$ is contained in W. On the other hand

$$||y-x|| \geq ||y-z|| - ||z-x|| \geq \alpha - r_x \geq \frac{\alpha}{2} \geq r_x \quad \text{for all } y \in W_\alpha.$$

This implies $B_{\widetilde{E}}(x, r_x) \cap W_\alpha = \emptyset$, which contradicts to the definition of T_α. Thus $\delta(x, \partial W) > \alpha/2$ for all $x \in T_\alpha$ or $T_\alpha \subset W_{\alpha/2}$.

Now we show that $\inf\{||DJ(x)|| : x \in T_\alpha\} > 0$. Indeed assume by contradiction that there is a sequence $\{x_m\}$ in T_α such that $\lim_{m\to\infty} ||DJ(x_m)|| = 0$. Note that $T_\alpha \subset B \subset \overline{J^{-1}([a,b])}$. Since J satisfies the $(\mathrm{PS})_c$ condition for any c in $[a,b]$, there is a subsequence $\{x_{m_k}\}$ of $\{x_m\}$ such that $\{x_{m_k}\}$ converges to y in $\widetilde{E}$.

According to Lemma 3.7 we see that

$$\lim_{m\to\infty} J(x_{m_k}) = \eta \in [a,b], \qquad \lim_{m\to\infty} ||DJ(x_{m_k})|| = ||DJ(y)|| = 0.$$

This implies y in K_η. By the hypotheses (b) of the theorem we see that $\eta \in \{a,b\}$ and $y \in \overline{W} \cap \{K_a \cup K_b\}$. Since $W \cap \{K_a \cup K_b\} = \emptyset$ and $T_\alpha \subset W_{\alpha/2} \subset W$, we have $y \in \overline{W_{\alpha/2}} \cap \partial W$, which is absurd because $\overline{W_{\alpha/2}} \cap \partial W = \emptyset$. Thus there is a positive real number β_α such that

$$||DJ(x)|| \geq \beta_\alpha \quad \text{for all } x \in T_\alpha.$$

By Lemma 3.8, $||h_x|| \leq \max\{1, 2\beta_\alpha^{-1}\}$ for any x in T_α. Thus

$$||g(y)|| \leq \sum_{x\in T_\alpha} \bigg(\sum_{x\in T_\alpha} \varphi_x(y)\bigg)^{-1} \varphi_x(y)||h_x|| \leq \max\{1, 2\beta_\alpha^{-1}\}$$

for all $y \in W_\alpha$. By Lemma 3.8 we have

$$DJ(y)(g(y)) < -1 \quad \text{for all } y \in W \cap E.$$

Since g satisfies all the hypotheses of Theorem 2.1, we have for any x in $W \cap E$, there is t_x in $(0,\infty]$ and a unique continuous mapping ν_x from $[0,t_x)$ into $W \cap E$ such that

$$\begin{cases} \nu_x'(t) = g(\nu_x(t)) & \text{for all } t \in (0,t_x), \\ \nu_x(0) = x. \end{cases}$$

Recall that $\inf\{\delta(\nu_x(t), \partial W) : t \in [0,t_x)\} = 0$ if t_x is finite. Fix z in $A \cap E$. By Lemma 2.5 and the continuously V-differentiability of J, $Jo\nu_z$ is differentiable on $(0,t_z)$ and

$$\begin{aligned} J(\nu_z(t)) - J(\nu_z(0)) &= \int_0^t DJ(\nu_z(s))(\nu_z'(s))\,ds \\ &= \int_0^t DJ(\nu_z(s))(g(\nu_z(s))\,ds < -t \end{aligned} \tag{3.1}$$

for all $t \in (0,t_z)$. Thus for any z in $J^{-1}([a,b]) \setminus (K_a \cup K_b)$, there is a biggest s_z in $(0, b-a\ )$ such that $\nu_z([0,s_z)) \subset A \cap E$.

We complete the proof of the theorem by the following steps.

Step 1. Fix z in $J^{-1}([a,b]) \setminus (K_a \cup K_b)$, we show that ν_z can be continuously extended at s_z, $s_z < t_z$ and $J(\nu_z(s_z)) = a$.

We consider two cases:

Case 1. Assume $\inf\{\delta(\nu_z(t), K_a) : t \in [0, s_z)\} = s_1 > 0$. By (3.1)

$$J(\nu_z(t)) < b - \frac{s_z}{2} \quad \text{for all } t \in [s_z/2, s_z).$$

Using the compactness of $K_b \subset E$, we have

$$\inf\{\delta(\nu_z(t), K_b) : t \in [s_z/2, s_z)\} = s_2 > 0.$$

Put $S_\alpha = \{x \in A : \delta(x, K_a \cup K_b) \geq \alpha\}$ for any positive real number α. Arguing as in the proof of boundedness of $g(W_\alpha)$, we find a positive real number M_α such that

$$||g(y)|| \leq M_\alpha \quad \text{for all } y \in S_\alpha. \tag{3.2}$$

Put $s = \min(s_1, s_2)$, we have $\nu_z([s_z/2, s_z)) \subset S_s$. It implies that

$$||\nu_z(t) - \nu_z(t')|| = \left\| \int_t^{t'} g(\nu_z(\xi))\, d\xi \right\| \leq M_s|t - t'|$$

for all $t, t' \in (s_z/2, s_z)$. Thus $\lim_{t \to s_z} \nu_z(t) = y \in \widetilde{E}$. On the other hand since $\nu_z([s_z/2, s_z))$ is contained in a closed set $\overline{J^{-1}([a,b])} \cap (\widetilde{E} \setminus \bigcup_{x \in K_a \cup K_b} B_{\widetilde{E}}(x, s))$, we have

$$y \in \overline{J^{-1}([a,b])} \cap \left(\widetilde{E} \setminus \bigcup_{x \in K_a} B_{\widetilde{E}}(x, s)\right) \subset W.$$

Thus $\delta(y, \partial W) > 0$, and by Theorem 2.1, y is in E. By (3.1) and the definition of s_z we see that $s_z < t_z$ and $J(y)$ should be a.

Case 2. Assume

$$\inf_{t \in [0, s_z)} \delta(\nu_z(t), K_a) = 0. \tag{3.3}$$

We shall prove

$$\lim_{t \to s_z} \delta(\nu_z(t), K_a) = 0. \tag{3.4}$$

Assume by contradiction that there are a positive real number ε and a sequence $\{t_n\}$ converging to s_z in $[0, s_z)$ such that

$$\delta(\nu_z(t_n), K_a) > 2\varepsilon > 0. \tag{3.5}$$

By (3.3) there is a sequence $\{s_n\}$ converging to s_z in $[0, s_z)$ such that

$$\lim \delta(\nu_z(s_n), K_a) = 0. \tag{3.6}$$

By (3.5), (3.6) and continuity of the mappings ν_z and δ, there are two sequence $\{\alpha_n\}$ and $\{\beta_n\}$ converging to s_z in $[0, s_z)$ such that

$$\begin{cases} \delta(\nu_z(\alpha_n), K_a) = 2\varepsilon, \\ \delta(\nu_z(\beta_n), K_a) = \varepsilon, \\ \delta(\nu_z(t), K_a) \in [\varepsilon, 2\varepsilon] \quad \text{for all } t \in [\alpha_n, \beta_n]. \end{cases} \tag{3.7}$$

We may assume $\alpha_n > s_z/2$ for any n in $\mathbb{N}$. Put $C = \bigcup_{n=1}^{\infty} \nu_z[\alpha_n, \beta_n]$. We can see that $C \subset S_{\varepsilon'}$ with $\varepsilon' = \min(s_2, \varepsilon)$. By (3.2),

$$||g(y)|| \leq M_{\varepsilon'} \quad \text{for all } y \in C.$$

Thus we have

$$||\nu_z(\alpha_n) - \nu_z(\beta_n)|| = \left\| \int_{\alpha_n}^{\beta_n} g(\nu_z(t))\, dt \right\| \leq M_{\varepsilon'} |\beta_n - \alpha_n|.$$

Since $\{\alpha_n\}$ and $\{\beta_n\}$ converge to s_z, $\{||\nu_z(\alpha_n) - \nu_z(\beta_n)||\}$ converges to 0, which contradicts to (3.7). Therefore $\lim_{t\to s_z} \delta(\nu_z(t), K_a) = 0$.

Put $T = \overline{\nu_z[0, s_z)} \cap K_a$. Since K_a is compact then T is compact. We prove T is connected. Indeed, suppose by contradiction that there are two non-empty disjoint compact subsets Q and R of E such that $Q \cup R = T$.

Therefore, there exist two disjoint open subsets U and U' in E such that $Q \subset U$ and $R \subset U'$. Let p and q be in $U \cap T$ and $U' \cap T$, respectively. There are two sequence $\{t_n\}$ and $\{s_n\}$ converging to s_z in $(0, s_z)$ such that the sequences $\{\nu_z(t_n)\}$ and $\{\nu_z(s_n)\}$ converge to p and q, respectively. We can suppose $t_n < s_n$ for any integer n.

By the continuity of ν_z, there is $k_n \in (t_n, s_n)$ such that

$$\nu_z(k_n) \notin U \cup U' \quad \text{for all } n \in \mathbb{N}.$$

By (3.4) we have $\lim_{n\to\infty} \delta(\nu_z(k_n), K_a) = 0$. Since K_a is compact, there is a subsequence of $\{\nu_z(z_{n_k})\}$ converging to an element of K_a. But $\nu_z(k_{n_k}) \notin U \cup U'$, which is an open set containing T. Thus we get a contradiction and T should be connected. By (d) the set T has a unique element v. Define $\nu_z(s_z) = v$, then ν_z may be continuously extended at s_z and $J(\nu_z(s_z)) = a$.

Step 2. We prove the map $z \mapsto s_z$ is continuous on $A \cap E$.

Assume by contradiction that there are a positive real number ε and a sequence $\{x_n\}$ converging to x in $A \cap E$ such that $|s_{x_n} - s_x| > \varepsilon$ for any integer n. Applying (b) of Theorem 2.1, we may assume:

$$s_{x_n} > s_x + \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

Note that, for all $t \in (s_x + \varepsilon/2, s_{x_n})$,

$$a < J(\nu_{x_n}(t)) = J(\nu_{x_n}(s_x)) + \int_{s_x}^{t} DJ(\nu_{x_n}(s))(g(\nu_{x_n}(s))\, ds$$
$$\leq J(\nu_{x_n}(s_x)) - (t - s_x) < J(\nu_{x_n}(s_x)) - \frac{\varepsilon}{2}.$$

By the continuous dependence to the initial values of the Cauchy problem, we have $\{\nu_{x_n}(s_x)\}$ converges to $\nu_x(s_x)$. Thus $\{J(\nu_{x_n}(s_x))\}$ converges to $J(\nu_x(s_x)) = a$, which contradicts to above inequality.

Step 3. Let $\{x_n\}$ be a sequence in $A \cap E$ such that $\{x_n\}$ converges to x in K_a. We shall prove s_{x_n} converges to 0. By the continuity of J, we have

$$\lim_{n\to\infty} J(\nu_{x_n}(0)) = \lim_{n\to\infty} J(x_n) = J(x) = a \ = J(\nu_{x_n}(s_{x_n})).$$

On the other hand

$$J(\nu_{x_n}(s_{x_n})) - J(\nu_{x_n}(0)) = \int_0^{s_{x_n}} DJ(\nu_{x_n}(s))(g(\nu_{x_n}(s)))\, ds < -s_{x_n} < 0.$$

Thus we get the third step.

Step 4. Let $\{x_n\}$ be a sequence in $A \cap E$ such that $\{x_n\}$ converges to x in K_a. We may assume $J(x_n) < a + 2^{-1}(b - a)$ for any n in $\mathbb{N}$. We shall prove that there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{\nu_{x_{n_k}}(s_{x_{n_k}})\}$ converges to a point in K_a.

Assume by contradiction that there are a positive real number s and a positive integer N_0 such that

$$\delta(x_n, K_a) < s/2 \quad \text{and} \quad \delta(\nu_{x_n}(s_{x_n}), K_a) > s \quad \text{for all } n \geq N_0.$$

Since $\{\nu_{x_n}(0) = x_n\}$ converges to x in K_α, by the continuity of ν_z and the distance function, there are two positive real numbers sequences $\{s_n\}$ and $\{k_n\}$ such that $s_n < k_n < s_{x_n}$ and

$$\begin{aligned} \delta(\nu_{x_n}(s_n), K_a) &= s/2, \\ \delta(\nu_{x_n}(k_n), K_a) &= s, \\ \delta(\nu_{x_n}(t), K_a) &\in [s/2, s] \quad \text{for all } t \in [s_n, k_n],\ n \geq N_0. \end{aligned}$$

This implies

$$||\nu_{x_n}(s_n) - \nu_{x_n}(k_n)|| \geq s/2 \quad \text{for all } n \geq N_0. \tag{3.8}$$

Since $\{s_{x_n}\}$ converges to 0 by Step 3, the sequence $\{k_n - s_n\}$ converges to 0. By (3.2) and arguing as in the first step, we see that $\{\nu_{x_n}(s_n) - \nu_{x_n}(k_n)\}$ converges to 0, which contradicts to (3.8). Thus we get the fourth step.

Step 5. Let $\{y_n\}$ and $\{z_n\}$ be sequences converging to x in K_a such that the sequences $\{\nu_{y_n}(s_{y_n})\}$ and $\{\nu_{z_n}(s_{z_n})\}$ converge to u and v in K_a, respectively. We shall prove $u = v$.

Assume by contradiction that there are two sequences $\{y_n\}$ and $\{z_n\}$ converging to x in $A \cap E$, and two different vector u and v in K_a, such that the sequences $\{\nu_{y_n}(s_{y_n})\}$ and $\{\nu_{z_n}(s_{z_n})\}$ converge to u and v respectively.

Since K_a is compact, by (d), there are two disjoint open subsets U and V such that $u \in U$, $v \in V$ and $K_a \subset U \cup V$. Arguing as in the fourth step we get a contradiction.

Step 6. Put

$$\eta(s,x) = \begin{cases} x & \text{for all } (s,x) \in [0,1] \times J_a, \\ \nu_x(ss_x) & \text{for all } (s,x) \in [0,1] \times (A \cap E). \end{cases}$$

We shall prove the deformation η is continuous. It is sufficient to prove that η is continuous on $\{1\} \times K_a$. Let $\{x_n\}$ be a sequence in $A \cap E$ converges to x in K_a. We may assume

$$J(x_n) < a + \frac{b-a}{2} \quad \text{for all } n \in \mathbb{N}.$$

We prove $\{\nu_{x_n}(s_{x_n})\}$ converges to x. Indeed, by the fourth and fifth step, the sequence $\{\nu_{x_n}(s_{x_n})\}$ converges to some u in K_a. It is sufficient to prove $u = x$. Assume by contradiction that $u \neq x$. Since $\{\nu_{x_n}(0) = x_n\}$ converges to x and $\{\nu_{x_n}(s_{x_n})\}$ converges to u, arguing as in the fourth step, we find a positive real number r and two sequences of positive real numbers $\{s_n\}$ and $\{k_n\}$ such that they converge to 0, $s_n < k_n < s_{x_n}$, $\nu_{x_n}(s_n) \in S_{r/2}$ and $\nu_{x_n}(k_n) \in S_r \setminus S_{r/2}$. Arguing as the fourth step we get a contradiction. Thus $\{\nu_{x_n}(s_{x_n})\}$ converges to x. $\square$

Using Theorem 3.5, we easily get the following result.

Theorem 3.10 (Mountain Pass Theorem). *Let V be a non-trivial vector subspace of a normed space $(E, \|\cdot\|)$, J be a continuous function from a normed space $(E, \|\cdot\|)$ into $\mathbb{R}$ such that J is continuously V-differentiable on E and satisfies the Palais–Smale condition* $(\mathrm{PS})_c$ *for every real number c. Assume that $J(0) = 0$ and there exist a positive real number r and $z_0 \in E$ such that $||z_0|| > r$ and $J(z_0) \leq 0$ and*

$$\alpha \equiv \inf\{J(u) : u \in E, ||u|| = r\} > 0.$$

Put $G = \{\varphi \in C([0,1], E) : \varphi(0) = 0, \varphi(1) = z_0\}$. Assume that $G \neq \emptyset$. Set

$$\beta = \inf\{\max J(\varphi([0,1])) : \varphi \in G\}.$$

Then $\beta \geq \alpha$ and β is a generalized critical value of J.

Remark 3.11. If E is a Banach space and $V = E$, by Lemma 3.6, β is a classical critical value of J and we get the classical version of Mountain-Pass Theorem in [1] without C^1-smoothness of J.

DEFINITION 3.12. Let $(E, \|\cdot\|)$ be a Banach space, J be a continuous function from an open subset U in E into $\mathbb{R}$ such that J is continuously E-differentiable on U. Let x be a critical point of J. Then x is said to be a critical point of mountain-pass type if there exists a neighbourhood U_x of x contained in U such that $W \cap J^{-1}(-\infty, J(x))$ is nonempty and not path-connected whenever W is an open neighborhood of x contained in U_x.

Using Theorem 3.5 and arguing as in the proof of Theorem 1 in [10] we get the following result.

THEOREM 3.13. *Let $(E, \|\cdot\|)$ be a Banach space, J be a continuous function from an open subset U in E into $\mathbb{R}$ such that J is continuously E-differentiable on U. Assume J satisfies all the hypotheses in Theorem* 3.10. *Assume in addition that the critical points in $J^{-1}(\beta)$ are isolated in E. Then there exists a critical point of mountain-pass type in $J^{-1}(\beta)$.*

4. Proof of Theorem 1.1

We can suppose $a = 0$, $J(a) = 0$. Let $\widetilde{H}$ be the completion of H and put

$$\begin{aligned} h_x &= \frac{-A(x)}{||A(x)||}, \\ r_x &= \min\left\{\frac{\alpha^2||x||}{4\Gamma^2+\alpha^2}, \frac{(\alpha^2-\Gamma\theta)\alpha||x||}{4\Gamma^2(\Gamma+\theta)}, \delta - ||x||\right\}, \\ \varphi_x(y) &= \delta(y, \widetilde{H} \setminus B_{\widetilde{H}}(x, r_x)) \quad \text{for all } x \in B_H(0,\delta) \setminus \{0\},\ y \in \widetilde{H}, \\ W_0 &= \bigcup_{x \in B_H(0,\delta)\setminus\{0\}} B_{\widetilde{H}}(x, r_x). \end{aligned}$$

We have $||h_x|| = 1$ and

$$\begin{aligned} ||h_x - h_y|| &= \left\|\frac{A(x)}{||A(x)||} - \frac{A(y)}{||A(y)||}\right\| = \left\|\frac{A(x)||A(y)|| - A(y)||A(x)||}{||A(x)||||A(y)||}\right\| \\ &= \left|\frac{A(x)(||A(y)|| - ||A(x)||) + ||A(x)||(A(x) - A(y))}{|A(x)||||A(y)||}\right\| \\ &\le \frac{2||A(x-y)||}{||A(y)||} \le \frac{2\Gamma||x-y||}{\alpha||y||} \end{aligned}$$

for all $x, y \in B_H(0,\delta) \setminus \{0\}$. Since $r_x \le \alpha^2||x||/(4\Gamma^2+\alpha^2)$, it follows that

$$\begin{aligned} \langle A(y), h_x\rangle &= \langle A(y), h_y\rangle + \langle A(y), h_x - h_y\rangle = \left\langle A(y), \frac{-A(y)}{||A(y)||}\right\rangle + \langle A(y), h_x - h_y\rangle \\ &\le -||A(y)|| + ||A(y)||||h_x - h_y|| \le -\alpha||y|| + \Gamma||y||||h_x-h_y|| \\ &\le -\alpha||y|| + \frac{2\Gamma^2}{\alpha}||x-y|| \le -\frac{\alpha}{2}||y|| \end{aligned}$$

for all $y \in B_H(x, r_x)$. On the other hand, by the mean value theorem there is t_y in $[0,1]$ such that

$$
\begin{aligned}
(4.1)\quad DJ(y)(h_x) &= DJ(y)(h_y) + DJ(y)(h_x - h_y) \\
&= -\frac{1}{||A(y)||} D^2J(t_y y)(y, A(y)) + D^2J(t_y y)((y, h_x - h_y) \\
&= -\frac{1}{||A(y)||}[D^2J(0)(y, A(y)) + (D^2J(t_y y) - D^2J(0))(y, A(y))] \\
&\quad + [D^2J(0)(y, h_x - h_y) + t_y^{-1}(D^2J(t_y y) - D^2J(0))(t_y y, h_x - h_y)] \\
&\le -\frac{\alpha^2}{\Gamma}||y|| + \theta||y|| + \Gamma||y||\frac{2\Gamma||x-y||}{\alpha||x||} + \theta||y||\frac{2\Gamma||x-y||}{\alpha||x||} \\
&\le -\frac{\alpha^2 - \Gamma\theta}{\Gamma}||y|| + \frac{2\Gamma^2 + 2\Gamma\theta}{\alpha}||y||\frac{r_x}{||x||} \\
&\le \left[-\frac{\alpha^2 - \Gamma\theta}{\Gamma} + \frac{\alpha^2 - \Gamma\theta}{2\Gamma}\right]||y|| = -\frac{(\alpha^2 - \Gamma\theta)}{2\Gamma}||y|| < 0
\end{aligned}
$$

for all $y \in B_H(x, r_x)$.

By definition, $B_H(0,\delta) \setminus \{0\} \subset W_0$. We show that $B_{\widetilde{H}}(0,\delta) \setminus \{0\}$ is contained in W_0. Let y be in $B_{\widetilde{H}}(0,\delta) \setminus \{0\}$. Put $r = ||y||$, $s = \alpha^2/(4\Gamma^2 + \alpha^2)$, $t = (\alpha^2 - \Gamma\theta)\alpha/(4\Gamma^2(\Gamma + \theta))$ and $\varepsilon = \min\{sr/(s+1), tr/(t+1), (\delta - r)/2\}/2$. There is x in $B_H(0,\delta) \setminus \{0\}$ such that $||x - y|| < \varepsilon$. Choosing ε we see that y is in $B_{\widetilde{H}}(x, r_x)$. Thus $B_{\widetilde{H}}(0,\delta) \setminus \{0\} \subset W_0$. By the Dugunji theorem, there is a subset B of $B_H(0,\delta) \setminus \{0\}$ such that

$$
B_{\widetilde{H}}(0,\delta) \setminus \{0\} \subset W = \bigcup_{x \in B} B_{\widetilde{H}}(x, r_x)
$$

and the set $\{x \in B : B_{\widetilde{H}}(x, r_x) \cap B_{\widetilde{H}}(y, s_y) \neq \emptyset\}$ is finite for any y in W and a sufficiently small positive real number s_y. Put

$$
g(y) = \sum_{x \in B} \left(\sum_{z \in B} \varphi_z(y)\right)^{-1} \varphi_x(y) h_x \quad \text{for all } y \in W.
$$

By choice of B, the mapping g is well defined, $g(W)$ is contained in H, and

$$
(4.2)\qquad ||g(y)|| \le 1 \quad \text{for all } y \in W.
$$

We can see that g satisfies all assumptions of Theorem 2.1. Thus for any u in $B_H(0,\delta) \setminus \{0\}$ there exist t_u in $(0,\infty]$ and a unique mapping ν_u from $(-t_u, t_u)$ into $W \cap H$ such that

$$
\begin{cases}
\nu_u'(t) = g(\nu_u(t)) & \text{for all } t \in (-t_u, t_u), \\
\nu_u(0) = u.
\end{cases}
$$

Fix $u \in B_H(0, \delta/3) \setminus \{0\}$ and put $s_u = \min\{||u||, t_u\}$. We have

$$\nu_u(t) = u + \int_0^t g(\nu_u(s))\,ds \quad \text{for all } t \in (0, t_u).$$

Thus, by (4.2)

$$0 < ||u|| - |t| \leq ||\nu_u(t)|| \leq ||u|| + |t| \leq 2||u|| \leq 2\delta/3 \quad \text{for all } t \in (-s_u, s_u).$$

By (a) of Theorem 2.1, it implies that $t_u \geq ||u||$. We define

$$\begin{aligned} \eta(t,u) &= \nu_u(t) && \text{for all } u \in B_H(0,\delta/3) \setminus \{0\},\ t \in (-||u||, ||u||), \\ \Lambda(u) &= \langle A(u), u\rangle/2 && \text{for all } u \in B_H(0,\delta/3) \setminus \{0\}. \end{aligned}$$

Thus

$$\begin{cases} \dfrac{\partial \eta}{\partial s}(s,u) = g(\eta(s,u)) \quad \text{for all } u \in B_H(0,\delta/3) \setminus \{0\},\ s \in (-||u||, ||u||), \\ \eta(0,u) = u \end{cases}$$

Since $\theta < \alpha/2$, by Taylor theorem there is a t_u in $(0,1)$ such that

$$\begin{aligned} |J(u) - \Lambda(u)| &= \left| J(0) + DJ(0)(u) + \frac{1}{2} D^2 J(t_u u)(u,u) - \frac{1}{2} D^2 J(0)(u,u) \right| \\ &= \frac{1}{2} |(D^2 J(t_u u) - D^2 J(0))(u,u)| \leq \frac{1}{2}\theta ||u||^2 < \frac{1}{4}\alpha ||u||^2. \end{aligned}$$

Thus

$$\Lambda(u) - \frac{1}{4}\alpha||u||^2 < J(u) < \Lambda(u) + \frac{1}{4}\alpha||u||^2. \tag{4.3}$$

Moreover,

$$\begin{aligned} \frac{\partial}{\partial t}\Lambda(\eta(t,u)) &= \frac{1}{2} \cdot \frac{\partial}{\partial t} \langle A(\eta(t,u)), \eta(t,u) \rangle \\ &= \langle A(\eta(t,u)), \eta'(t,u) \rangle = \langle A(\eta(t,u)), g(\eta(t,u)) \rangle \\ &= \left\langle A(\eta(t,u)), \frac{\sum_{x \in B} \varphi_x(\eta(t,u)) h_x}{\sum_{x \in B} \varphi_x(\eta(t,u))} \right\rangle \\ &= \frac{1}{\sum_{x \in B} \varphi_x(\eta(t,u))} \left[\sum_{x \in B} \varphi_x(\eta(t,u)) \langle A(\eta(t,u)), h_x \rangle \right] \\ &\leq \frac{1}{\sum_{x \in B} \varphi_x(\eta(t,u))} \left[\sum_{x \in B} -\frac{\alpha}{2} \varphi_x(\eta(t,u)) ||\eta(t,u)|| \right] \\ &= -\frac{\alpha}{2} ||\eta(t,u)|| < 0. \end{aligned} \tag{4.4}$$

Thus $\Lambda(\eta(\,\cdot\,,u))$ is strictly decreasing on $(-||u||,||u||)$. On the other hand, for any t in $(-||u||,||u||)$,

$$
\tag{4.5}
\begin{aligned}
|\Lambda(\eta(t,u))-\Lambda(u)| &= |\Lambda(\eta(t,u))-\Lambda(\eta(0,u)| \\
&= \bigg|\int_0^t \frac{d}{ds}\Lambda(\eta(s,u))\,ds\bigg| \geq \bigg|\int_0^t \frac{\alpha}{2}||\eta(s,u)||\,ds\bigg| \\
&\geq \bigg|\frac{\alpha}{2}\int_0^t (||u||-s)\,ds\bigg| = \frac{\alpha}{2}\bigg(|t|||u||-\frac{1}{2}t^2\bigg).
\end{aligned}
$$

When $|t|=||u||$, we have $\alpha(|t|||u||-t^2/2)/2=\alpha||u||^2/4$. Thus by (4.3), there exists a positive real number $r_u<||u||$ such that $||u||-r_u$ is small and

$$
\begin{aligned}
\Lambda(u)-\frac{\alpha}{4}||u||^2 &< \Lambda(u)-\frac{\alpha}{2}\bigg(r_u||u||-\frac{1}{2}r_u^2\bigg) < J(u) \\
&< \Lambda(u)+\frac{\alpha}{2}\bigg(r_u||u||-\frac{1}{2}r_u^2\bigg) < \Lambda(u)+\frac{\alpha}{4}||u||^2.
\end{aligned}
$$

Applying (4.5) we obtain

$$
\begin{aligned}
\Lambda(\eta(r_u,u)) &\leq \Lambda(u)-\frac{\alpha}{2}\bigg(r_u||u||-\frac{1}{2}r_u^2\bigg) < J(u) \\
&< \Lambda(u)+\frac{\alpha}{2}\bigg(r_u||u||-\frac{1}{2}r_u^2\bigg) \leq \Lambda(\eta(-r_u,u)).
\end{aligned}
$$

This implies there is a unique s_u in $(-r_u,r_u)\subset(-||u||,||u||)$ such that

$$J(u)=\Lambda((\eta(s_u,u))=\frac{1}{2}\langle A((\eta(s_u,u)),\eta(s_u,u)\rangle.$$

We shall prove that $J(\eta(\,\cdot\,,u))$ is strictly decreasing on $(-||u||,||u||)$. By Lemma 2.4, $J(\eta(\,\cdot\,,u))$ is differentiable on $(-||u||,||u||)$. Furthermore, by (4.1),

$$
\begin{aligned}
\frac{\partial}{\partial t}J(\eta(s,u)) &= DJ(\eta(s,u))\circ\frac{\partial\eta}{\partial t}(s,u) = DJ(\eta(s,u))\circ g(\eta(s,u)) \\
&= DJ(\eta(s,u))\left(\frac{\sum_{x\in B}\varphi_x(\eta(s,u))h_x}{\sum_{x\in B}\varphi_x(\eta(s,u))}\right) \\
&= \frac{\sum_{x\in B}\varphi_x(\eta(s,u))DJ(\eta(s,u)(h_x))}{\sum_{x\in B}\varphi_x(\eta(s,u))} < 0
\end{aligned}
$$

for all $s\in(-||u||,||u||)$, $u\in B_H(0,\delta)\setminus\{0\}$. It implies the strictly decreasing of $J(\eta(\,\cdot\,,u))$ on $(-||u||,||u||)$ for any u in $B_H(0,\delta)\setminus\{0\}$. Put

$$
\begin{cases}
\phi(x)=\eta(s_x,x) & \text{for all } x\in B_H(0,\delta/3)\setminus\{0\}, \\
\phi(0)=0.
\end{cases}
$$

We have

$$J(x)=\frac{1}{2}\langle A(\phi(x),\phi(x)\rangle \quad \text{for all } x\in B_H(0,\delta/3).$$

From the above properties, ϕ is an one-to-one mapping from $B_H(0,\delta/3)$ into $\phi(B_H(0,\delta/3))$.

Now we show that ϕ is continuous on $B_H(0,\delta/3)$. First we note that

$$||\eta(s_x,x)-x|| = ||\eta(s_x,x)-\eta(0,x)|| = \left\| \int_0^{s_x} g(\eta(t,x))\,dt \right\| \le |s_x| \le ||x||.$$

It implies the continuity of ϕ at 0. Fix x in $B_H(0,\delta/3)\setminus\{0\}$. Let $\{x_n\}$ be a sequence converging to x in $B_H(0,\delta/3)$. Firstly, we show that the sequence $\{s_{x_n}\}$ converges to s_x.

Assume by contradiction that there are a positive real number d and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $|s_{x_{n_k}} - s_x| > d$ for any integer k. Since $\eta(t,x)$ is defined on $(-||x||,||x||)$ and s_x is in $(-||x||,||x||)$, by (b) of Theorem 2.1 we can (and shall) suppose $\eta(t,x_{n_k})$ is defined for any t in $[-|s_x|-d, s_x+d]$ and $s_{x_{n_k}} - s_x > d$ for any integer k. By (2.9) the map $y \to \eta(s_x+d), y)$ is continuous at x. Since $\Lambda(\eta(\,\cdot\,,u))$ is strictly decreasing,

$$J(x_{n_k}) = \Lambda(\eta(s_{x_{n_k}}), x_{n_k}) \le \Lambda(\eta(s_x+d), x_{n_k}).$$

Taking the limits of the two sides, we have

$$J(x) = \Lambda(\eta(s_x,x) \le \Lambda(\eta(s_x+d),x),$$

which contradicts to the strictly decrease of $\Lambda(\eta(\,\cdot\,,x))$. Therefore we should have $\lim_{n\to\infty} s_{x_n} = s_x$. On the other hand

$$\begin{aligned} ||\eta(s_{x_n},x_n)-\eta(s_x,x)|| &\le ||\eta(s_{x_n},x_n)-\eta(s_x,x_n)|| + ||\eta(s_x,x_n)-\eta(s_x,x)|| \\ &= \left\| \int_{s_x}^{s_{x_n}} g(\eta(t,x_n))\,dt \right\| + ||\eta(s_x,x_n)-\eta(s_x,x)|| \\ &\le |s_{x_n}-s_x| + ||\eta(s_x,x_n)-\eta(s_x,x)||. \end{aligned}$$

This implies $\lim_{n\to\infty}\phi(x_n)=\phi(x)$, so that ϕ is continuous on $B_H(0,\delta/3)$. The path $\eta(t,x)$ carries x to $\phi(x)$. We can use the same path to move $\phi(x)$ to x in order to define the inverse map $\Psi = \phi^{-1}$ of ϕ. Arguing as above we see that ϕ is a homeomorphism from $B_H(0,\delta/3)$ onto $\phi(B_H(0,\delta/3))$.

Corollary 4.1. *Assume J satisfies all hypotheses in Theorem* 1.1. *Let x_0 be a nondegenerate critical point of J with Morse index j. Then*

$$C_q(J,x_0) = \begin{cases} G & \text{if } q=j, \\ 0 & \text{if } q\ne j. \end{cases}$$

Proof. Without lost of any generality we may assume $x_0=0$. Let E and F be the positive and negative subspaces of the operator $D^2J(0)$ respectively. According to Theorem 1.1, it is sufficient to consider the case where

$$J(y+z) = \langle y,y\rangle - \langle z,z\rangle \quad \text{for all } x=y+z\in H=E\oplus F.$$

Put $\nu_0 = J^{-1}((-\infty, 0])$, note that

$$B(0,\varepsilon) \cap \nu_0 = \{x = y + z \in H : ||x|| \leq \varepsilon, ||y|| \leq ||z||\}.$$

We defined a deformation

$$\eta(t,x) = z + ty \quad \text{for all } t \in [0,1];\ x \in B(0,\varepsilon) \cap \nu_0.$$

It is a strong deformation retract from $(B(0,\varepsilon) \cap \nu_0, B(0,\varepsilon) \cap (\nu_0 \setminus \{0\}))$ to $(B(0,\varepsilon) \cap F, B(0,\varepsilon) \cap (F \setminus \{0\}))$. Thus

$$\begin{aligned} C_q(J,0) &\cong H_q(B(0,\varepsilon) \cap \nu_0, B(0,\varepsilon) \cap (\nu_0 \setminus \{0\})) \\ &\cong H_q(B(0,\varepsilon) \cap F, B(0,\varepsilon) \cap (F \setminus \{0\})) \\ &\cong H_q(B^j, S^{j-1}) \cong \begin{cases} G & \text{if } q = j, \\ 0 & \text{if } q \neq j. \end{cases} \end{aligned} \qquad \square$$

Remark 4.2. If H is a Hilbert space and $J \in C^2(H, \mathbb{R})$, the above corollary was proved in [5] (see Theorem 4.1, p. 34).

Corollary 4.3. *If $K_c = \{z_1, \ldots, z_m\}$, then*

$$H_q(\nu_{c+\varepsilon}, \nu_{c-\varepsilon}, G) \cong H_q(\nu_c, \nu_c \setminus K_c, G) \cong \bigoplus_{j=1}^{m} C_q(J, z_j).$$

Proof. By Theorem 3.5 and the homotopy invariance of singular homology group, we have

$$\begin{aligned} H_q(\nu_{c+\varepsilon}, \nu_{c-\varepsilon}) &\cong H_q(\nu_c, \nu_{c-\varepsilon}), \\ H_q(\nu_c \setminus K_c, \nu_{c-\varepsilon}) &\cong H_q(\nu_{c-\varepsilon}, \nu_{c-\varepsilon}) \cong 0. \end{aligned}$$

Applying the exactness of singular homology groups to the triple $(\nu_c, \nu_c \setminus K_c, \nu_{c-\varepsilon})$ we have

$$\begin{aligned} \ldots \to H_q(\nu_c \setminus K_c, \nu_{c-\varepsilon}) \to H_q(\nu_c, \nu_{c-\varepsilon}) \\ \to H_q(\nu_c, \nu_c \setminus K_c) \to H_{q-1}(\nu_c \setminus K_c, \nu_{c-\varepsilon}) \to \ldots \end{aligned}$$

we get

$$0 \to H_q(\nu_c, \nu_{c-\varepsilon}) \to H_q(\nu_c, \nu_c \setminus K_c) \to 0.$$

It implies $H_q(\nu_c, \nu_{c-\varepsilon}) \cong H_q(\nu_c, \nu_c \setminus K_c)$.

Using the excision property, we may decompose the relative singular homology groups into critical groups

$$H_q(\nu_c, \nu_c \setminus K_c) \cong H_q\bigg(\nu_c \cap \bigcup_{j=1}^{m} B(z_j, \varepsilon), \nu_c \cap \bigcup_{j=1}^{m} B(z_j, \varepsilon) \setminus \{z_j\}\bigg) \cong \bigoplus_{j=1}^{m} C_q(J, z_j)$$

for any sufficiently small positive real number ε, which completes the proof of the corollary. $\square$

DEFINITION 4.4. Assume J satisfies all the hypotheses of Theorem 1.1. Let q be a nonnegative integer, a and b be two regular values of J such that $a < b$. Assume that J has only isolated critical values $\{c_i\}_{i\in\mathbb{Z}}$ in (a, b), $c_i < c_{i+1}$ and $K_{c_i} = \{z_1^i, \dots, z_{m_i}^i\}$ with m_i in $\mathbb{N}$. Choose ε_i in $(0, \min\{c_{i+1} - c_i, c_i - c_{i-1}\})$ for any i in $\mathbb{Z}$. Put

$$\begin{aligned} \beta_q &\equiv \beta_q(a,b) \equiv \operatorname{rank} H_p(\nu_b, \nu_a, G), \\ M_q &\equiv M_q(a,b) \equiv \sum_{a<c_i<b} \operatorname{rank} H_p(\nu_{c_i+\varepsilon_i}, \nu_{c_i-\varepsilon_i}, G) \end{aligned}$$

We call M_q the *q-th Morse type number of the function J with respect to* (a, b).

By Theorem 3.5, M_q is well defined and independent of the choice of $\{\varepsilon_i\}$. Using Corollary 4.3 and arguing as in [5], we have the following Morse inequality

$$\sum_{j=0}^{q} (-1)^{q-j} M_j(a,b) \ge \sum_{j=0}^{q} (-1)^{q-j} \beta_j(a,b).$$

5. Proof of Gromoll–Meyer Splitting Theorem

In this section, H is a Hilbert space with the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle$ and its dual space H^* is always identified with H. We denote by $B(x_0, r)$ the set $\{x \in H \mid ||x - x_0|| < r\}$ for any x_0 in H. Now, we recall the class $(S)_+$ introduced by Browder (see [3], [4]).

DEFINITION 5.1. Let A be a subset of H and h be a mapping of A into H. We say:

(a) h is *demicontinuous* if the sequence $\{h(x_n)\}$ converges weakly to $h(x)$ in H for any sequence $\{x_n\}$ converging strongly to x in H.

(b) h is of *class* $(S)_+$ if h is demicontinuous and has the following property: Let $\{x_n\}$ be a sequence in A such that $\{x_n\}$ converges weakly to x in H. Then $\{x_n\}$ converges strongly to x in H if

$$\limsup_{n\to\infty} \langle h(x_n), x_n - x\rangle \le 0.$$

Let D be a bounded open subset in H with boundary ∂D and closure $\overline{D}$. Let h be a mapping of class $(S)_+$ on $\overline{D}$ and p be in $H \setminus h(\partial D)$. By Theorems 4 and 5 in [3], the topological degree of h on D at p is defined as a family of integers and is denoted by $\deg(h, D, p)$. In [18] Skrypnik showed that this topological degree is single-valued (see also [4]).

To prove the Gromoll–Meyer splitting theorem we need the following lemmas, in which the notations and assumptions of Theorem 1.3 are used.

Lemma 5.2. *Assume DJ is of class $(S)_+$. Let $\{x_n\}$ be a bounded sequence in H such that the sequence $\{DJ(x_n)\}$ converges to c in H. Then $\{x_n\}$ has a subsequence converging in H.*

Proof. We can suppose $\{x_n\}$ weakly converges to x_0 in H. We have

$$\begin{aligned}\limsup_{n\to\infty} \langle DJ(x_n), x_n - x_0\rangle &\le \limsup_{n\to\infty} |\langle DJ(x_n), x_n - x_0\rangle| \\ &\le \limsup_{n\to\infty} |\langle DJ(x_n) - c, x_n - x_0\rangle| + \limsup_{n\to\infty} |\langle c, x_n - x_0\rangle| \\ &\le \limsup_{n\to\infty} ||DJ(x_n) - c\| \cdot \|x_n - x_0|| + \limsup_{n\to\infty} |\langle c, x_n - x_0\rangle| = 0.\end{aligned}$$

Since DJ is of class $(S)_+$, we have the lemma. □

Lemma 5.3. *There are positive real numbers C, α and Γ, a closed vector subspace H^+, two finite-dimensional vector subspaces H^0 and H^- of H such that $H^- \oplus H^0 \oplus H^+$ is a orthogonally direct decomposition of H, $H^0 = \ker A$,*

$$\begin{aligned}\langle Ax, x\rangle &\ge C||x||^2 && \textit{for all } x \in H^+ \\ \langle Ax, x\rangle &\le -C||x||^2 && \textit{for all } x \in H^-, \\ \Gamma||y|| &\ge ||A(y)|| \ge \alpha||y|| && \textit{for all } y \in Y \equiv H^+ \oplus H^-.\end{aligned}$$

Proof. Let E be the orthogonal complement of H^0 in H. We see that

$$\langle Au, v\rangle = \langle u, Av\rangle = 0 \quad \text{for all } u \in E,\ v \in H^0.$$

Therefore $A(E) \subset E$. Denote by B the restriction of A on E. We see that B is a bounded self-adjoint linear operator on E. It is clear that B is one-to-one.

We claim that $B(E)$ is a closed subspace of E. Let $\{x_n\}$ be a sequence in E such that $\{B(x_n)\}$ converges to y in E, we will prove that $y \in B(E)$. First, we show that $\{x_n\}$ is bounded. Suppose by contradiction that $\{||x_n||\}$ tends to ∞. Put $v_n = (||x_n|| + 1)^{-1}x_n$ for any integer n, then $\{||v_n||\}$ converges to 1 and $\{B(v_n)\}$ converges to 0. Without loss of generality, we can (and shall) suppose that $\{v_n\}$ converges weakly to a vector v_0 in E. Since A is of class $(S)_+$, and

$$\limsup_{n\to\infty}\langle A(v_n), v_n - v_0\rangle = \limsup_{n\to\infty}\langle B(v_n), v_n - v_0\rangle = 0,$$

$\{v_n\}$ converges to v_0. Thus, $A(v_0) = 0$ and $||v_0|| = 1$, which is a contradiction. Therefore $\{x_n\}$ is bounded and we can suppose that it converges weakly to a vector x_0 in E. Since $\{A(x_n)\}$ converges to y, by the definition of class $(S)_+$, the sequence $\{x_n\}$ converges to x_0 in E. Therefore $A(x_0) = y$ and $B(E)$ is closed.

Next we show that $B(E) = E$. Otherwise, there is x in $E \setminus \{0\}$ such that

$$\begin{aligned}&\langle B(z), x\rangle = 0 \quad \text{for all } z \in E \\ \text{or}\quad &\langle z, A(x)\rangle = 0 \quad \text{for all } z \in E.\end{aligned}$$

Thus, $A(x)$ is in H^0. It implies that $A(A(x))$ is also in H^0 and

$$\langle A(x), A(x)\rangle = \langle x, A(A(x))\rangle = 0.$$

It follows that x is in $H^0 \cap E$, and therefore $x = 0$, which is impossible. This contradiction shows that $B(E) = E$.

We have proved that B is an one-to-one mapping from E onto E. Thus, by the open mapping theorem, B is an invertible self-adjoint bounded operator on E and we can find the desired real positive real numbers α and Γ. By a result on self-adjoint operators (see [13, p. 172]), there exist a positive real number C and an orthogonal decomposition $H^- \oplus H^+$ of E such that H^- and H^+ are A-invariant closed subspaces of E and

$$\begin{aligned} \langle A(x), x\rangle &\le -C||x||^2 \quad \text{for all } x \in H^-, \\ \langle A(x), x\rangle &\ge C||x||^2 \quad \text{for all } x \in H^+. \end{aligned}$$

Finally we prove that $H^- \oplus H^0$ is finite dimensional. It is sufficient to show that $H^- \oplus H^0$ is locally compact. Let $\{x_n\}$ be a sequence weakly converging to x in $H^- \oplus H^0$. We see that

$$\lim_{n\to\infty} \langle A(x), x_n - x\rangle = 0 \quad \text{and} \quad \langle A(x_n - x), x_n - x\rangle \le 0$$

for all $n \in \mathbb{N}$. Thus, $\limsup_{n\to\infty} \langle A(x_n), x_n - x\rangle \le 0$ and $\{x_n\}$ converges to x. Therefore $H^- \oplus H^0$ is locally compact. □

LEMMA 5.4. *If* 0 *is an isolated critical point of* J *and* $x \in B'(0, \delta)$*, we have*

$$||DJ(x) - A(x)|| \le \frac{\alpha}{2}||x||.$$

PROOF. For any $v \in H$, by the mean value theorem there is t_x in $(0, 1)$ such that

$$DJ(x)v = DJ(x)v - DJ(0)v = D^2J(t_x x)(x, v).$$

Thus

$$\begin{aligned} |\langle DJ(x) - A(x), v\rangle| &= |D^2J(t_x x)(x, v) - D^2J(0)(x, v)| \\ &= t_x^{-1}|(D^2J(t_x x) - D^2J(0))(t_x x, v)| \le \frac{\alpha}{2}||x||||v||. \end{aligned}$$

This implies $||DJ(x) - A(x)|| \le \alpha||x||/2$. □

LEMMA 5.5. *If* x_0 *is a nondegenerate isolated critical point of* J*, we have*

$$i(DJ, x_0) = \sum_{q=0}^{\infty} (-1)^q \operatorname{rank} C_q(J, x_0).$$

Proof. We can suppose $x_0 = 0$. By Lemma 5.5, $H = H^- \oplus H^+$, H^+ is closed and $\dim H^- = j < \infty$. By Corollary 4.1, it is sufficient to prove $i(DJ,0) = (-1)^j$. We need two following steps

Step 1. Prove $\deg(DJ, B'(0,\delta), 0) = \deg(A, B'(0,\delta), 0)$.

Define

$$k(t,x) = (1-t)DJ(x) + tA(x) \quad \text{for all } t \in [0,1],\ x \in B'(0,\delta).$$

We shall prove that $k(t,x) \neq 0$ for all (t,x) in $[0,1] \times \partial B(0,\delta)$. Suppose by contradiction that $k(t,x) = 0$ for some (t,x) in $[0,1] \times \partial B(0,\delta)$. We have $A(x) + (1-t)(DJ(x) - A(x)) = 0$ and by Lemma 5.4

$$\begin{aligned} 0 &= ||A(x) + (1-t)(DJ(x) - A(x))|| \\ &\geq ||A(x)|| - (1-t)||DJ(x) - A(x)|| \geq \alpha||x|| - (1-t)\frac{\alpha}{2}||x|| \geq \frac{\alpha}{2}||x|| > 0, \end{aligned}$$

which is absurd. Thus by the homotopy invariance of the topological degree, $\deg(DJ, B'(0,\delta), 0) = \deg(A, B'(0,\delta), 0)$, which implies $i(DJ,0) = i(A,0)$.

Step 2. Define

$$A_1(y+z) = y + A(z) \quad \text{for all } y \in H^+,\ z \in H^-.$$

We prove $\deg(A, B'(0,\delta), 0) = \deg(A_1, B'(0,\delta), 0)$. Indeed, put

$$h(t,x) = (1-t)A(x) + tA_1(x) \quad \text{for all } t \in [0,1],\ x \in B'(0,\delta).$$

We shall prove $h(t,x) \neq 0$ for all $(t,(y+z))$ in $[0,1] \times \partial B(0,\delta)$.

Assume by contradiction that $h(t,x) = 0$ for some $(t,(y+z))$ in $[0,1] \times \partial B(0,\delta)$. Thus $(ty + (1-t)A(y)) + A(z) = 0$. Since $A(H^+) \subset H^+$ and $A(H^-) \subset H^-$, it follows that $ty + (1-t)A(y) = 0$ and $A(z) = 0$. By the definitions of H^+ and H^-, we see that $y = 0$ and $z = 0$, which is absurd. Thus $\deg(A, B'(0,\delta), 0) = \deg(A_1, B'(0,\delta), 0)$, which implies

$$i(A,0) = i(A_1,0).$$

Note that A_1 is a compact vector field, therefore by the homology invariance of the Leray–Schauder topological degree we have

$$i(A_1,0) = (-1)^j.$$

Combining these two steps we get the lemma. □

Lemma 5.6. *Assume $H^- = \{0\}$. Put $\delta_1 = \delta/2$, $Y = H^+$ and $Z = H^0$. Assume that for any z in $B'_Z(0,\delta_1)$*

$$\langle DJ(z+y_1) - DJ(z+y_2), y_1 - y_2\rangle > 0 \quad \textit{for all } y_1 \neq y_2 \in B'_Y(0,\delta_1). \tag{5.1}$$

We have

(a) *There is a mapping* ψ *from* $B_Z(0,\delta_1)$ *into* $B_Y(0,\delta_1)$ *such that* $DJ(z+\psi(z))|_Y = 0$ *and*

$$J(z+\psi(z)) = \min_{v\in B'_Y(0,\delta_1)} J(z+v) \quad \textit{for all } z \in B_Z(0,\delta_1).$$

(b) ψ *is continuous on* $B_Z(0,\delta_1)$.

(c) *Put* $j(z) = J(z+\psi(z))$ *for any* z *in* $B_Z(0,\delta_1)$. *Then* j *is of class* C^1 *and*

$$Dj(z)h = DJ(z+\psi(z))h \quad \textit{for all } h \in Z.$$

Proof. (a) Fix z in $B_Z(0,\delta_1)$. We shall denote $A|_Y$ and $DJ(\,\cdot\,+z)|_Y$ by A and $DJ(\,\cdot\,+z)$. By Lemma 5.5 we have $\deg(A, B'_Y(0,\delta_1), 0) = (-1)^{\dim H^-} = 1$. We shall prove

$$\deg(DJ(\,\cdot\,+z), B'_Y(0,\delta_1), 0) = \deg(A, B'_Y(0,\delta_1), 0). \tag{5.2}$$

Put

$$h(t,y) = (1-t)DJ(y+z) + tA(y) \quad \text{for all } (t,y) \in [0,1]\times B'_Y(0,\delta_1).$$

We shall prove $(1-t)DJ(y+z)+tA(y) \neq 0$ for all (t,y) in $[0,1]\times \partial B_Y(0,\delta_1)$.

Assume by contradiction that $(1-t)DJ(y+z)+tA(y) = 0$ for some (t,y) in $[0,1]\times\partial B_Y(0,\delta_1)$. It follows that

$$A(y) + (1-t)(DJ(y+z) - A(y)) = 0.$$

Since Z is the kernel of A, $A(y+z) = A(y)$. Thus

$$A(y) + (1-t)(DJ(y+z) - A(y+z)) = 0. \tag{5.3}$$

On the other hand since $||y|| = \delta_1$, $||z|| < \delta_1$ and by Lemma 5.4

$$\begin{aligned} ||A(y) + (1-t)(DJ(y+z) - A(y+z))|| &\geq ||A(y)|| - (1-t)\frac{\alpha}{2}||y+z|| \\ &\geq \alpha||y|| - \frac{\alpha}{2}(||y||+||z||) = \frac{\alpha}{2}(\delta_1 - ||z||) > 0, \end{aligned}$$

which contradicts to (5.3). Thus by the homotopy invariance of the topological degree, we have (5.2). This implies

$$\deg(DJ(\,\cdot\,+z)|_Y, B'_Y(0,\delta_1), 0) = (-1)^{\dim H^-} = 1.$$

By the property of the topological degree and (5.1) there is a unique y_0 in $B_Y(0,\delta_1)$ such that

$$DJ(z+y_0)|_Y = 0. \tag{5.4}$$

Thus, we define the function ψ such that $\psi(z) = y_0$.

Now we show $J(z+\psi(z)) = \min_{v\in B'_Y(0,\delta_1)} J(z+v)$. Indeed, by (5.1) and (5.4), if $y \neq \psi(z)$, we have

$$\begin{aligned} J(z+y) &- J(z+\psi(z)) \\ &= \int_0^1 DJ(z+\psi(z)+s(y-\psi(z)))(y-\psi(z))\,ds \\ &= \int_0^1 [DJ(z+\psi(z)+s(y-\psi(z)))-DJ(z+\psi(z))](y-\psi(z))ds > 0 \end{aligned}$$

and we get (a).

(b) Let $\{z_n\}$ be a sequence converging to z_0 in Z. We prove that $\{\psi(z_n)\}$ converges to $\psi(z_0)$. Indeed, since $\{\psi(z_n)\}$ is contained in $B_Y(0,\delta_1)$ for any integer n, we can suppose $\{\psi(z_n)\}$ converges weakly to y_0 in $B'_Y(0,\delta_1)$. Now we show $\{\psi(z_n)\}$ converges to y_0. Since $||z_n+\psi(z_n)|| < 2\delta_1 = \delta$, by Lemma 5,4 we see that

$$\begin{aligned} ||DJ(z_n &+\psi(z_n))|| \\ &= ||DJ(z_n+\psi(z_n)) - A(z_n+\psi(z_n)) + A(z_n+\psi(z_n))|| \\ &\leq ||A(z_n+\psi(z_n))|| + ||DJ(z_n+\psi(z_n)) - A(z_n+\psi(z_n))|| \\ &\leq \Gamma||z_n+\psi(z_n)|| + \frac{\alpha}{2}||z_n+\psi(z_n)|| \leq \left(\Gamma+\frac{\alpha}{2}\right)\delta. \end{aligned} \tag{5.5}$$

By (5.4) nd (5.5), we have

$$\begin{aligned} &|\langle DJ(z_n+\psi(z_n)), (z_n+\psi(z_n)) - (z_0+y_0)\rangle| \\ &\qquad = |\langle DJ(z_n+\psi(z_n)), z_n - z_0\rangle| \leq ||DJ(z_n+\psi(z_n))||||z_n - z_0||, \end{aligned}$$

which implies that

$$\lim_{n\to\infty} \langle DJ(z_n+\psi(z_n)), (z_n+\psi(z_n)) - (z_0+y_0)\rangle = 0.$$

Since DJ is of class $(S)_+$, it implies that $\lim_{n\to\infty}\psi(z_n) = y_0$. Note that $DJ(z_n+\psi(z_n))|_Y = 0$. By condition (b) of Theorem 1.3 $DJ(z_0+y_0))|_Y = 0$. Now the uniqueness of $\psi(z_0)$ implies $\psi(z_0) = y_0$ and ψ is continuous.

(c) We prove j is of class C^1. Fix h in Z. Using (a) of this lemma,

$$\begin{aligned} J(z+th+\psi(z&+th)) - J(z+\psi(z+th)) \\ &\leq J(z+th+\psi(z+th)) - J(z+\psi(z)) \\ &\leq J(z+th+\psi(z)) - J(z+\psi(z)), \end{aligned} \tag{5.6}$$

By condition (b) of Theorem 1.3 and the continuity of ψ, we have

$$\begin{aligned} \lim_{t\to 0} &\frac{J(z+th+\psi(z+th)) - J(z+\psi(z+th))}{t} \\ &= \lim_{t\to 0}\int_0^1 DJ(z+sth+\psi(z+th))(h)ds = DJ(z+\psi(z))(h). \end{aligned} \tag{5.7}$$

Similarly,

$$\lim_{t\to 0} \frac{J(z+th+\psi(z)) - J(z+\psi(z))}{t} = DJ(z+\psi(z))(h). \tag{5.8}$$

By (5.6)–(5.8) we get

$$\lim_{t\to 0} \frac{J(z+th+\psi(z+th)) - J(z+\psi(z))}{t} = DJ(z+\psi(z))(h).$$

Therefore $Dj(z)h = DJ(z+\psi(z))(h)$ and the map $z \mapsto Dj(z)h$ is continuous on $B_Z(0,\delta_1)$. Since Z is finite-dimensional, j is of class C^1 on $B_Z(0,\delta_1)$. This completes the proof of the lemma. □

DEFINITION 5.7. Let Y be $H^+ \oplus H^-$. We denote by P and Q the orthogonal projections of Y into H^+ and H^-, respectively.

LEMMA 5.8. *Put $\delta_1 = \delta/2$, $Y = H^+ \oplus H^-$ and $Z = H^0$. Assume that for any z in $B'_Z(0,\delta_1)$, $x_1 + y_1 \neq x_2 + y_2 \in B'_Y(0,\delta_1)$*

$$\langle DJ(z+x_1+y_1) - DJ(z+x_2+y_2), (x_1 - x_2) - (y_1 - y_2)\rangle > 0. \tag{5.9}$$

We have:

(a) *There is a mapping ψ from $B_Z(0,\delta_1)$ into $B_Y(0,\delta_1)$ such that*

$$DJ(z+\psi(z))|_Y = 0,$$

$$\begin{aligned} J(z+\psi(z)) &= \min\{J(z+Q\psi(z)+x) : x \in H^+;\ Q\psi(z)+x \in B_Y(0,\delta_1)\}, \\ J(z+\psi(z)) &= \max\{J(z+P\psi(z)+t) : t \in H^-;\ P\psi(z)+t \in B_Y(0,\delta_1)\}. \end{aligned}$$

(b) *ψ is continuous on $B_Z(0,\delta_1)$.*

(c) *Put $j(z) = J(z+\psi(z))$ for any z in $B_Z(0,\delta_1)$. Then j is of class C^1 and*

$$Dj(z)h = DJ(z+\psi(z))h \quad \textit{for all } h \in Z.$$

PROOF. (a) Fix z in $B_Z(0,\delta_1)$. We shall denote $A|_Y$ and $DJ(\,\cdot\,+z)|_Y$ by A and $DJ(\,\cdot\,+z)$. By Lemma 5.5 we have $\deg(A, B'_Y(0,\delta_1), 0) = (-1)^{\dim H^-}$. We shall prove

$$\deg(DJ(\,\cdot\,+z), B'_Y(0,\delta_1), 0) = \deg(A, B'_Y(0,\delta_1), 0). \tag{5.10}$$

Put

$$h(t,y) = (1-t)DJ(y+z) + tA(y) \quad \text{for all } (t,y) \in [0,1] \times B'_Y(0,\delta_1).$$

We shall prove $(1-t)DJ(y+z) + tA(y) \neq 0$ for all (t,y) in $[0,1] \times B'_Y(0,\delta_1)$.

Assume by contradiction that $(1-t)DJ(y+z) + tA(y) = 0$ for some (t,y) in $[0,1] \times \partial B_Y(0,\delta_1)$. It follows that

$$A(y) + (1-t)(DJ(y+z) - A(y)) = 0.$$

Since Z is the kernel of A, $A(y+z) = A(y)$. Thus

$$A(y) + (1-t)(DJ(y+z) - A(y+z)) = 0. \tag{5.11}$$

On the other hand since $||y|| = \delta_1$, $||z|| < \delta_1$ and by Lemma 5.4,

$$\begin{aligned}||A(y) + (1-t)(DJ(y+z) - A(y+z))|| &\geq ||A(y)|| - (1-t)\frac{\alpha}{2}||y+z|| \\ &\geq \alpha||y|| - \frac{\alpha}{2}(||y|| + ||z||) = \frac{\alpha}{2}(\delta_1 - ||z||) > 0,\end{aligned}$$

which contradicts to (5.3). Thus by the homotopy invariance of the topological degree, we have (5.2) and $\deg(DJ(y+z)|_Y, B'_Y(0,\delta_1), 0) = (-1)^{\dim H^-}$. By the property of the topological degree and (5.1) there is a unique y_0 in $B_Y(0,\delta_1)$ such that

$$DJ(z+y_0)|_Y = 0. \tag{5.12}$$

Put $\psi(z) = y_0$. Now we show

$$J(z+\psi(z)) = \min\{J(z + Q\psi(z) + x) : x \in H^+;\ Q\psi(z) + x \in B_Y(0,\delta_1)\}.$$

Indeed, let x be in H^+ such that $Q\psi(z) + x \in B_Y(0,\delta_1)$, by (5.10) and (5.12)

$$\begin{aligned}&J(z + Q\psi(z) + x) - J(z+\psi(z)) \\ &\quad= \int_0^1 DJ(z+\psi(z) + t(x - P\psi(z)))(x - P\psi(z))\,dt \\ &\quad= \int_0^1 [DJ(z+\psi(z) + t(x - P\psi(z))) - DJ(z+\psi(z))](x - P\psi(z))\,dt > 0.\end{aligned}$$

Similarly we have

$$J(z+\psi(z)) = \max\{J(z + P\psi(z) + t) : t \in H^-;\ P\psi(z) + t \in B_Y(0,\delta_1)\}.$$

(b) Let $\{z_n\}$ be a sequence converging to z_0 in Z. We prove that $\{\psi(z_n)\}$ converges to $\psi(z_0)$. Indeed, since $\{\psi(z_n)\}$ is contained in $B_Y(0,\delta_1)$ for any integer n, we can suppose $\{\psi(z_n)\}$ converges weakly to y_0 in $B'_Y(0,\delta_1)$. Now we show $\{\psi(z_n)\}$ converges to y_0. Since $||z_n + \psi(z_n)|| < 2\delta_1 = \delta$, by Lemma 5,4 we see that

$$\begin{aligned}||DJ(z_n &+ \psi(z_n))|| \\ &= ||DJ(z_n + \psi(z_n)) - A(z_n + \psi(z_n)) + A(z_n + \psi(z_n))|| \\ &\leq ||A(z_n + \psi(z_n))|| + ||DJ(z_n + \psi(z_n)) - A(z_n + \psi(z_n))|| \\ &\leq \Gamma||z_n + \psi(z_n)|| + \frac{\alpha}{2}||z_n + \psi(z_n)|| \leq \left(\Gamma + \frac{\alpha}{2}\right)\delta.\end{aligned} \tag{5.13}$$

By (5.12) and (5.13), we have

$$\begin{aligned}&|\langle DJ(z_n + \psi(z_n)), (z_n + \psi(z_n)) - (z_0 + y_0)\rangle| \\ &\qquad= |\langle DJ(z_n + \psi(z_n)), z_n - z_0\rangle| \leq ||DJ(z_n + \psi(z_n))||||z_n - z_0||,\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \langle DJ(z_n + \psi(z_n)), (z_n + \psi(z_n)) - (z_0 + y_0)\rangle = 0.$$

Since DJ is of class $(S)_+$, it implies that $\lim_{n\to\infty} \psi(z_n) = y_0$. Note that $DJ(z_n + \psi(z_n))|_Y = 0$. By condition (a) of Theorem 1.3, $DJ(z_0 + y_0))|_Y = 0$. Now the uniqueness of $\psi(z_0)$ implies $\psi(z_0) = y_0$ and ψ is continuous.

(c) We prove j is of class C^1. Fix h in Z. Using (a) of this lemma,

$$\begin{aligned} J(z + th + P\psi(z + th) + Q\psi(z)) &- J(z + P\psi(z + th) + Q\psi(z)) \\ &\le J(z + th + \psi(z + th)) - J(z + \psi(z)) \\ &\le J(z + th + P\psi(z) + Q\psi(z + th)) - J(z + P\psi(z) + Q\psi(z + th)). \end{aligned} \tag{5.14}$$

By condition (b) of Theorem 1.3 and the continuity of ψ, we have

$$\begin{aligned} &\lim_{t\to 0} \frac{J(z + th + P\psi(z + th) + Q\psi(z)) - J(z + P\psi(z + th) + Q\psi(z))}{t} \\ &= \lim_{t\to 0} \int_0^1 DJ(z + sth + P\psi(z + th) + Q\psi(z))(h) ds = DJ(z + \psi(z))(h). \end{aligned} \tag{5.15}$$

Similarly,

$$\begin{aligned} \lim_{t\to 0} &\frac{J(z + th + P\psi(z) + Q\psi(z + th)) - J(z + P\psi(z) + Q\psi(z + th))}{t} \\ &= DJ(z + \psi(z))(h), \end{aligned}$$

By (5.14)–(5.16) we get

$$\lim_{t\to 0} \frac{J(z + th + \psi(z + th)) - J(z + \psi(z))}{t} = DJ(z + \psi(z))(h).$$

Therefore $Dj(z)h = DJ(z+\psi(z))(h)$ and the mapping $z \mapsto Dj(z)h$ is continuous on $B_Z(0, \delta_1)$.

Since Z is finite-dimensional, j is of class C^1 on $B_Z(0, \delta_1)$ and we get (c). □

PROOF OF THEOREM 1.3. Using Lemma 5.8 and Theorem 1.1 and arguing as the proof of Theorem 5.1 in [5], we get the theorem. □

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