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CONLEY INDEX IN HILBERT SPACES AND THE LERAY–SCHAUDER DEGREE

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Abstract. Let H be a real infinite dimensional and separable Hilbert space. With an isolated invariant set $\mathrm{inv}(N)$ of a flow ϕ^t generated by an $\mathcal{LS}$-vector field $f\colon H \supseteq \Omega \to H$, $f(x) = Lx + K(x)$, where $L\colon H \to H$ is strongly indefinite linear operator and $K\colon H \supseteq \Omega \to H$ is completely continuous, one can associate a homotopy invariant $h_{\mathcal{LS}}(\mathrm{inv}(N), \phi^t)$ called the $\mathcal{LS}$-Conley index. In fact, this is a homotopy type of a finite CW-complex. We define the Betti numbers and hence the Euler characteristic of such index and prove the formula relating these numbers to the Leray–Schauder degree $\deg_{\mathcal{LS}}(\widehat{f}, N, 0)$, where $\widehat{f} : H \supseteq \Omega \to H$ is defined as $\widehat{f}(x) = x + L^{-1}K(x)$.

1. Introduction

The purpose of this paper is to present certain generalization of the Poincaré–Hopf index theorem. This generalization is concerned with the infinite dimensional nonlinear analysis and occurs when we are working with infinite dimensional Conley-type invariant for flows. Let H be a real, infinite dimensional Hilbert space. With a locally Lipschitz vector field $f\colon H \supseteq \Omega \to H$, which is completely continuous perturbation of an isomorphism $L\colon H \to H$, $f(x) = Lx+K(x)$ we can associate a local flow $\phi^t_f\colon \Omega \to \Omega$ satisfying

$$\frac{d}{dt}\phi^t_f = -f \circ \phi^t_f, \qquad \phi^0_f = \mathrm{id}.$$

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Under certain assumptions we prove the formula

$$\deg_{\mathcal{LS}}(\widehat{f}, \mathrm{int}(N), 0) = \chi(h_{\mathcal{LS}}(\mathrm{inv}(N), \phi_f^t)). \tag{1.1}$$

The left-hand side of above equality stands for the standard Leray–Schauder degree with respect to a bounded set $\mathrm{int}(N)$ and 0. The map $\widehat{f}$ is defined by $\widehat{f}(x) = x + L^{-1}K(x)$. On the other hand we have the Euler characteristic. Here $h_{\mathcal{LS}}(\mathrm{inv}(N), \phi_f^t)$ is the Conley index of an isolated invariant set $\mathrm{inv}(N)$ of the flow ϕ_f^t.

The extension of the classical Conley's theory (for flows on locally compact metric spaces) we are going to work with, was introduced by K. Gęba, M. Izydorek and A. Pruszko in [6]. They considered so-called $\mathcal{LS}$-vector fields i.e. completely continuous perturbation of an isomorphism $L: H \to H$, and have defined Conley index for flows induced by such maps. One of the most important facts is that this index admits situations, where L is strongly indefinite, i.e. both stable and unstable eigenspaces of L are infinite dimensional. This property makes this theory applicable to many variational problems occurring in Hamiltonian dynamics.

Further development of this homotopy invariant was presented by Izydorek in [7]. He defined a cohomological Conley index in Hilbert spaces in order to obtain existance results for various strongly indefinite problems (variational problems, where gradient of action functional is $\mathcal{LS}$-vector field with strongly indefinite linear part). We briefly sketch out this definition. A cohomological version of the $\mathcal{LS}$-index allows us to define the Betti numbers and next the Euler characteristic of the index in the most natural way.

The proof of our theorem is based on finite dimensional case of the Poincaré–Hopf relation (1.1). It has been first proved by C. McCord [10] in terms of local indices of zeros of a vector field. Earlier N. Dancer in [4] proved this kind of relation for considerably smaller class of isolated invariant sets, precisely for degenerate critical points. A simple proof can be found in the book by K. Rybakowski [14] (See Chapter 3, Theorem 3.8). We present an elegant proof of this fact given by M. Razvan and M. Fotouhi in [11] based on Morse inequalities and Reineck continuation theorem [12]. Similar result was obtained by W. Kryszewski and A. Szulkin in [8] in terms of critical groups of smooth functional $\Phi: H \to \mathbb{R}$.

To gain insight into classical homotopy index theory we refer the reader to famous Conley's book [2] or Salamon's paper [Sal].

2. Classical Conley's theory

2.1 Finite dimensional case. First of all, we collect basic facts from Conley index theory for flows on a locally compact metric space X. Recall, that a continuous map $\phi\colon \mathbb{R}\times X \to X$ is called a *flow* on X if the following properties are satisfied:

(Fl.1) $\phi(0,x) = x$;
(Fl.2) $\phi(s,\phi(t,x)) = \phi(s+t,x)$ for all $s,t\in\mathbb{R}$.

We will interchangably write $\phi^t(x)$ and $\phi(t,x)$. Thus we have $\phi^0 = \mathrm{id}_X$ and $\phi^{t+s} = \phi^t\circ\phi^s$. The main objects of this theory are isolated invariant sets and associated with them isolating neighbourhoods. Let ϕ^t be a flow on X. A subset S of X is called an *invariant set*, if $S = \bigcup_{t\in\mathbb{R}}\phi^t(S)$. For $N\subset X$ we define

$$\mathrm{inv}(N) := \{x\in N : \phi^t(x)\in N,\ t\in\mathbb{R}\},$$

the maximal invariant set contained in N. If N is compact and $\mathrm{inv}(N)\subset\mathrm{int}(N)$, then N is called an *isolating neighbourhood* and $S = \mathrm{inv}(N)$ is an *isolated invariant set*.

Let N be a compact subset of X. We say that $L\subset N$ is *positively invariant relative to* N, if for any $x\in L$ the inclusion $\phi^{[0,t]}(x)\subset N$ implies that $\phi^{[0,t]}(x)\subset L$.

Definition 2.1 (Index pair). A compact pair (N,L) is called an *index pair* for S, if:

(IP.1) $N\setminus L$ is a neighbourhood of S and $S = \mathrm{inv}(\mathrm{cl}(N\setminus L))$;
(IP.2) L positively invariant relative to N;
(IP.3) if $x\in N$ and there exists $t>0$, such that $\phi^t(x)\notin N$, then there exists $s\in[0,t]$, such that $\phi^s(x)\in L$.

The next two theorems are crucial in the definition of homotopy Conley index. The proofs can be found in Salamon's paper [Sal].

Theorem 2.2. *Every isolated invariant set S admits an index pair (N,L).*

If (N,L) is a pair of spaces, $L\subset N$, then the quotient N/L is obtained from N by collapsing L to a single point denoted by $[L]$, the base point of N/L. A set $X\subset N/L$ is open if either X is open in L and $X\cap L=\emptyset$ or the set $(X\cap N\setminus L)\cup L$ is open in N.

Recall that $f\colon(X,x_0)\to(Y,y_0)$ is a *homotopy equivalence* if there exists a map $g\colon(Y,y_0)\to(X,x_0)$ such that $g\circ f$ is homotopic to $\mathrm{id}_{|X}$ rel. x_0 and $f\circ g$ is homotopic to $\mathrm{id}_{|Y}$ rel. y_0. If there is a homotopy equivalence $f\colon(X,x_0)\to(Y,y_0)$ we say that the pairs (X,x_0) and (Y,y_0) are *homotopy equivalent* or they have the same *homotopy type*. The homotopy type of (X,x_0) is denoted by $[X,x_0]$.

Theorem 2.3. *Let (N_0,L_0) and (N_1,L_1) be two index pairs for the isolated invariant set S. Then the pointed topological spaces N_0/L_0 and N_1/L_1 are homotopy equivalent.*

Definition 2.4. If (N, L) is any index pair for the isolated invariant set S, then the homotopy type $h(S, \phi^t) = [N/L]$ is said to be the Conley (homotopy) index of S.

Theorem 2.3 says that $h(S, \phi^t)$ is independent of the choice of index pair. Let us illustrate the concept of Conley index by the following simple example.

Example 2.5. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set and $f\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that $\nabla f^{-1}(0) \notin \partial\Omega$. Assume that f generates a flow ϕ_f^t on $\mathbb{R}^n$ defined by

$$\frac{d}{dt}\phi_f^t = \nabla f \circ \phi_f^t, \quad \phi_f^0 = \mathrm{id}.$$

The rest points of ϕ_f^t are the critical points of f. They are hyperbolic if f is a Morse function i.e. the Hessian of f is nonsingular at every $x \in \mathrm{Crit}(f)$, where $\mathrm{Crit}(f) = \{x \in \mathbb{R}^n : Df(x) = 0\}$. In this case the number

$$\mathrm{ind}_f(x) = \#\{\text{negative eigenvalues of the Hessian } \nabla^2 f(x)\}$$

is well defined. The Conley index of an isolated invariant set $S = \{x\}$, where $x \in \mathrm{Crit}(f)$, is the homotopy type of pointed k-sphere, where $k = n - \mathrm{ind}_f(x)$.

A *Morse decomposition* of an isolated invariant set S is a finite collection

$$\mathcal{M}(S) = \{M_i : 1 \le i \le l\}$$

of subsets $M_i \subset S$, which are disjoint, compact and invariant, and which can be ordered $(M_1, \dots, M_l)$ so that for every $x \in S \setminus \bigcup_{1 \le j \le l} M_j$ there are indices $i < j$ such that

$$\omega(x) \subset M_i, \quad \alpha(x) \subset M_j.$$

Notice that in the previous example the set $\mathrm{Crit}(f)$, of all critical points of f, forms a Morse decomposition of $\mathrm{inv}(\Omega)$.

The formal power series

$$\mathcal{P}(t, A, B) = \sum_{q \in \mathbb{Z}} \mathrm{rank}\, H^q(A, B) \cdot t^q$$

is called the *Poincaré series of a pair* (A, B). One can prove, that for an isolated invariant set there is an index pair (N, L) for which the isomorphism $H^*(N, L) \cong H^*(N/L)$ holds. Such an index pair is called *regular* (See Section 5 of [Sal]). We can therefore define the Poincaré polynomial for S as

$$\mathcal{P}(t, h(S, \phi^t)) := \mathcal{P}(t, N, L)$$

where (N, L) is any regular index pair for S. The next theorem is a generalization of the classical Morse inequalities.

Theorem 2.6 (cf. [3], [7]). *Let S be an isolated invariant set with a Morse decomposition $\mathcal{M}(S) = \{M_i : 1 \le i \le l\}$. Then there is a polynomial $\mathcal{Q}$ with nonnegative coefficients such that*

$$\sum_{i=1}^{l} \mathcal{P}(t, h(M_i)) = \mathcal{P}(t, h(S)) + (1+t)\mathcal{Q}(t).$$

2.2. Continuation to a gradient. Let $\phi: \mathbb{R} \times X \times [0,1] \to X$ be a continuous family of flows on X, i.e. $\phi^t_\lambda := \phi(t, \cdot, \lambda): X \to X$ is a flow on X. Suppose that $N \subset X$ is compact and $S_i = \operatorname{inv}(N, \phi^t_i)$, $i = 0, 1$. We say that two isolated invariant sets S_0 and S_1 are *related by continuation* or S_0 *continues to* S_1 if N is an isolating neighbourhood for all ϕ^t_λ for $\lambda \in [0,1]$. The notion of continuation is essential in the Conley index theory because of the following statement.

Theorem 2.7 ([2]). *If S_0 and S_1 are related by continuation, then their Conley indices coincide.*

Recall that a Morse–Smale gradient flow is one where (i) all bounded orbits are either critical points of the potential function or orbits connecting two critical points; (ii) stable and unstable manifolds of the rest points intersect transversally.

Let $F: \Omega \to \mathbb{R}^n$ be a smooth vector field with $\Omega \subset \mathbb{R}^n$ open. Without loss of generality we can assume that F generates the flow $\phi^t_F: \Omega \to \Omega$ by a differential equation $\dot{x}(t) = -F(x(t))$. Let N be an isolating neighbourhood and $S = \operatorname{inv}(N)$.

Theorem 2.8 (Reineck [12]). *The set S can be continued to an isolated invariant set of a positive gradient flow of ∇f, without changing F on $\Omega \setminus N$. Moreover, this can be done that the new flow is Morse–Smale.*

Remarks 2.9. The fact that such function f exists has been proved by Robbin and Salamon in [13]. They showed that for an isolated invariant set $S = \operatorname{inv} N$ there exists a smooth function $f: U \to \mathbb{R}$ defined on a neighbourhood of N such that

(a) $f(x) = 0$ if and only if $x \in S$ and
(b) $(d/dt)|_{t=0} f(\phi^t(x)) < 0$ for all $x \in \Omega \setminus S$.

The function which fulfils those properties is called the Lyaponov function. In general we cannot expect that for an isolated invariant set its Lyaponov function would have only nondegenerate critical points, i.e. the rest points of gradient flow are hyperbolic. But this can be obtained via arbitrary small perturbation of ∇f. So without loss of generality we can assume that the gradient flow is Morse–Smale.

Following Reineck, we can explicitly write the homotopy joining $-F$ and ∇f. We define $h\colon \Omega \times [0,1] \to \mathbb{R}^n$ as

$$h(x,\lambda) = \rho(x)[\lambda \nabla f(x) + (\lambda - 1)F(x)] + (\rho(x) - 1)F(x), \tag{2.1}$$

where $\rho\colon \Omega \to [0,1]$ is smooth function equal 1 on a compact neighbourhood of S, say M ($\mathrm{cl}(M) \subset \mathrm{int}(N)$) and ρ is zero on $\Omega \setminus N$.

2.3. Euler characteristic of $h(\mathrm{inv}(N), \phi^t)$. Recall that the Euler characteristic of a pair (E, E') is defined as

$$\chi(E,E') = \sum_{q\in\mathbb{Z}} (-1)^q \mathrm{rank}\, H^q(E,E').$$

Notice that $\chi(E,E') = \mathcal{P}(-1,E,E')$. If both $H^q(E)$ and $H^q(E')$ are finitely generated (e.g. if E and E' are CW-complexes) the integer $\chi(E,E')$ is well defined. In particular if E' is a point in E (that is, E is a pointed space), then we have $\chi(E,*) = \sum_{q\in\mathbb{Z}}(-1)^q \mathrm{rank}\, \widetilde{H}^q(E)$, where $\widetilde{H}^q(E)$ stands for the reduced cohomology. Note that χ is independent of principal ideal domain used for define cohomology groups.

The Euler characteristic is defined especially for the Conley index of an isolated invariant set for flows generated by equation $\dot{x} = -F(x)$.

The next proposition is due to Gęba (see Proposition 5.6 of [5]).

PROPOSITION 2.10. *Let N be an isolating neighbourhood for gradient Morse–Smale flow ϕ^t. Then $h(\mathrm{inv}(N), \phi^t)$ is a homotopy type of finite* CW*-complex.*

COROLLARY 2.11. *Let N be an isolating neighbourhood for flow ϕ^t generated by $\dot{x} = -F(x)$. Then $h(\mathrm{inv}(N), \phi^t)$ is a homotopy type of finite* CW*-complex.*

PROOF. Since $\mathrm{inv}(N)$ is related by continuation to some isolated invariant set of gradient Morse–Smale flow, the result follows from Proposition 2.10. □

2.4. Mapping degree. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set. If $f\colon \mathrm{cl}\Omega \to \mathbb{R}^n$ is continuous map and does not vanish on the boundary $\partial\Omega$, then it is well known, that there is defined an integer $\deg(f,\Omega,0) \in \mathbb{Z}$ called the Brouwer degree ([9]). Now, we will formulate only a few fundamental facts about the degree:

(a) (Nontriviality) If $0 \in \Omega$ then $\deg(I,\Omega,0) = 1$;
(b) (Existence) If $\deg(f,\Omega,0) \neq 0$ then $f^{-1}(0) \cap \Omega$ is nonempty;
(c) (Additivity) If Ω_1, Ω_2 are open, disjoint subsets of Ω and there is no zeros of f in the completion $\Omega \setminus (\Omega_1 \cup \Omega_2)$, then

$$\deg(f,\Omega,0) = \deg(f,\Omega_1,0) + \deg(f,\Omega_2,0);$$

(d) (Homotopy invariance) If $h\colon \mathrm{cl}\Omega \times [0,1] \to \mathbb{R}^n$ is a continuous map such that $h(x,t) \neq 0$ for all $(x,t) \in \partial\Omega \times [0,1]$, then

$$\deg(h(\,\cdot\,,0),\Omega,0) = \deg(h(\,\cdot\,,1),\Omega,0).$$

There is a generic situation, when the degree is easy to calculate. If $\varphi\colon \mathrm{cl}\Omega \to \mathbb{R}$ is a Morse function of class C^1, such that $\deg(\nabla\varphi,\Omega,0)$ is defined, then

$$\deg(\nabla\varphi,\Omega,0) = \sum_{x\in\nabla\varphi^{-1}(0)\cap\Omega} (-1)^{\mathrm{ind}_\varphi(x)}.$$

Theorem 2.12 (cf. [11]). *Suppose that N is an isolating neighbourhood for the flow ϕ_F^t generated by $\dot{x} = -F(x)$, where $F\colon \Omega \to \mathbb{R}^n$ is locally Lipschitz map. Then*

$$\chi(h(\mathrm{inv}(N),\phi_F^t)) = \deg(F,\mathrm{int}(N),0). \tag{2.2}$$

Notation. Now and subsequently we will at times write $\deg(F,N,0)$ instead of $\deg(F,\mathrm{int}(N),0)$.

Proof. By the Reineck continuation theorem we can deform $-F$ to ∇f on N using (2.1) to obtain the isolated invariant set of gradient flow, which consists of only non-degenerate critical points of f and connecting orbits between them. Denote this set by $\mathrm{inv}_{\phi_f^t}(N)$. By the continuation property of the Conley index we have $h(\mathrm{inv}(N),\phi_F^t) = h(\mathrm{inv}_{\phi_f^t}(N),\phi_f^t)$. The set of critical points $\{x_1,\ldots,x_m\}$ forms a Morse decomposition of $\mathrm{inv}_{\phi_f^t}(N)$ and we can apply Theorem 2.6. We know that $h(\{x_i\},\phi_f^t)$ has a homotopy type of pointed k-sphere, where $k = n - \mathrm{ind}_f(x_i)$. So the Poincaré polynomial of $h(\{x_i\},\phi_f^t)$ is of the form

$$\mathcal{P}(t,h(\{x_i\},\phi_f^t)) = t^{n-\mathrm{ind}_f(x_i)}.$$

From the Morse inequalities we have

$$\begin{aligned}\chi(h(\mathrm{inv}(N),\phi_F^t)) &= \chi(h(\mathrm{inv}_{\phi_f^t}(N),\phi_f^t)) = \mathcal{P}(-1,h(\mathrm{inv}_{\phi_f^t}(N),\phi_f^t)) \\ &= \sum_{i=1}^m \mathcal{P}(-1,h(\{x_i\},\phi_f^t)) = (-1)^n \sum_{i=1}^m (-1)^{\mathrm{ind}_f(x_i)}.\end{aligned} \tag{2.3}$$

For $1 \le i \le m$, let Ω_i be the neighbourhood of x_i i N such that $\Omega_i \cap \Omega_j = \emptyset$. By the homotopy invariance of the Brouwer degree and additive property we can write

$$\deg(-F,N,0) = \deg(\nabla f,N,0) = \sum_{i=1}^m \deg(\nabla f,\Omega_i,0). \tag{2.4}$$

Now it is easy to compute $\deg(\nabla f,\Omega_i,0)$. Since f is Morse, the hessian $\nabla^2 f(x_i)$ is non-degenerate linear operator. The degree of ∇f with respect to Ω_i is just

$(-1)^\mu$, where μ is the number of negative eigenvalues of $\nabla^2 f(x_i)$. So we have $\deg(\nabla f, \Omega_i, 0) = (-1)^{\mathrm{ind}_f(x_i)}$, and by (2.4)

$$\deg(F, N, 0) = (-1)^n \deg(-F, N, 0) = (-1)^n \sum_{i=1}^{m} (-1)^{\mathrm{ind}_f(x_i)} \tag{2.5}$$

Comparing (2.3) and (2.5) we obtain the formula (2.2). □

3. $\mathcal{LS}$-index

3.1. $\mathcal{LS}$-flows and the index. Let H be a real, separable Hilbert space and $L: H \to H$ be a linear bounded operator which satisfies following assumptions:

- (L.1) L gives a splitting $H = \bigoplus_{n=0}^{\infty} H_n$ onto finite dimensional, mutually orthogonal L-invariant subspaces;
- (L.2) $\dim H_0 < \infty$, where H_0 is subspace corresponding to the part of spectrum on imaginary axis, i.e. $\sigma_0(L) := \sigma(L_{|H_0}) = \sigma(L) \cap i\mathbb{R}$;
- (L.3) $\sigma_0(L)$ is isolated in $\sigma(L)$.

We do not preclude the case $\dim H_\pm = \infty$, where H_- (resp. H_+) is invariant subspace corresponding to those part of spectrum of L which lies on the left (resp. right) half complex plane. Operators with above property are called *strongly indefinite*.

Let Λ be a compact metric space. A *family of flows indexed by* Λ is a continuous map $\phi: \mathbb{R} \times H \times \Lambda \to H$ such that $\phi_\lambda: \mathbb{R} \times H \to H$ defined by $\phi_\lambda(t, x) = \phi(t, x, \lambda)$ is a flow on H. As before we write $\phi^t(x, \lambda)$ instead of $\phi(t, x, \lambda)$. If $X \subset H$ and ϕ is a family of flows indexed by Λ then we define

$$\mathrm{inv}(X \times \Lambda) = \mathrm{inv}(X \times \Lambda, \phi) := \{(x, \lambda) \in X \times \Lambda : \phi^t(x, \lambda) \in X,\ t \in \mathbb{R}\}.$$

Definition 3.1. A family of flows $\phi^t: H \times \Lambda \to H$ is called a *family of $\mathcal{LS}$-flows* if

$$\phi^t(x, \lambda) = e^{tL} x + U(t, x, \lambda),$$

where $U: \mathbb{R} \times H \times \Lambda \to H$ is completely continuous.

Recall, that a map is completely continuous if it is continuous and maps bounded sets to relatively compact sets.

Definition 3.2. We say that a map $f: H \times \Lambda \to H$ is a *family of $\mathcal{LS}$-vector fields*, if f is of the form

$$f(x) = Lx + K(x, \lambda), \quad (x, \lambda) \in H \times \Lambda,$$

where $K: H \times \Lambda \to H$ is completely continuous and locally Lipschitz map.

If in the above definitions $\Lambda = \{\lambda_0\}$, we drop the parameter space out from notation, and we are talking about *$\mathcal{LS}$-flows* or *$\mathcal{LS}$-vector fields*.

Suppose that $f: H \to H$ is an $\mathcal{LS}$-vector field, $f(x) = Lx + K(x)$. We say that f is *subquadratic* if $|\langle K(x), x\rangle| \le a\|x\|^2 + b$ for some $a, b > 0$. One can prove that if f is subquadratic then f generates an $\mathcal{LS}$-flow (see [7] and references therein). That is for all $x \in H$, there exists a C^1-curve $\phi^{(\cdot)}(x): \mathbb{R} \to H$ satisfying

$$\frac{d}{dt}\phi^t(x) = -f \circ \phi^t(x), \quad \phi^0(x) = x,$$

and is of the form $\phi^t(x) = e^{-tL}x + U(t, x)$, where $U: \mathbb{R} \times H \to H$ is completely continuous. Without loss of generality we will restrict our consideration to subquadratic $\mathcal{LS}$-vector fields.

An isolating neighbourhood for a flow ϕ^t on infinite dimensional space is defined similarly to finite dimensional case. The difference lies in the fact that we cannot expect compactness of that set.

Definition 3.3. A bounded and closed set N is an isolating neighbourhood for a flow ϕ^t if and only if $\operatorname{inv}(N) \subset \operatorname{int}(N)$.

The isolating neighbourhoods are stable with respect to small perturbation of the flow. The sense of this concept is given by the following.

Proposition 3.4 (Gęba et al. [6]). *Let $\phi: \mathbb{R} \times H \times \Lambda \to H$ be a family of $\mathcal{LS}$-flows. For any bounded and closed $N \subset H$ the set*

$$\Lambda(N) = \{\lambda \in \Lambda : \operatorname{inv}(N, \phi_\lambda) \subset \operatorname{int}(N)\}$$

is open in Λ.

We are going to work in the category of compact metrizable spaces with a base point. The notion $f: (X, x_0) \to (Y, y_0)$ means that f is a continuous map preserving base points, i.e. $f(x_0) = y_0$. The product is defined in this category by $(X, x_0) \times (Y, y_0) = (X \times Y, (x_0, y_0))$. The *wedge* of two pointed spaces i.e. the space $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y$ is closed in $X \times Y$. Hence, the *smash product* $X \wedge Y = (X \times Y)/(X \vee Y)$ is also an object in that category.

Consider the circle as the unit interval modulo its end points $S^1 = [0, 1]/\{0, 1\}$. The suspension functor is defined to be the smash product

$$S(X, x_0) := S^1 \wedge (X, x_0).$$

For any $m \in \mathbb{N}$ we define

$$S^m(X, x_0) := S(S^{m-1}(X, x_0)).$$

$\mathcal{LS}$-index is defined as a sequence of pointed spaces with an extra information added. This leads us to a notion of spectra. Let $\nu: \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ be a fixed map and suppose that $(E_n)_{n=n(E)}^{\infty}$ is a sequence of pointed spaces and $(\varepsilon_n: S^{\nu(n)}E_n \to E_{n+1})_{n=n(E)}^{\infty}$ is a sequence of maps.

Definition 3.5. We say that a pair $E = ((E_n)_{n=n(E)}^{\infty}, (\varepsilon_n)_{n=n(E)}^{\infty})$ is a *spectrum* if there exists $n_0 \geq n(E)$ such that $\varepsilon_n: S^{\nu(n)} E_n \to E_{n+1}$ is a homotopy equivalence for all $n \geq n_0$.

One can define the notion of maps of spectra, homotopy of spectra, their homotopy type etc. For us it is sufficient to know that a homotopy type $[E]$ of a spectrum E is uniquely determined by a homotopy type of a pointed space E_n for n sufficiently large. Moreover, in order to define the homotopy type $[E]$ one only needs a sequence $(E_n)_{n=n(E)}^{\infty}$ such that $S^{\nu(n)} E_n$ is homotopy equivalent to E_{n+1} for n sufficiently large.

Assume that $f: H \to H$ is an $\mathcal{LS}$-vector field, $f(x) = Lx + K(x)$. Let $\phi^t: H \to H$ be the $\mathcal{LS}$-flow generated by f and assume that $N \subset H$ is an isolating neighbourhood for ϕ^t. Denote by $P_n: H \to H$ the orthogonal projection onto $H^n = \bigoplus_{i=1}^n H_i$. Define

$$f_n: H^n \to H^n, \quad f_n(x) = Lx + P_n K(x).$$

Let $\phi_n^t: H \to H$ be a flow induced by f_n. The definition of $\mathcal{LS}$-Conley index is based on the following.

Lemma 3.6 (Gęba et al. [6]). *There exists $n_0 \in \mathbb{N}$ such that $N^n = N \cap H^n$ is an isolating neighbourhood for a flow ϕ_n^t provided that $n \geq n_0$.*

By the above lemma the set $\operatorname{inv}(N^n, \phi_n^t)$ is an isolated and invariant (by definition) and thus admits an index pair (Y_n, Z_n) by Theorem 2.2. The Conley index of $\operatorname{inv}(N^n)$ is the homotopy type $[Y_n/Z_n]$. Fix a map $\nu: \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ by setting $\nu(n) := \dim H_{n+1}^-$. Using the continuation property of the Conley index one can prove that the pointed space Y_{n+1}/Z_{n+1} is in fact homotopy equivalent to $\nu(n)$-fold suspension of Y_n/Z_n, that is

$$[Y_{n+1}/Z_{n+1}] = [S^{\nu(n)}(Y_n/Z_n)]$$

for all $n \geq n_0$. In the light of earlier observation the sequence $(E_n)_{n=n_0}^{\infty} = (Y_n/Z_n)_{n=n_0}^{\infty}$ represents the spectrum, say E and uniquely determines its homotopy type $[E]$. This leads us to the definition.

Definition 3.7. Let ϕ^t be an $\mathcal{LS}$-flow generated by an $\mathcal{LS}$-vector field. If N is an isolating neighbourhood for ϕ^t, then the homotopy type of spectrum

$$h_{\mathcal{LS}}(\operatorname{inv}(N), \phi^t) := [E]$$

is well defined and we call it the $\mathcal{LS}$-Conley index of $\operatorname{inv}(N)$ with respect to ϕ^t.

Let $\mathbf{0}$ represents the homotopy type of spectrum such that for all $n \geq 0$ E_n is just a point and ε_n maps this point into the point in E_{n+1}.

Proposition 3.8 (Gęba et al. [6]). *The $\mathcal{LS}$-Conley index has the following properties:*

(a) (Nontriviality) *Let $\phi^t\colon H \to H$ be an $\mathcal{LS}$-flow and $N \subset H$ be an isolating neighbourhood for ϕ^t. If $h_{\mathcal{LS}}(\mathrm{inv}(N), \phi^t) \neq \mathbf{0}$, then* $\mathrm{inv}(N, \phi^t) \neq \emptyset$*;*

(b) (Continuation) *Let Λ be a compact, connected and locally contractible metric space. Assume that $\phi^t\colon H \times \Lambda \to H$ is a family of $\mathcal{LS}$-flows. Let N be an isolating neighbourhood for a flow ϕ^t_λ for some $\lambda \in \Lambda$. Then there is a compact neighbourhood $\mathcal{U}_\lambda \subset \Lambda$ such that*

$$h_{\mathcal{LS}}(\mathrm{inv}(N), \phi^t_\mu) = h_{\mathcal{LS}}(\mathrm{inv}(N), \phi^t_\nu) \quad \textit{for all } \mu, \nu \in \mathcal{U}_\lambda.$$

3.2. Cohomological $\mathcal{LS}$-Conley index. The main reference for this section is [7]. Now and subsequently H denotes the Alexander–Spanier cohomology functor. Let $E = (E_n, \varepsilon_n)_{n=n(E)}^{\infty}$ be a spectrum. Define $\rho\colon \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ by setting $\rho(0) = 0$ and $\rho(n) = \sum_{i=0}^{n-1} \nu(i)$ for $n \geq 1$. For a fixed $q \in \mathbb{Z}$ consider a sequence of cohomology groups

$$H^{q+\rho(n)}(E_n), \quad n \geq n(E).$$

Define a sequence of homomorphisms h_n by a composition in the following diagram

$$\begin{array}{ccc} H^{q+\rho(n+1)}(E_{n+1}) & \xrightarrow{\quad h_n \quad} & H^{q+\rho(n)}(E_n) \\ & {\scriptstyle \varepsilon_n^{q+\rho(n+1)}} \searrow \qquad \nearrow {\scriptstyle (S^*)^{-\nu(n)}} & \\ & H^{q+\rho(n+1)}(S^{\nu(n)}E_n) & \end{array}$$

where S^* denotes the suspension isomorphism. Thus we see that $\{H^{q+\rho(n)}(E_n), h_n\}$ forms an inverse system and we are ready to make the following definition.

Definition 3.9. The q^{th} cohomology group of a spectrum E is defined to be

$$CH^q(E) := \varprojlim \{H^{q+\rho(n)}(E_n), h_n\}.$$

Since E_{n+1} is homotopically equivalent to $S^{\nu(n)}E_n$ for $n \geq n_0$, we see that

$$h_n\colon H^{q+\rho(n+1)}(E_{n+1}) \to H^{q+\rho(n)}(E_n)$$

is an isomorphism for $n \geq n_0$ and the sequence of groups $H^{q+\rho(n)}(E_n)$ stabilizes. This simply observation implies that:

- $CH^q(E) \cong H^{q+\rho(n)}(E_n)$ for $n \geq n_0$;
- the graded group $CH^*(E)$ is finitely generated if $H^*(E_{n_0})$ is finitely generated;
- the spectrum E is of finite type if the space E_{n_0} is of finite type.

These groups may be nonzero for positive and also negative integers (see [7]).

Now we are able to define Betti numbers and Euler characteristic of an $\mathcal{LS}$-Conley index represented by the spectrum E in the obvious way.

Definition 3.10. Let E be a fixed spectrum. The q-th Betti number of E is defined as

$$\beta_q(E) := \operatorname{rank} CH^q(E),$$

and the Euler characteristic is given by

$$\chi(E) := \sum_{q\in\mathbb{Z}} (-1)^q \beta_q(E).$$

Remark 3.11. There exist n_0 such that for all $n \geq n_0$ we have $\chi(E) = (-1)^{\rho(n)}\chi(E_n)$.

Proof. Since $CH^q(E) \cong H^{q+\rho(n)}(E_n)$ for $n \geq n_0$ we have

$$(-1)^{\rho(n)}\chi(E) = (-1)^{\rho(n)} \sum_{q\in\mathbb{Z}} (-1)^q \beta_q(E) = \sum_{q\in\mathbb{Z}} (-1)^{q+\rho(n)} \beta_{q+\rho(n)}(E_n) = \chi(E_n).$$

□

4. Relationship between $\mathcal{LS}$-index and degree

4.1. The Leray–Schauder degree with respect to L. Let U be an open and bounded subset of H. Denote by $\deg_{\mathcal{LS}}(f, U, 0)$ the Leray–Schauder degree, defined for completely continuous perturbation of identity. For more details about degree theory see [9]. Consider an $\mathcal{LS}$-vector field f in H, $f(x) = Lx + K(x)$, where L is strongly indefinite linear bounded and invertible operator and K is completely continuous map. Suppose that f does not vanish on ∂U. We will define degree for the class of such maps in the following manner:

$$\deg_L(f, U, 0) := \deg_{\mathcal{LS}}(I + L^{-1}K, U, 0).$$

Since the zero sets for both f and $I + L^{-1}K$ are the same and $L^{-1}K$ is completely continuous map, the above definition works. The $\deg_L$ inherits all the properties of the Leray–Schauder degree. In particular one has:

(a) (Nontriviality) If $0 \in U$ then $\deg_L(L, U, 0) = 1$;
(b) (Existence) If $\deg_L(f, U, 0) \neq 0$ then f has a zero inside U;
(c) (Additivity) If U_1, U_2 are open, disjoint subsets of U and there are no zeros of f in the completion $U \setminus (U_1 \cup U_2)$, then

$$\deg_L(f, U, 0) = \deg_L(f, U_1, 0) + \deg_L(f, U_2, 0);$$

(d) (Homotopy invariance) If $h\colon H \times [0,1] \to H$ is an $\mathcal{LS}$-vector field for all $t \in [0,1]$ such that $h(x,t) \neq 0$ for all $(x,t) \in \partial U \times [0,1]$, then $\deg_L(h(\,\cdot\,, t), U, 0))$ is independent of $t \in [0,1]$.

4.2. Main theorem.

Theorem 4.1. *Let H be a real Hilbert space and $L\colon H \to H$ be a linear isomorphism satisfying assumptions* (L.1)–(L.4). *Assume that $\Omega \subseteq H$ is open and bounded, and $f\colon \Omega \to H$ is of the form $f(x) = Lx + K(x)$, where $K\colon \Omega \to H$ is completely continuous map of class C^1; ϕ^t is the local flow of equation $\dot{x} = -f(x)$ and N is an isolating neighbourhood for ϕ^t. Then we have*

$$\chi(h_{\mathcal{LS}}(\mathrm{inv}(N), \phi^t)) = \deg_L(f, \mathrm{int}(N), 0). \tag{4.1}$$

Proof. Suppose that $h_{\mathcal{LS}}(\mathrm{inv}(N), \phi^t)$ is represented by spectrum

$$E = (E_n, \varepsilon_n)_{n \geq n(E)}.$$

Assume that n_0 is chosen such that $\chi(E) = (-1)^{\rho(n)}\chi(E_n)$ and

$$\deg_{\mathcal{LS}}(I + L^{-1}K, N, 0) = \deg(I + P_n L^{-1}K, N^n, 0)$$

for all $n \geq n_0$. According to finite dimensional formula (2.2)

$$(-1)^{\rho(n)}\chi(E_n) = (-1)^{\rho(n)} \deg(L + P_n K, N^n, 0).$$

thus

$$\begin{aligned}
\chi(E) &= (-1)^{\rho(n)}\chi(E_n) = (-1)^{\rho(n)} \deg(L + P_n K, N^n, 0) \\
&= (-1)^{\rho(n)} \deg L_{|H^n} \cdot \deg(I + P_n L^{-1}K, N^n, 0) \\
&= \deg(I + P_n L^{-1}K, N^n, 0) = \deg_{\mathcal{LS}}(I + L^{-1}K, N, 0) = \deg_L(f, N, 0),
\end{aligned}$$

since the degree of linear isomorphism $L_{|H^n}$ with respect to 0 is $(-1)^{\nu}$, where ν is the number of negative eigenvalues of L. But in this case it is exactly $\dim H^n_- = \sum_{i=1}^{n} \dim H_i^- = \sum_{i=0}^{n-1} \nu(i) = \rho(n)$. This completes the proof. □

4.3. L is not an isomorphism. Now consider weaker assumption about an operator $L\colon H \to H$. We would like to admit the case, when L is not invertible operator but is selfadjoint, i.e. $\langle Lx, y\rangle = \langle x, Ly\rangle$ for all $x, y \in H$. Let $P_0\colon H \to H$ denote the orthogonal projection onto H_0, the kernel of L. Define a map $\widehat{L}\colon H \to H$ by $\widehat{L}x := Lx + P_0 x$. Since the kernel of L is orthogonal to the image of L we see, that $\widehat{L}$ is an isomorphism. In particular, if L is invertible, then $\widehat{L} = L$. If f is a vector field of the form $Lx + K(x)$, where K is completely continuous, we can write it equivalently as

$$f(x) = \widehat{L}x + \widehat{K}(x),$$

where $\widehat{K}(x) = K(x) - P_0 x$. Note that $\widehat{K}$ is completely continuous as well, since $\dim H_0 < \infty$. As before for an open bounded subset $U \subset H$ and $\mathcal{LS}$-vector field $f = L + K$ such that $0 \notin f(\partial U)$ we set

$$\deg_L(f, U, 0) := \deg_{\mathcal{LS}}(I + \widehat{L}^{-1}\widehat{K}, U, 0).$$

Proposition 4.2. *Let assumptions of Theorem* 4.1 *be satisfied. Suppose that* $L\colon H \to H$ *is selfadjoint* (*instead of isomorphism*). *Then the equality* (4.1) *is valid.*

Proof. If L is selfadjoint then

$$\deg(L + P_n K, N^n, 0) = \deg(\widehat{L} + P_n \widehat{K}, N^n, 0),$$

since $P_n P_0 = P_0$ and L preserves the splitting of $H = \bigoplus_{n=1}^{\infty} H_n$. Next

$$\deg(\widehat{L} + P_n \widehat{K}, N^n, 0) = \deg \widehat{L}_{|H^n} \cdot \deg(I + P_n \widehat{L}^{-1} \widehat{K}, N^n, 0).$$

Observe that $\deg \widehat{L}_{|H^n} = (-1)^{\rho(n)}$. Indeed, the number of negative eigenvalues of L and $\widehat{L}$ coincide, because $\widehat{L}$ differs from L only on the kernel of L by identity. That is there are only the $\lambda = 1$ of multiplicity $\dim H_0$ added to spectrum of L in places of zeros. The $\deg(I + P_n \widehat{L}^{-1} \widehat{K}, N^n, 0)$ stabilizes for large n and represents $\deg_{\mathcal{LS}}(I + \widehat{L}^{-1} \widehat{K}, N, 0)$. In the light of the proof of preceding theorem it gives us the required result. □

In fact this theorem can be formulated for much bigger class of operators L. It is easy to see that L is admissible if $H = \ker L \oplus \operatorname{im} L$, where $\oplus$ means a direct sum (not orthogonal). This condition allows us to define the $\deg_L$.

5. Particular case

5.1. Finite-dimensional approximation. In this section the equality (4.1) will be obtained via direct calculation, in the case when $L = (-I, I)\colon H_- \oplus H_+ \to H_- \oplus H_+$, and S is an isolated zero of a given vector field.

We say, that an operator sequence $\{P_n\}_{n=1}^{\infty}$, $P_n\colon H \to H$ is *strongly convergent* to the identity operator $I\colon H \to H$, if $\lim_{n\to\infty} P_n x = x$ for all $x \in H$.

Lemma 5.1. *If* $K\colon H \to H$ *is compact operator and* $P_n\colon H \to H$, $n = 1, 2, \ldots$ *is a sequence of orthogonal projections onto* H^n *strongly convergent to the identity, then*

(a) $\lim_{n\to\infty} \|P_n K - K\| = 0$;
(b) $\lim_{n\to\infty} \|P_n K P_n - K\| = 0$;
(c) $\lim_{n\to\infty} \|Q_n K\| = 0$, *where* $Q_n\colon H \to H$ *is an orthogonal projection onto* H_n.

Proof. Statement (a) is a well known fact from the Riesz–Schauder theory. Since

$$\|P_n K P_n - K\| \le \|P_n K P_n - P_n K\| + \|P_n K - K\|$$

and $P_n K$ is compact, in order to prove (b) it is enough to show that for any compact A we have $\lim_n \|A P_n - A\| = 0$. If A is compact, then the adjoint operator A^* is compact as well and we may write $\|AP_n - A\| = \|(AP_n - A)^*\| = \|P_n A^* - A^*\| \to 0$. Finally, we have an estimation

$$0 \le \|Q_n K\| \le \left\| \left(\sum_{i=n}^{\infty} Q_i \right) K \right\| = \|(I - P_{n-1})K\| < \varepsilon$$

provided $n \ge n_0$. This proves (c). □

Definition 5.2. We say that $A \in \mathfrak{B}(H)$ is *hyperbolic*, if

$$\operatorname{dist}(\sigma(A), i\mathbb{R}) := \inf_{\lambda \in \sigma(A),\, x \in i\mathbb{R}} d(x, \lambda) > 0.$$

The set of all hyperbolic operators will be denoted by $\mathfrak{B}_{\mathrm{hip}}(H)$.

Here $d(\,\cdot\,,\,\cdot\,)$ stands for the distance function on $\mathbb{C}$.

Recall, that the multivalued map $\mathfrak{B}(H) \ni A \mapsto \sigma(A) \subset \mathbb{C}$ is *upper semi continuous*, that is for all $A \in \mathfrak{B}(H)$ and $\epsilon > 0$, there exists $\delta > 0$, such that inequality $\|A - B\| < \delta$ implies $\sup_{\lambda \in \sigma(B)} \operatorname{dist}(\lambda, \sigma(A)) < \epsilon$.

Lemma 5.3. *$\mathfrak{B}_{\mathrm{hip}}(H)$ is an open subset of $\mathfrak{B}(H)$.*

Proof. Set $\rho := \operatorname{dist}(\sigma(A), i\mathbb{R})$. There exists $\delta > 0$ such that for all B in δ-neighbourhood of A

$$\sup_{\lambda \in \sigma(B)} \operatorname{dist}(\lambda, \sigma(A)) < \rho/2.$$

Thus, the triangle inequality gives us the following estimation

$$\begin{aligned} \operatorname{dist}(\sigma(B), i\mathbb{R}) &= \inf_{\mu \in \sigma(B),\, x \in i\mathbb{R}} d(\mu, x) \ge \inf_{\mu \in \sigma(B),\, \lambda \in \sigma(A),\, x \in i\mathbb{R}} (d(x, \lambda) - d(\lambda, \mu)) \\ &\ge \inf_{\lambda \in \sigma(A),\, x \in i\mathbb{R}} d(x, \lambda) - \sup_{\mu \in \sigma(B)} \big(\inf_{\lambda \in \sigma(A)} d(\lambda, \mu)\big) > \rho - \frac{\rho}{2} = \frac{\rho}{2} > 0, \end{aligned}$$

which completes the proof. □

5.2. Conley index and the $\mathcal{LS}$-degree.

Theorem 5.4. *Let H be a real Hilbert space and let L satisfy all the assumptions* (L.1)–(L.4). *Moreover, assume that:*

- (a) *L is of the form $(-I, I)\colon H_- \oplus H_+ \to H_- \oplus H_+$, where both $H_\pm$ are of infinite dimension;*
- (b) *$\Omega \subseteq H$ is a neighbourhood of the origin in H and $f\colon \Omega \to H$ is an $\mathcal{LS}$-vector field, with $K\colon \Omega \to H$ being continuously differentiable;*
- (c) *$f(0) = 0$ and $Df(0) \in \mathfrak{B}_{\mathrm{hip}}(H)$.*

Let ϕ^t be a flow generated by equation $\dot{x} = -f(x)$. Then there exists $\rho > 0$, such that

$$\chi(h_{\mathcal{LS}}(\{0\}, \phi^t)) = \deg_L(f, B(0, \rho), 0). \tag{5.1}$$

REMARK 5.5. Assumption (c) guarantees that $S = \{0\}$ is an isolated invariant set and $x_0 = 0$ is isolated in the set $f^{-1}(0)$ (cf. Remark 1.11 of [1]).

In order to compute the index on the left-hand side of (5.1) consider a sequence of finite dimensional approximations $f_n: H^n \to H^n$, $f_n(x) = Lx + P_n K(x)$. Since the derivative $Df(0) = L + DK(0)$ is a hyperbolic operator, then by Lemmas 5.1 and 5.3 there exists $n_0 \in \mathbb{N}$ such that $Df_n(0) = L + P_n DK(0)$ is hyperbolic, provided $n \geq n_0$. Let us note that $DK(0)$ is a compact linear operator.

The set $\mathrm{cl}(B(0,\rho)) \cap H^n$ is an isolating neighbourhood for the invariant set $\{0\} \in H^n$ for $n \geq n_1$ (comp. Lemma 3.6). Assume that n_0 is chosen such that $n_0 \geq n_1$. We have a splitting $H^{n_0} = \widehat{H}_-^{n_0} \oplus \widehat{H}_+^{n_0}$ where $\widehat{H}_-^{n_0}$ (resp. $\widehat{H}_+^{n_0}$) stands for unstable (resp. stable) subspace of the linear equation $\dot{x} = -Df_{n_0}(0)x$. In the hyperbolic case, the Conley index is exactly the homotopy type of pointed sphere: $h(\{0\}, \phi_{n_0}^t) = [S^{\dim \widehat{H}_-^{n_0}}, *]$.

Denote by E_{n_0} the space that is homotopy equivalent to $(S^{\dim \widehat{H}_-^{n_0}}, *)$. In order to establish the relation between E_{n_0} and E_{n_0+1}, we have to compute the index of the flow generated by $f_{n_0+1}: H^{n_0+1} \to H^{n_0+1}$. Note that the derivative $Df_{n_0+1}(0) = L + P_{n_0+1}DK(0)$ preserve the splitting $H^{n_0+1} = H^{n_0} \oplus H_{n_0+1}$. It is easy to see if we write it as

$$L_{|H^{n_0}} + P_{n_0}DK(0) + L_{|H_{n_0+1}} + Q_{n_0+1}DK(0): H^{n_0} \oplus H_{n_0+1} \to H^{n_0} \oplus H_{n_0+1}.$$

In this situation we have the formula

$$h(\{0\}, \phi_{n_0+1}^t) = h(\{0\}, \phi_{n_0}^t) \wedge h(\{0\}, \eta),$$

where $h(\{0\}, \eta)$ is an index of $\{0\}$ with respect to flow generated by

$$\dot{x} = -L_{|H_{n_0+1}}x - Q_{n_0+1}DK(0)x.$$

Since $\|Q_n DK(0)\| \to 0$, the maps $L_{|H_{n_0+1}}$ and $L_{|H_{n_0+1}} + Q_{n_0+1}DK(0)$ are homotopic for sufficiently large n_0 and the index $h(\{0\}, \eta)$ is determined by dimension of the unstable subspace of linear equation

$$\dot{x} = -L_{|H_{n_0+1}}x.$$

Set $H_{n_0+1} = H_{n_0+1}^- \oplus H_{n_0+1}^+$, where $H_{n_0+1}^-$ (resp. $H_{n_0+1}^+$) is the unstable (resp. stable) subspace of L and define $\nu: \mathbb{N}^* \to \mathbb{N}^*$ by $\nu(n) = \dim H_{n+1}^-$. We have

$$h(\{0\}, \phi_{n_0+1}^t) = [S^{\dim \widehat{H}_-^{n_0}}, *] \wedge [S^{\nu(n_0)}, *] = [S^{\nu(n_0)} S^{\dim \widehat{H}_-^{n_0}}, *].$$

COROLLARY 5.6. *E_{n+1} is the $\nu(n)$-fold suspension of E_n, provided that n is sufficiently large.*

Define $\rho\colon \mathbb{N}^* \to \mathbb{N}^*$ by $\rho(0) = 0$ and $\rho(n) = \sum_{i=0}^{n-1} \nu(i)$. According to definition of cohomological Conley index we have an isomorphism

$$CH^q(h_{\mathcal{LS}}(\{0\}, \phi^t)) \cong H^{q+\rho(n)}(h(\{0\}, \phi^t_n)), \quad n \geq n_0.$$

It follows that $CH^q(h_{\mathcal{LS}}(\{0\}, \phi^t)) \cong H^{q+\rho(n)}(S^{\dim \widehat{H}^n_-}, *) \cong \mathbb{Z}$ for $q = \dim \widehat{H}^n_- - \rho(n)$ and hence

$$\chi(h_{\mathcal{LS}}(\{0\}, \phi^t)) = (-1)^{\dim \widehat{H}^n_- - \rho(n)}, \quad n \geq n_0.$$

In particular we have $\chi(h_{\mathcal{LS}}(\{0\}, \phi^t)) = (-1)^{\dim \widehat{H}^{n_0}_- - \rho(n_0)}$.

By the stability of an $\mathcal{LS}$-degree we have

$$\deg_L(f, B(0,\rho), 0) = \deg_{\mathcal{LS}}(I + L^{-1}K, B(0,\rho), 0) = \deg(I + L^{-1}P_nK, B^n(0,\rho), 0)$$

for $n \geq n_0$. From the fact that $\deg(L_{|H^n}, B^n, 0) = (-1)^{\rho(n)}$ and $\deg(L + P_nK, B^n, 0) = (-1)^{\dim \widehat{H}^n_-}$ we conclude that

$$\begin{aligned}\deg(I + P_nL^{-1}K, B^n, 0) &= \deg(L + P_nK, B^n, 0) \cdot [\deg(L_{|H^n}, B^n, 0)]^{-1} \\ &= (-1)^{\dim \widehat{H}^n_- - \rho(n)}\end{aligned}$$

for $n \geq n_0$ and the proof of (5.1) is completed.

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