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ON A GENERALIZATION OF LAZER–LEACH CONDITIONS FOR A SYSTEM OF SECOND ORDER ODE'S

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Abstract. We study the existence of periodic solutions for a nonlinear second order system of ordinary differential equations. Assuming suitable Lazer–Leach type conditions, we prove the existence of at least one solution applying topological degree methods.

1. Introduction

We study the nonlinear system of second order differential equations for a vector function $u\colon [0,2\pi] \to \mathbb{R}^N$ satisfying

$$u'' + m^2 u + g(u) = p(t), \quad 0 < t < 2\pi \tag{1.1}$$

under periodic boundary conditions:

$$u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \tag{1.2}$$

We shall assume that $m \neq 0$ is an integer, $p \in L^2(0,2\pi)$, and that the nonlinearity g is continuous and bounded. Thus, (1.1)–(1.2) is a resonant problem, since the kernel of the operator $L_m u := u'' + m^2 u$ over the space of 2π-periodic functions is non-trivial. This situation is referred in the literature as *resonance at a higher order eigenvalue*: indeed, if one considers the eigenvalue problem $-u'' = \lambda u$ under periodic conditions, a simple computation shows that $\lambda_m = m^2 \in \mathbb{N}_0$. Let us recall that the case $m = 0$ for a scalar equation has

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a solution if one assumes the well known Landesman–Lazer conditions, which have been firstly obtained in [2] for a resonant elliptic second order scalar equation under Dirichlet conditions (for a survey on Landesman–Lazer conditions see e.g. [5]). Roughly speaking, these conditions state that if $\overline{p}$ (the average of p) lies between the limits at $\pm\infty$ of the nonlinearity g, then the problem admits at least one solution. This condition may be regarded as a degree condition over the sphere $S^0 = \{-1, 1\}$, in the following sense: if for $v = \pm 1$ we define $g_{\pm 1} = g(\pm\infty)$, then the function $\theta\colon S^0 \to S^0$ given by $\theta(v) = (g_v - \overline{p})/|g_v - \overline{p}|$ is well defined and changes sign, and in consequence it has non-zero degree.

Thus, the following result, adapted from a theorem given by Nirenberg in [6] for elliptic systems, may be regarded as a natural extension of Landesman–Lazer theorem:

Theorem 1.1. *Assume that the radial limits $g_v := \lim_{r\to\infty} g(rv)$ exist uniformly with respect to $v \in S^{N-1}$, the unit sphere of $\mathbb{R}^N$. Then* (1.1)–(1.2) *with $m = 0$ has at least one T-periodic solution if the following conditions hold:*

(a) *$g_v \neq \overline{p} := (1/T)\int_0^T p(t)\,dt$ for any $v \in S^{N-1}$.*

(b) *The degree of the mapping $\theta\colon S^{N-1} \to S^{N-1}$ given by*

$$\theta(v) = \frac{g_v - \overline{p}}{|g_v - \overline{p}|}$$

is different from 0.

We remark that the average $\overline{p}$ can be regarded as the projection of the forcing term p into the kernel of the linear operator L_0, which consists in the set of constant functions, naturally identified with $\mathbb{R}^N$.

In contrast with the above described case, the situation when $m \neq 0$ makes it necessary to deal with a $2N$-dimensional kernel, namely:

$$\mathrm{Ker}(L_m) = \{\cos(mt)\alpha + \sin(mt)\beta : (\alpha, \beta) \in \mathbb{R}^{2N}\} := V_m.$$

One might expect that a Landesman–Lazer type condition corresponding to this case can be expressed in terms of the projection of p into V_m or, equivalently, in terms of the m-th Fourier coefficients of p. For $N = 1$, it has been shown by D. E. Leach and A. Lazer that this is, indeed, the case (see [3]):

Theorem 1.2. *Let $N = 1$ and assume that $g \in C(\mathbb{R})$ has limits at infinity. Moreover, let α_p and β_p denote the m-th Fourier coefficients of p. Then, if*

$$\sqrt{\alpha_p^2 + \beta_p^2} < \frac{2}{\pi}|g(\infty) - g(-\infty)|, \tag{1.3}$$

problem (1.1)–(1.2) *admits at least one 2π-periodic solution.*

The aim of this paper is to obtain a generalization of Lazer–Leach theorem for $N > 1$. It is worthy to observe that some extra difficulty should be expected when one attempts to extend the result to a system of equations. For example, when $m = 0$, it is not necessary to assume in the scalar case that the limits $g(\pm\infty)$ exist; however, the same argument cannot be implemented for a system. An interesting example has been given in [7], showing that the existence of radial limits of g is in some sense necessary. More precisely, the authors have presented a system for which no periodic solution exists, although the following conditions are fulfilled for some $R > 0$:

(i) $g(u) \neq \overline{p}$ for $|u| \geq R$.

(ii) The degree of the mapping $\theta_R\colon S^{N-1} \to S^{N-1}$ given by

$$\theta_R(v) = \frac{g(Rv) - \overline{p}}{|g(Rv) - \overline{p}|}$$

is different from 0.

Despite this example, we shall show that the assumption on the existence of radial limits can be replaced by a weaker condition (see condition (G1) below).

Applying topological degree methods [4], we shall obtain solutions of (1.1)–(1.2) under appropriate conditions of Lazer–Leach type. In particular, if the nonlinearity g has uniform radial limits at infinity, these conditions involve the m-th Fourier coefficients of some suitable extension of g to the infinite sphere. However, unlike in Nirenberg's result, our condition (G1) below does not assume that *all* radial limits exist: we shall assume instead the existence of *upper* limits, and only in some specific directions. This kind of condition has been introduced in [1] in the case of resonance at the first eigenvalue for a ϕ-Laplacian system.

2. Preliminaries

Let H be the space of absolutely continuous 2π-periodic vector functions $u\colon [0, 2\pi] \to \mathbb{R}^N$, namely

$$H = H^1_{\text{per}}(0, 2\pi) := \{u \in H^1([0, 2\pi], \mathbb{R}^N) : u(0) = u(2\pi)\}$$

provided with the usual norm $\|u\| := \|u\|_{H^1}$, and let

$$D = H^2_{\text{per}}(0, 2\pi) := \{u \in H \cap H^2([0, 2\pi], \mathbb{R}^N) : u'(0) = u'(2\pi)\}.$$

The operator $L_m\colon D \to L^2([0, 2\pi], \mathbb{R}^N)$ is defined as in the introduction, and its kernel V_m may be described as

$$V_m := \operatorname{Ker}(L_m) = \{u_w : w = (\alpha, \beta) \in \mathbb{R}^{2N}\},$$

where $u_w(t) := \cos(mt)\alpha + \sin(mt)\beta$. For convenience, let $J\colon \mathbb{R}^{2N} \to V_m$ denote the isomorphism given by $J(w) = u_w$. The m-th Fourier coefficients of a function $u \in L^1([0, 2\pi], \mathbb{R}^N)$ shall be denoted respectively by α_u and β_u, i.e.

$$\alpha_u = \frac{1}{\pi}\int_0^{2\pi} \cos(mt)u(t)\,dt, \qquad \beta_u = \frac{1}{\pi}\int_0^{2\pi} \sin(mt)u(t)\,dt.$$

Furthermore, if $w(u) = (\alpha_u, \beta_u)$, then the orthogonal projection $\mathcal{P}$ of the space $L^2([0,2\pi],\mathbb{R}^N)$ onto V_m can be defined as $\mathcal{P}u = J(w(u)) = u_{w(u)}$.

In particular, the projection of p is given by

$$u_{w(p)} = \cos(mt)\alpha_p + \sin(mt)\beta_p.$$

A straightforward computation (or, equivalently, the fact that L_m is symmetric with respect to the inner product of L^2) shows that the range of L_m is the orthogonal complement of V_m, namely:

$$R(L_m) = \left\{\varphi \in L^2([0,2\pi],\mathbb{R}^N) : \int_0^{2\pi} \cos(mt)\varphi(t)\,dt = \int_0^{2\pi} \sin(mt)\varphi(t)\,dt = 0\right\}.$$

Thus, we may define a right inverse $\mathcal{K}\colon R(L) \to H$ of the operator L_m, given by $\mathcal{K}\varphi = u$, where $u \in D$ is the unique solution of the problem

$$\begin{cases} u'' + m^2 u = \varphi, \\ \mathcal{P}u = 0. \end{cases}$$

Moreover, we have the following standard estimate:

Lemma 2.1. *There exists a constant c such that*

$$\|u - \mathcal{P}u\|_{H^2} \le c\|L_m(u)\|_{L^2} \quad \textit{for each } u \in D.$$

Remark 2.2. From the previous lemma and the embedding $H^2(0,2\pi) \hookrightarrow H$ it becomes immediate that $\mathcal{K}$ is compact.

3. Main result

In the sequel, we shall assume that the following condition is satisfied:

(G1) There exists an open covering $\{U_j\}_{j=1\dots,K}$ of the unit sphere $S^{2N-1} \subset \mathbb{R}^{2N}$, and vectors $w_j = (\alpha^j, \beta^j) \in S^{2N-1}$ such that for each $w \in U_j$ the limit

$$\overline{g}_{w,j}(t) := \limsup_{s\to\infty} \langle g(su_w(t)), u_{w_j}(t)\rangle$$

is upper semi-continuous in w for almost every t, where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the usual inner product of $\mathbb{R}^N$.

Under this condition, our abstract version of the Lazer–Leach result for a system reads as follows:

Theorem 3.1. *Assume that condition* (G1) *holds, and that:*

(a) *For each $w \in S^{2N-1}$ there exists $j \in \{1,\dots,K\}$ such that*

$$\frac{1}{\pi}\int_0^{2\pi} \overline{g}_{w,j}(t)\,dt < \langle \alpha_p, \alpha^j\rangle + \langle \beta_p, \beta^j\rangle.$$

(b) *For every* $R \gg 0$ *the Brouwer degree* $\deg_B(G, B_R(0), 0)$ *is different from zero, where* $G\colon \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ *is the mapping defined by*

$$G(w) = w(p) - J^{-1}\mathcal{P}(g \circ u_w).$$

Then problem (1.1)–(1.2) *admits at least one solution.*

Remark 3.2. It follows from the above definitions that G may be expressed in terms of the m-th Fourier coefficients of the function $\varphi(t) = g(u_w(t))$, namely:

$$G(w) = (\alpha_p - \alpha_{g\circ u_w}, \beta_p - \beta_{g\circ u_w}).$$

Theorem 3.1 has an immediate consequence if we assume that radial limits $g_v = \lim_{s\to\infty} g(sv)$ exist uniformly for $v \in S^{N-1}$. Indeed, in this case, we may define for each $t \in [0, 2\pi]$ and each $w \in S^{2N-1}$ the limit

$$g_w(t) := \lim_{s\to\infty} g(su_w(t)). \tag{3.1}$$

Note that $u_w(t)$ might eventually be 0 for a finite number of values of t, in which case $g_w(t) = g(0)$. However, this "singular set" of values of t does not play any role when using the standard Lebesgue convergence theorems for the integral. On the other hand, if $u_w(t) \neq 0$, then $g_w(t)$ is continuous as a function of w: in order to prove this, it suffices to fix a constant $c > 0$ such that $|u_{\widetilde{w}}(t)| \geq c > 0$ for $\widetilde{w}$ in a neighbourhood W of w. Then, $g(su_{\widetilde{w}}(t)) = g(s|u_{\widetilde{w}}(t)|\widetilde{v}) \to g_{\widetilde{v}}$ as $s \to \infty$ for $\widetilde{v} = u_{\widetilde{w}}(t)/|u_{\widetilde{w}}(t)|$. Given $\varepsilon > 0$, fix s such that $|g(su_{\widetilde{w}}(t)) - g_{\widetilde{v}}| < \varepsilon/3$ for $\widetilde{w} \in W$, then

$$|g_{\widetilde{w}}(t) - g_w(t)| = |g_{\widetilde{v}} - g_v| < \frac{2\varepsilon}{3} + |g(su_{\widetilde{w}}(t)) - g(su_w(t))| < \varepsilon$$

for $\widetilde{w}$ close enough to w. Thus, condition (G1) is clearly satisfied for any family $\{(U_j, w_j)\}_{j=1,\dots,K}$ such that $\{U_j\}$ covers S^{2N-1} and $w_j \in S^{2N-1}$. Furthermore, the inequality in condition (a) of Theorem 3.1 is equivalent to:

$$\int_0^{2\pi} \langle g_w(t) - p(t), u_{w_j}(t)\rangle\, dt < 0.$$

Hence, if $g_w - p$ is not orthogonal to the kernel of L_m, that is to say

$$(\alpha_{g_w}, \beta_{g_w}) \neq (\alpha_p, \beta_p), \tag{3.2}$$

then there exists a vector $w_j \in S^{2N-1}$ such that the previous inequality holds in a neighbourhood of w. By compactness, if (3.2) holds for any $w \in S^{2N-1}$, then condition (a) is satisfied.

In this setting, the previous theorem can be formulated, as in Nirenberg's result, in terms of a condition on the extension of g to the infinite sphere or, more precisely, in terms of the m-th Fourier coefficient of this extension. Indeed, for $w \in S^{2N-1}$ we have:

$$\lim_{s\to\infty} G(sw) = (\alpha_p - \alpha_{g_w}, \beta_p - \beta_{g_w}) \neq 0,$$

and thus the mapping $\theta: S^{2N-1} \to S^{2N-1}$ given by

$$\theta(w) = \frac{(\alpha_p - \alpha_{g_w}, \beta_p - \beta_{g_w})}{|(\alpha_p - \alpha_{g_w}, \beta_p - \beta_{g_w})|}$$

is well defined. From the properties of the degree, we obtain:

Corollary 3.3. *Assume that the radial limits g_v exist uniformly for $v \in S^{N-1}$, and for each $w \in S^{2N-1}$ define the function $g_w(t)$ by* (3.1). *Further, assume that:*

(a) $(\alpha_{g_w}, \beta_{g_w}) \neq (\alpha_p, \beta_p)$ *for any* $w \in S^{2N-1}$.
(b) $\deg(\theta) \neq 0$.

Then (1.1)–(1.2) *admits at least one solution.*

Remark 3.4. In the particular case $N = 1$, if $w = (\alpha, \beta) \in S^1$ one has that $u_w(t) = \cos(mt - \omega)$, where $\alpha = \cos(\omega)$ and $\beta = \sin(\omega)$. It follows that

$$g(su_w(t)) \to \begin{cases} g(\infty) & \text{if } t \in I_\omega^+, \\ g(-\infty) & \text{if } t \in I_\omega^-, \end{cases}$$

where $I_\omega^+ = \{t \in [0, 2\pi] : \cos(mt - \omega) > 0\}$, $I_\omega^- = \{t \in [0, 2\pi] : \cos(mt - \omega) < 0\}$. Hence

$$g_w(t) = g(\infty)\chi_{I_\omega^+}(t) + g(-\infty)\chi_{I_\omega^-}(t),$$

except for a finite number of values of t. After computation, it follows that

$$\int_{I_\omega^+} e^{imt}\,dt = e^{i\omega}\int_{I_\omega^+} e^{i(mt-\omega)}\,dt = e^{i\omega}\int_{I_0^+} e^{imt}\,dt = e^{i\omega}\int_{-\pi/2}^{\pi/2} e^{it}\,dt = 2e^{i\omega},$$

and thus

$$\int_{I_\omega^\pm} \cos(mt)\,dt = \pm 2\cos(\omega) = \pm 2\alpha,$$
$$\int_{I_\omega^\pm} \sin(mt)\,dt = \pm 2\sin(\omega) = \pm 2\beta.$$

Hence

$$\lim_{s\to\infty} G(sw) = (\alpha_p, \beta_p) - \frac{2}{\pi}[g(\infty) - g(-\infty)](\alpha, \beta),$$

from which the original result by Lazer and Leach is retrieved.

It is worthy to notice that Corollary 3.3 allows a natural interpretation of Lazer–Leach Theorem in terms of a complex integral. Indeed, from the previous computations it is clear that the degree of the function $\theta: S^1 \to S^1$ given by

$\lim_{s\to\infty} G(sw)/|G(sw)|$ is equivalent to the index of the curve γ defined over the complex plane by

$$\gamma(t) = \frac{2}{\pi}[g(\infty) - g(-\infty)]e^{it}$$

at the point $z_0 = \alpha_p + i\beta_p$. From condition (1.3), it is seen that $|z_0| < |\gamma(t)|$, and hence

$$I(\gamma, z_0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\gamma'(t)}{\gamma(t) - z_0}\,dt = \pm 1.$$

4. Proof of the main result

From the classical continuation theorems in coincidence degree theory (see e.g. [4]), it suffices to prove that the following conditions are satisfied over some bounded domain $\Omega \subset H$:

(a) $L_m u \neq \lambda(p - g(u))$ for $\lambda \in (0,1]$ and $u \in \partial\Omega$.

(b) $Fu \neq 0$ for $u \in \partial\Omega \cap V_m$, where $F\colon V_m \to V_m$ is defined by $Fu := \mathcal{P}(p - g(u))$.

(c) $\deg_B(F, \Omega \cap V_m, 0) \neq 0$.

We shall verify the previous conditions for $\Omega = B_R(0)$, with R large enough. In order to prove that (i) holds for $R \gg 0$, let us suppose that $L_m(u^n) = \lambda_n(p - g(u^n))$ for some $\lambda_n \in (0,1]$ and $\|u^n\|_H \to \infty$. From Lemma 2.1, we deduce that

$$\|u^n - \mathcal{P}u^n\| \leq c\|p - g(u^n)\|_{L^2} \leq C$$

for some constant C, whence $\|\mathcal{P}u^n\|_H \to \infty$.

Writing $\mathcal{P}u^n = \cos(mt)\alpha^n + \sin(mt)\beta^n = u_{w^n}(t)$, with $w^n = (\alpha^n, \beta^n) \to \infty$ in $\mathbb{R}^{2N}$, and passing to a subsequence if necessary, we may assume that $w^n/|w^n| \to w \in S^{2N-1}$.

Let $j \in \{1, \dots, K\}$ be chosen as in condition (i), and let $z^n(t) := u^n(t)/|w^n|$. Then we may write

$$z^n(t) = \frac{u^n(t) - \mathcal{P}u^n(t)}{|w^n|} + \frac{\mathcal{P}u^n(t)}{|w^n|},$$

and using the embedding of $H^1([0,2\pi], \mathbb{R}^N)$ into $C([0,2\pi], \mathbb{R}^N)$ and the continuity of $\mathcal{P}$, we conclude that if $n \to \infty$, then $z^n(t) \to u_w(t)$. From the upper semi-continuity of $\overline{g}_{w,j}$ with respect to w, for almost every t we have:

$$\limsup_{n\to\infty} \langle g(u^n(t)), u_{w_j}(t)\rangle = \limsup_{n\to\infty} \langle g(|w^n| z^n(t)), u_{w_j}(t)\rangle \leq \overline{g}_{w,j}(t).$$

Moreover, as $L_m(u^n) = \lambda_n(p - g(u^n))$,

$$0 = \int_0^{2\pi} \langle L_m(u^n), u_{w_j}\rangle = \lambda_n \int_0^{2\pi} \langle p - g(u^n), u_{w_j}\rangle.$$

This implies that

$$\pi(\langle \alpha_p, \alpha^j \rangle + \langle \beta_p, \beta^j \rangle) = \limsup_{n\to\infty} \int_0^{2\pi} \langle g(u^n), u_{w_j} \rangle \le \int_0^{2\pi} \overline{g}_{w,j},$$

a contradiction.

On the other hand, if $Fu^n = 0$ for $u^n \in V_m$ such that $\|u^n\|_H \to \infty$, then $u^n = u_{w^n} \in V_m$, with $w^n \to \infty$ in $\mathbb{R}^{2N}$. Using the fact that $\mathcal{P}(p - g(u^n)) = 0$, a contradiction yields as before. Thus, (b) is proved.

Finally, for $u = u_w$ with $w \in \mathbb{R}^{2N}$ we have:

$$J^{-1}FJ(w) = (\alpha_p, \beta_p) - J^{-1}\mathcal{P}(g(u_w)) = G(w).$$

Hence, the degree of F at 0 over $\Omega \cap V_m$ can be identified with the Brouwer degree of G at 0 over a large ball of $\mathbb{R}^{2N}$. In consequence, condition (c) follows from assumption (ii), and the proof is complete. □

5. An example: a weakly coupled system

As an application of Theorem 3.1, consider the system

$$u_i'' + m^2 u_i + \widetilde{g}_i(u_i) + h_i(u) = p_i(t), \quad i = 1, \dots, N,$$

where $\widetilde{g}_i$ has limits at infinity, and $h_i(u) \to 0$ uniformly as $|u_i| \to \infty$. We remark that, in this case, radial limits of $g = \widetilde{g} + h$ do not necessarily exist for those $v \in S^{N-1}$ such that $v_i = 0$ for some i, since $h_i(sv)$ does not necessarily converge as $s \to \infty$.

However, condition (G1) is satisfied: for $w = (\alpha, \beta) \in S^{2N-1}$, fix i such that the i-th coordinate of α or β is different from 0. Then taking $z \in S^{2N-1} \cap span\{e_i, e_{N+i}\}$, where e_k is the k-th canonical vector of $\mathbb{R}^N$, it follows that

$$\langle g(su_w(t)), u_z(t) \rangle = \cos(mt - \omega)[\widetilde{g}_i(su_i(t)) + h_i(su_w(t))],$$

with $u_i = \alpha_i \cos(mt) + \beta_i \sin(mt) = \rho_i \cos(mt - \omega_i)$ for some $\rho_i > 0$, and some $\omega, \omega_i \in [0, 2\pi)$. As in Remark 3.4,

$$\widetilde{g}_i(su_i(t)) \to \widetilde{g}_i(\infty)\chi_{I^+_{\omega_i}}(t) + \widetilde{g}_i(-\infty)\chi_{I^-_{\omega_i}}(t) \quad \text{a.e. in } t$$

as $s \to \infty$, and as $h_i(su_w(t)) \to 0$ for almost every t, it is easy to see that condition (G1) holds, as well as condition (a) in Theorem 3.1.

Furthermore, if $w = (\alpha, \beta)$ satisfies as before that $\alpha_i \neq 0$ or $\beta_i \neq 0$, then

$$G_i(sw) \to (\alpha_p)_i - \frac{2}{\pi}[\widetilde{g}_i(\infty) - \widetilde{g}_i(-\infty)]\alpha_i$$

and

$$G_{N+i}(sw) \to (\beta_p)_i - \frac{2}{\pi}[\widetilde{g}_i(\infty) - \widetilde{g}_i(-\infty)]\beta_i$$

as $s \to \infty$. Thus, if we define the mapping $T\colon \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ given by

$$T(w) := (\alpha_p, \beta_p) - \frac{2}{\pi}\sum_{i=1}^{N}[\widetilde{g}_i(\infty) - \widetilde{g}_i(-\infty)](\alpha_i e_i + \beta_i e_{N+i})$$

it follows that under the assumption

$$\sqrt{(\alpha_p)_i^2 + (\beta_p)_i^2} < \frac{2}{\pi}|\widetilde{g}_i(\infty) - \widetilde{g}_i(-\infty)| \quad \text{for } i = 1, \ldots, N,$$

the homotopy $h(\lambda, w) = \lambda G(w) + (1-\lambda)T(w)$ does not vanish on ∂B_R for $R \gg 0$. From the product property of the degree,

$$\deg_B(G, B_R, 0) = \deg_B(T, B_R, 0) = \pm 1.$$

Thus, condition (b) in Theorem 3.1 is satisfied.

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