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TWO POSITIVE SOLUTIONS FOR ONE-DIMENSIONAL p-LAPLACIAN WITH A SINGULAR WEIGHT

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Abstract. We investigate a bifurcation problem for one-dimensional p-Laplace equation with a singular weight under Dirichlet boundary condition. Using super-subsolution method and mountain pass lemma, we prove the existence of at least two positive solutions, at least one positive solution and no positive solution according to the range of a bifurcation parameter.

1. Introduction

In this paper, we study one-dimensional p-Laplacian with a singular weight

$$(\mathrm{P}_\lambda) \qquad \begin{cases} \varphi_p(u'(t))' + \lambda h(t) f(u(t)) = 0 \quad \text{a.e. in } (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, λ is a nonnegative parameter, h is a nonnegative measurable function on $(0,1)$, $h \not\equiv 0$ on any open subinterval in $(0,1)$ which may be singular at $t = 0, 1$ and $f \in C(\mathbb{R}, \mathbb{R})$.

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We will prove the existence of two positive solutions, one positive solution and no positive solution according to different ranges of λ. As for the singularity of h, we suppose:

(A1) $$\int_0^1 t^{p-1}(1-t)^{p-1}h(t)\,dt < \infty.$$

We also assume conditions on f as follows:

- (A2) $sf(s) > 0$ for $s \neq 0$ and f is odd,
- (A3) $0 < f_0 \equiv \lim_{s\to 0^+} f(s)/s^{p-1} < \infty$,
- (A4) $\lim_{s\to\infty} f(s)/s^{p-1} = \infty$,
- (A5) there exists $s_* > 0$ such that $f(s) < f_0 s^{p-1}$ for $s \in (0, s_*)$.

It is easy to see that if $h \in L^1(0,1)$, then all solutions of (P_λ) are in $C^1[0,1]$. On the other hand, if $h \notin L^1(0,1)$, then this regularity of solutions is not true in general, for example if we take $h(t) = (p-1)t^{-1}|1+\ln t|^{p-2}$, $p > 2$ and $\lambda = 1$, $f \equiv 1$, then $h \notin L^1(0,1)$ but h satisfies (A1) and the solution u is given by $u(t) = -t\ln t$ which is not in $C^1[0,1]$. But Proposition 2.6 in [11] guarantees to consider $C^1[0,1]$-solution of (P_λ) when h holds (A1) and specially f holds (A2) and (A3). Therefore, in this paper we will study the existence of $C^1[0,1]$-solution for (P_λ). We call u a *solution* of (P_λ) if $u \in C^1_0[0,1]$, $\varphi_p(u') \in W^{1,1}(0,1)$ and u satisfies (P_λ). Here $W^{1,1}(0,1)$ denotes the usual L^1 Sobolev space of the first order and $C^1_0[0,1]$ is defined by

$$C^1_0[0,1] := \{u \in C^1[0,1] : u(0) = u(1) = 0\}.$$

By (A3), (P_λ) near $u = 0$ is approximated by

$$(\mathrm{E}_\lambda) \qquad \begin{cases} \varphi_p(u'(t))' + \lambda f_0 h(t)\varphi_p(u(t)) = 0 & \text{a.e. in } (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

This is an eigenvalue problem with a singular weight. We call u an *eigenfunction* with an *eigenvalue* λ if $u \in C^1_0[0,1]$, $\varphi_p(u') \in W^{1,1}(0,1)$ and $u \not\equiv 0$ satisfies (E_λ). Assumption (A1) is essential for the existence of discrete eigenvalues of (E_λ) with C^1-eigenspace, indeed, we have the following known result.

Theorem 1.1 ([11, Theorem 2.1]). *Assume* (A1). *Then the eigenvalues of* (E_λ) *consist of a countable set* $\{\mu_k : k \in \mathbb{N}\}$ *which satisfies the following assertions*:

- (a) μ_k *is strictly increasing on k and diverges to ∞ as $k \to \infty$.*
- (b) *Each eigenspace is one dimensional in* $C^1_0[0,1]$.
- (c) *Any eigenfunction corresponding to μ_k has exactly $k-1$ simple zeros in* $(0,1)$.
- (d) *If $\mu(\neq \mu_k,\ k \in \mathbb{N})$ is an eigenvalue, then its eigenspace is not of* C^1.

In [13], among other results about sign-changing solutions, we have proved the existence of positive solutions for (P_λ).

THEOREM 1.2 ([13, Theorem 2.3]). *Assume* (A1)–(A5). *Then there exist* $\overline{\lambda}_1, \underline{\lambda}_1 \in (\mu_1, \infty)$ *with* $\overline{\lambda}_1 \geq \underline{\lambda}_1$ *such that* (P_λ) *has at least two positive solutions for* $\lambda \in (\mu_1, \underline{\lambda}_1)$*, at least one positive solution for* $\lambda \in (0, \mu_1]$ *and no positive solutions for* $\lambda \in (\overline{\lambda}_1, \infty)$.

The above theorem has been proved by using Rabinowitz's global bifurcation method in the space $(\lambda, u) \in (0, \infty) \times C_0^1[0,1]$ mainly due to the following proposition.

PROPOSITION 1.3 ([12, Theorem 1.1] and [13]). *Assume* (A1)–(A5). *Let* $\mathcal{S}$ *denote the closure of the set of nontrivial solutions* $(\lambda, u) \in (0, \infty) \times C_0^1[0,1]$ *for* (P_λ). *Then there exists an unbounded subcontinuum* $\mathcal{C}_1$ *in* $\mathcal{S}$ *bifurcating from* $(\mu_1, 0)$ *which satisfies the following*:

(a) $\mathcal{C}_1 \cap (\mathbb{R} \times \{0\}) = \{(\mu_1, 0)\}$ *and each* $(\lambda, u) \in \mathcal{C}_1 \setminus \{(\mu_1, 0)\}$ *is a positive solution.*
(b) $\mathcal{C}_1$ *intersects* $\{\lambda\} \times (C_0^1[0,1] \setminus \{0\})$ *for any* $\lambda \in (0, \mu_1)$.
(c) *There exists* $\Lambda_1 > 0$ *such that* (P_λ) *has no positive solution when* $\lambda > \Lambda_1$.
(d) *Let* (λ, u) *be a positive solution with* $\|u\|_\infty < s_*$. *Then* $\lambda > \mu_1$*, where* s_* *is given in* (A5).

Here $\|\cdot\|_\infty$ denotes the $L^\infty(0,1)$-norm. By (d), we see that the bifurcation branch starting from $(\mu_1, 0)$ grows to the right in a small range, which with (b) guarantees the existence of at least two positive solutions for λ slightly larger than μ_1. In Theorem 1.2, there may be a gap between $\underline{\lambda}_1$ and $\overline{\lambda}_1$. The purpose of this paper is to study when $\underline{\lambda}_1 = \overline{\lambda}_1$. The proof of this assertion is not obvious and to this end, we assume an additional condition, so called Ambrosetti–Rabinowitz condition (see [2, p. 363] or [16, p. 19] for $p = 2$) such as:

(A6) There exist $\alpha \in (p, \infty)$ and $M > 0$ such that

$$\alpha F(s) \leq s f(s) \quad \text{for } s \geq M, \tag{1.1}$$

where $F(s)$ is defined by

$$F(s) := \int_0^s f(\tau)\, d\tau. \tag{1.2}$$

We now state our main result.

THEOREM 1.4. *Assume* (A1)–(A6). *Then there is* $\lambda_1 \in (\mu_1, \infty)$ *such that* (P_λ) *has at least one positive solution for* $\lambda \in (0, \mu_1]$*, at least two distinct positive* *solutions u_0 and u_1 satisfying $u_0 \le u_1$ on $(0,1)$ for $\lambda \in (\mu_1, \lambda_1)$, at least one positive solution for $\lambda = \lambda_1$, and no positive solutions for $\lambda \in (\lambda_1, \infty)$.*

Our approach for proof is based on combining a super-subsolution method and a mountain pass lemma. Such a combination is originated by Brezis and Nirenberg [4]. See [1] and [6] also. As for p-Laplacian problems, one may refer to [7] and [9]. These papers focused on problems of the form

$$f_\lambda(x,u) = \lambda a(x) u^{q-1} + b(x) u^{r-1},$$

where $1 < q < p < r$ and weight functions a, b are continuous. We see that $f_\lambda(x,u)$ is sublinear near $u = 0$ and superlinear near $u = \infty$ and in this case, we know that the bifurcation occurs at $\lambda = 0$. We notice that our nonlinear term is more generalized than previous ones. Indeed, examples of f satisfying (A1)–(A6) are

$$f(u) = u^{p-1} - u^{q-1} + u^{r-1}, \qquad f(u) = u^{p-1}(1 + u \log u),$$

where $1 < p < q < r$. Moreover, in our case weight function h may be singular at $t = 0$ and/or 1. Because of this singularity, we have to prove additional lemmas and theorems which are somehow obvious when h is continuous on $[0,1]$, e.g. the strong comparison principle, the C^1-regularity theorem of a $W_0^{1,p}(0,1)$-weak solution, and the theorem that a $C_0^1[0,1]$-local minimizer of a Lagrangian functional becomes a $W_0^{1,p}(0,1)$-local minimizer etc.

To complete these arguments, we organize this paper as follows. In Section 2, we state several preliminary lemmas. In Section 3, we prove the existence of a positive solution by using the super-subsolution method. In Section 4, we obtain the second positive solution by using the mountain pass lemma. In Section 5, we prove that the first solution is different from the second one.

2. Preliminary lemmas

In this section, we give some lemmas, which will be useful in the later sections. The first one is related to the Sobolev imbedding of $W_0^{1,p}(0,1)$.

Lemma 2.1 ([12, Lemma 2.2]). *For any $u \in W_0^{1,p}(0,1)$ and $t \in [0,1]$, we have*

$$|u(t)|^p \le (2t(1-t))^{p-1} \int_0^1 |u'(s)|^p \, ds. \tag{2.1}$$

The next lemma says that any nontrivial solution of (P_λ) has no double zeros.

Lemma 2.2 ([12, Lemma 4.1]). *Assume* (A1)–(A3). *If (λ, u) is a solution of (P_λ) with $u(t_0) = u'(t_0) = 0$ at some $t_0 \in [0,1]$, then u identically vanishes on $[0,1]$.*

In the next lemma, we see that a bounded subset of solutions in $\mathbb{R} \times C_0^1[0,1]$ is relatively compact.

LEMMA 2.3 ([12, Corollary 2.5]). *Assume* (A1)–(A3). *If a sequence of solutions* $\{(\lambda_n, u_n)\}$ *for* (P_λ) *is bounded in* $(0,\infty)\times C_0^1[0,1]$, *then it has a subsequence converging to a solution of* (P_λ).

The next lemma implies that when a sequence $\{u_n\}$ of solutions converges in $C_0^1[0,1]$, the number of their zeros is invariant for all large enough n.

LEMMA 2.4 ([12, Lemma 4.5]). *Assume* (A1)–(A3). *Let* $\{(\lambda_n, u_n)\}$ *and* (λ, u) *be solutions of* (P_λ) *such that* $\lambda_n > 0$, $\lambda > 0$, $u_n \not\equiv 0$ *and* $\{(\lambda_n, u_n)\}$ *converges to* (λ, u) *strongly in* $\mathbb{R} \times C_0^1[0,1]$. *Then there exists* $n_0 \in \mathbb{N}$ *such that all* u_n *with* $n \geq n_0$ *have the same number of zeros in* $(0,1)$. *Moreover, if* $u \not\equiv 0$, *then the number of zeros of* u_n *for* $n \geq n_0$ *coincides with that of* u. *If* $u \equiv 0$, *then* λ *is equal to a certain eigenvalue* μ_k *and* u_n *with* $n \geq n_0$ *has exactly* $k-1$ *zeros in* $(0,1)$.

For a solution (λ, u) of (P_λ), a boundedness of λ implies that of u.

LEMMA 2.5 ([13, Lemma 4.4]). *Assume* (A1)–(A4). *Then for any compact interval* $J \subset (0,\infty)$, *there exists* $M(J) > 0$ *such that if* (λ, u) *is a positive solution of* (P_λ) *with* $\lambda \in J$, *then* $\|u\|_{C^1} \leq M(J)$.

3. Existence of first solution u_0

In what is to follow, we assume conditions (A1)–(A6) for all lemmas and theorems. The key step to complete the proof of our main result (Theorem 1.4) can be stated as follows:

THEOREM 3.1. *Suppose that* (P_λ) *has a positive solution* $\overline{u}$ *at* $\overline{\lambda} \in (\mu_1, \infty)$ *and let* $\lambda \in (\mu_1, \overline{\lambda})$. *Then* (P_λ) *has at least two distinct positive solutions* u_0 *and* u_1 *satisfying* $u_0(t) \leq u_1(t)$, *for all* $t \in (0,1)$.

Since $\underline{\lambda}_1$ in Theorem 1.2 does not give sufficient information for λ_1 in Theorem 1.4, we need to find two positive solutions of (P_λ) for all $\lambda \in (\mu_1, \overline{\lambda})$, where $\overline{\lambda}$ is in Theorem 3.1. In this section, we find first solution u_0 by using the super-subsolution method. For reader's convenience, we give the fundamental theorem of super-subsolution.

THEOREM 3.2. *Assume* (A1). *Let* α *and* β *be a subsolution and a supersolution of problem* (P_λ), *respectively such that* $\alpha(t) \leq \beta(t)$ *for all* $t \in [0,1]$. *Then problem* (P_λ) *has at least one solution* u *such that* $\alpha(t) \leq u(t) \leq \beta(t)$ *for all* $t \in [0,1]$.

The proof can be done by obvious combination from Lee [14] and Lü–O'Regan [15].

Lemma 3.3. *Let ϕ be the first eigenfunction of* (E_λ) *such that $\phi > 0$ in $(0,1)$ and $\|\phi\|_\infty = 1$. Let $\overline{u}$ be as in Theorem* 3.1. *Then $\underline{u}(t) := \varepsilon\phi(t)$ with $\varepsilon > 0$ small enough is a subsolution of* (P_λ) *which satisfies $\underline{u}(t) \le \overline{u}(t)$, for $t \in (0,1)$. Therefore there exists a positive solution u_0 of* (P_λ) *such that $\underline{u}(t) \le u_0(t) \le \overline{u}(t)$, for $t \in (0,1)$.*

Proof. Although this lemma has been proved in paper [13], we give a proof for the reader's convenience. Since $\mu_1 < \lambda$, we choose $c_0 > 0$ satisfying $\mu_1 f_0/\lambda < c_0 < f_0$ and $\varepsilon > 0$ so small that

$$f(s)/s^{p-1} > c_0 \quad \text{for } 0 < s \le \varepsilon.$$

Since $\|\underline{u}\|_\infty = \varepsilon$, we have $c_0\varphi_p(\underline{u}) \le f(\underline{u})$. Since ϕ is an eigenfunction for μ_1, so is $\underline{u}$. Thus we have

$$-\varphi_p(\underline{u}')' = \mu_1 f_0 h(t)\varphi_p(\underline{u}) \le \lambda h(t) f(\underline{u}),$$

i.e. $\underline{u}$ is a subsolution of (P_λ). It holds that $\underline{u}(t) < \overline{u}(t)$ after $\varepsilon > 0$ is replaced by a smaller constant. Therefore by the fundamental theorem of super-subsolution, there exists a positive solution u_0 such that $\underline{u}(t) \le u_0(t) \le \overline{u}(t)$. □

For the p-Laplacian, it is proved in [3] and [8] that the strong comparison theorem holds under assumptions different from ours. Because of the singularity of h at $t = 0, 1$, we must prove the strong comparison theorem for our equation, i.e. $\underline{u}(t) < u_0(t) < \overline{u}(t)$ as follows.

Lemma 3.4. *Let $\underline{u}$, $\overline{u}$ and u_0 be as in Lemma* 3.3. *Then $\underline{u}(t) < u_0(t) < \overline{u}(t)$ for all $t \in (0,1)$ and $0 < \underline{u}'(0) < u_0'(0) < \overline{u}'(0)$, $\overline{u}'(1) < u_0'(1) < \underline{u}'(1) < 0$.*

Proof. Let us show $u_0(t) < \overline{u}(t)$ for $t \in (0,1)$. Suppose on the contrary that $u_0(t_0) = \overline{u}(t_0)$ at some $t_0 \in (0,1)$. Since $\overline{u}(t) - u_0(t)$ attains a minimum at $t = t_0$, we have $u_0'(t_0) = \overline{u}'(t_0)$. Integrating (P_λ) over (t_0, t), we have

$$\varphi_p(u_0'(t)) - \varphi_p(u_0'(t_0)) = -\lambda \int_{t_0}^t h f(u_0)\, ds.$$

Since $\overline{u}$ is a solution of (P_λ) at $\overline{\lambda}$, we have the same identity as above with λ replaced by $\overline{\lambda}$ and u_0 by $\overline{u}$. Subtracting these identities, we get

$$\varphi_p(u_0'(t)) - \varphi_p(\overline{u}'(t)) = (\overline{\lambda} - \lambda) \int_{t_0}^t h f(\overline{u})\, ds + \lambda \int_{t_0}^t h(f(\overline{u}) - f(u_0))\, ds. \tag{3.1}$$

Put $a = \overline{u}(t_0)$ and choose $\varepsilon > 0$ so small that $(\overline{\lambda} - \lambda)(f(a)/2) > \varepsilon\lambda$. Then there is $\delta > 0$ such that

$$\frac{1}{2} f(a) < f(\overline{u}(s)) < 2f(a), \qquad |f(\overline{u}(s)) - f(u_0(s))| < \varepsilon, \tag{3.2}$$

for $s \in (t_0 - \delta, t_0 + \delta)$. The right-hand side of (3.1) can be estimated as

$$[(\overline{\lambda} - \lambda)(f(a)/2) - \varepsilon\lambda] \int_{t_0}^{t} h(s)\,ds > 0 \quad \text{for } t \in (t_0, t_0 + \delta).$$

Thus $u_0'(t) > \overline{u}'(t)$ for $t \in (t_0, t_0 + \delta)$. Since $u_0(t_0) = \overline{u}(t_0)$, we obtain $u_0(t) > \overline{u}(t)$ for $t \in (t_0, t_0 + \delta)$. This contradicts the fact that $u_0(t) \le \overline{u}(t)$. Consequently, it holds that $u_0(t) < \overline{u}(t)$ for $t \in (0, 1)$. In the proof of Lemma 3.3, we have already obtained

$$\varphi_p(\underline{u}')' + (\mu_1 f_0/c_0)h(t)f(\underline{u}) \ge 0.$$

Using this inequality with the similar argument as in the proof of $u_0(t) < \overline{u}(t)$, we can prove that $\underline{u}(t) < u_0(t)$.

Next, we shall show that $u_0'(0) < \overline{u}'(0)$. Suppose on the contrary that $u_0'(0) = \overline{u}'(0)$. Then (3.1) with $t_0 = 0$ is valid, i.e.

$$\varphi_p(u_0'(t)) - \varphi_p(\overline{u}'(t)) = (\overline{\lambda} - \lambda) \int_0^t hf(\overline{u})\,ds + \lambda \int_0^t h(f(\overline{u}) - f(u_0))\,ds. \tag{3.3}$$

Putting $a = u_0'(0)$, which is positive by Lemma 2.2, we may write $\overline{u}$ and u_0 of the form

$$\overline{u}(t) = at + \xi(t)t, \qquad u_0(t) = at + \eta(t)t,$$

for some functions ξ and η satisfying $\xi(t), \eta(t) \to 0$ as $t \to 0$. Let $0 < \varepsilon < (1/2)\min\{a, f_0\}$. We shall determine ε later on. By (A3), there is $\delta > 0$ such that

$$|f(s) - f_0 s^{p-1}| \le \varepsilon s^{p-1} \quad \text{for } s \in (0, \delta). \tag{3.4}$$

Choose $t_0 > 0$ so small that

$$0 < \overline{u}(t),\ u_0(t) < \delta, \quad |\xi(t)|, |\eta(t)| < \varepsilon \quad \text{for } t \in (0, t_0).$$

We will estimate the right-hand side of (3.3). By (3.4) and the fact $\overline{u}(t) = at + \xi(t)t \ge at - at/2 = at/2$, we have

$$f(\overline{u}(t)) \ge (f_0/2)\overline{u}(t)^{p-1} \ge (f_0/2)(at/2)^{p-1} = bt^{p-1}, \tag{3.5}$$

where $b = (f_0/2)(a/2)^{p-1}$. We use (3.4) and the fact $\overline{u}(t) \le 2at$, to get

$$\begin{aligned} |f(\overline{u}(t)) - f(u_0(t))| &\le |f(\overline{u}) - f_0\overline{u}^{p-1}| + f_0|\overline{u}^{p-1} - u_0^{p-1}| + |f_0 u_0^{p-1} - f(u_0)| \\ &\le 2^p a^{p-1}\varepsilon t^{p-1} + f_0|\overline{u}(t)^{p-1} - u_0(t)^{p-1}|. \end{aligned} \tag{3.6}$$

Let us estimate the last term. Let $p \ge 2$. Then the mean-value theorem guarantees an $x \in (u_0(t), \overline{u}(t))$ such that

$$\begin{aligned} |\overline{u}(t)^{p-1} - u_0(t)^{p-1}| &= (p-1)x^{p-2}|\overline{u}(t) - u_0(t)| \\ &\le (p-1)\overline{u}(t)^{p-2}|\xi(t) - \eta(t)|t \le \varepsilon c t^{p-1}, \end{aligned} \tag{3.7}$$

where $c = 2(2a)^{p-2}(p-1)$. Let $1 < p < 2$. Then we have

$$|\overline{u}(t)^{p-1} - u_0(t)^{p-1}| \le |\overline{u}(t) - u_0(t)|^{p-1} \le (2\varepsilon t)^{p-1}. \tag{3.8}$$

By (3.6)–(3.8), we obtain

$$|f(\overline{u}(t)) - f(u_0(t))| \le C(\varepsilon + \varepsilon^{p-1})t^{p-1} \quad \text{for } t \in (0, t_0), \tag{3.9}$$

where $C > 0$ is independent of ε and t. By (3.3), (3.5) and (3.9), we have

$$\varphi_p(u_0'(t)) - \varphi_p(\overline{u}'(t)) \ge [(\overline{\lambda} - \lambda)b - \lambda C(\varepsilon + \varepsilon^{p-1})] \int_0^t h(s)s^{p-1}\,ds > 0,$$

where we have chosen $\varepsilon > 0$ so small that the coefficient of the integral is positive. Hence $u_0'(t) > \overline{u}'(t)$ for $t \in (0, t_0)$. However, this contradicts the fact that $u_0(t) \le \overline{u}(t)$ and $u_0(0) = \overline{u}(0) = 0$. Accordingly, we get $u_0'(0) < \overline{u}'(0)$. By the similar argument, we can prove that $\underline{u}'(0) < u_0'(0)$ and $\overline{u}'(1) < u_0'(1) < \underline{u}'(1) < 0$. □

4. Existence of second solution u_1

In this section, we find the second solution in Theorem 3.1 by using variational method. We define norm $\|\cdot\|_{1,p}$ of $W_0^{1,p}(0,1)$ by

$$\|u\|_{1,p} := \left(\int_0^1 |u'|^p\,dt \right)^{1/p}.$$

We put

$$\rho(t) := (t(1-t))^{p-1}. \tag{4.1}$$

Then assumption (A1) is equivalent to $h\rho \in L^1(0,1)$. By (2.1), we have

$$|u(t)| \le 2^{(p-1)/p}\rho(t)^{1/p}\|u\|_{1,p}, \tag{4.2}$$

for $t \in [0,1]$ and $u \in W_0^{1,p}(0,1)$. To obtain another positive solution, we define

$$\widetilde{f}(t,s) := \begin{cases} f(u_0(t)) & \text{if } s < u_0(t), \\ f(s) & \text{if } s \ge u_0(t), \end{cases}$$

$$\widetilde{g}(t,s) := \begin{cases} f(u_0(t)) & \text{if } s < u_0(t), \\ f(s) & \text{if } u_0(t) \le s \le \overline{u}(t), \\ f(\overline{u}(t)) & \text{if } \overline{u}(t) < s, \end{cases}$$

$$\widetilde{F}(t,u) := \int_0^u \widetilde{f}(t,s)\,ds, \qquad \widetilde{G}(t,u) := \int_0^u \widetilde{g}(t,s)\,ds,$$

where $\overline{u}$ and u_0 are given in Lemma 3.3 Moreover, we define

$$I(u) := \int_0^1 \left(\frac{1}{p}|u'|^p - \lambda h(t)\widetilde{F}(t,u) \right) dt,$$

$$J(u) := \int_0^1 \left(\frac{1}{p}|u'|^p - \lambda h(t)\widetilde{G}(t,u) \right) dt.$$

LEMMA 4.1. *$I(u)$ and $J(u)$ are C^1-functionals on $W_0^{1,p}(0,1)$ and satisfy the Palais-Smale condition.*

PROOF. We deal with $I(u)$ only because the same argument can be applied for $J(u)$ also. We prove that $h(\cdot)\widetilde{F}(\cdot,u(\cdot))$ is integrable on $(0,1)$ when $u \in W_0^{1,p}(0,1)$. Recall that $W_0^{1,p}(0,1)$ is imbedded in $L^\infty(0,1)$. Let $u \in W_0^{1,p}(0,1)$. Then by (A3), there is a constant $A > 0$ depending on $\|u\|_\infty$ such that

$$|\widetilde{F}(t,s)| \le A(|s|^p + u_0(t)^p) \quad \text{for } |s| \le \|u\|_\infty.$$

This inequality with (4.2) shows

$$|h(t)\widetilde{F}(t,u(t))| \le 2^{p-1}Ah(t)\rho(t)(\|u\|_{1,p}^p + \|u_0\|_{1,p}^p).$$

Therefore $h(\cdot)\widetilde{F}(\cdot,u(\cdot))$ is integrable and $I(u)$ is well-defined in $W_0^{1,p}(0,1)$. In the usual way, it can be proved that $I(u)$ is of class C^1.

To show the Palais–Smale condition, let $\{u_n\}$ be a sequence in $W_0^{1,p}(0,1)$ such that $|I(u_n)|$ is bounded and $\|I'(u_n)\|$ converges to zero, where $\|\cdot\|$ means the norm of dual space of $W_0^{1,p}(0,1)$. Since

$$I'(u)v = \int_0^1 (|u'|^{p-2}u'v' - \lambda h(t)\widetilde{f}(t,u)v)\,dt, \tag{4.3}$$

we have a relation

$$\alpha I(u_n) - I'(u_n)u_n = \beta\|u_n\|_{1,p}^p - \lambda\int_0^1 h(t)(\alpha\widetilde{F}(t,u_n) - u_n\widetilde{f}(t,u_n))\,dt, \tag{4.4}$$

where α is the constant in (A6) and we put $\beta = (\alpha - p)/p$. We will estimate the integral on the right-hand side. Choose $\varepsilon > 0$ so small that

$$2^{p-1}\lambda\varepsilon\int_0^1 h\rho\,dt \le \beta/4. \tag{4.5}$$

Observe that $\widetilde{F}(t,s) = f(u_0(t))s$, if $s < u_0(t)$ and $\widetilde{F}(t,s) = F(s)+f(u_0(t))u_0(t)-F(u_0(t))$, if $s \ge u_0(t)$, where F is defined by (1.2). We choose a constant $M > 0$ which satisfies (A6), $M > \|u_0\|_\infty$ and moreover

$$\alpha(f(u_0)u_0 - F(u_0)) \le \varepsilon M^p.$$

Then we use (A6) to get

$$\alpha\widetilde{F}(t,s) - s\widetilde{f}(t,s) = \alpha F(s) + \alpha(f(u_0)u_0 - F(u_0)) - sf(s) \le \varepsilon|s|^p,$$

for $s \ge M$, and

$$\alpha\widetilde{F}(t,s) - s\widetilde{f}(t,s) = (\alpha - 1)f(u_0)s \le 0, \quad \text{for } s \le -M.$$

In any case, we obtain

$$\alpha\widetilde{F}(t,s) - s\widetilde{f}(t,s) \le \varepsilon|s|^p, \quad \text{for } |s| \ge M. \tag{4.6}$$

By (A3), there is a constant $B > 0$ such that

$$|\alpha\widetilde{F}(t,s) - s\widetilde{f}(t,s)| \le B(|s|^p + u_0(t)^p), \quad \text{for } |s| \le M. \tag{4.7}$$

We choose $\delta > 0$ so small that

$$2^{p-1}\lambda B\bigg(\int_0^\delta h\rho\,dt + \int_{1-\delta}^1 h\rho\,dt\bigg) \le \beta/4. \tag{4.8}$$

Fix $n \in \mathbb{N}$ arbitrarily. We define

$$\begin{aligned} P &:= \{t \in [0,1] : |u_n(t)| \ge M\}, \\ Q &:= \{t \in (\delta, 1-\delta) : |u_n(t)| < M\}, \\ R &:= \{t \in [0,\delta] \cup [1-\delta, 1] : |u_n(t)| < M\}. \end{aligned}$$

First, by (4.6), (4.2) and (4.5), we have

$$\begin{aligned} \lambda\int_P h(t)(\alpha\widetilde{F}(t,u_n) - u_n\widetilde{f}(t,u_n))\,dt \\ \le \lambda\varepsilon\int_P h|u_n|^p\,dt \le 2^{p-1}\lambda\varepsilon\|u_n\|_{1,p}^p\int_0^1 h\rho\,dt \le \frac{\beta}{4}\|u_n\|_{1,p}^p. \end{aligned} \tag{4.9}$$

Second, we get

$$\begin{aligned} \lambda\int_Q h(t)(\alpha\widetilde{F}(t,u_n) - u_n\widetilde{f}(t,u_n))\,dt \\ \le \lambda\sup_{|s|\le M}|\alpha\widetilde{F}(t,s) - s\widetilde{f}(t,s)|\int_\delta^{1-\delta} h(t)\,dt. \end{aligned} \tag{4.10}$$

Finally, using (4.7) and (4.8), we have

$$\begin{aligned} \lambda\int_R h(t)(\alpha\widetilde{F}(t,u_n) - u_n\widetilde{f}(t,u_n))\,dt \\ \le \lambda B\int_R h|u_n|^p\,dt + C_0 \le (\beta/4)\|u_n\|_{1,p}^p + C_0. \end{aligned} \tag{4.11}$$

Here $C_0 = \int_0^1 \lambda Bh(t)u_0(t)^p\,dt$. Denoting the right-hand side of (4.10) by C_1 and combining (4.9)–(4.11), we obtain

$$\lambda\int_0^1 h(t)(\alpha\widetilde{F}(t,u_n) - u_n\widetilde{f}(t,u_n))\,dt \le (\beta/2)\|u_n\|_{1,p}^p + C_0 + C_1.$$

Substituting the above inequality into (4.4) and using the boundedness of $|I(u_n)|$ and $\|I'(u_n)\|$, we have a constant $C > 0$ such that

$$C + C\|u_n\|_{1,p} \ge (\beta/2)\|u_n\|_{1,p}^p - C_0 - C_1.$$

Thus $\|u_n\|_{1,p}$ is bounded and it has a subsequence (denoted by u_n again) converging to a limit u_∞ weakly in $W_0^{1,p}(0,1)$. The compact imbedding assures that u_n converges to u_∞ strongly in $C[0,1]$. To prove the strong convergence in $W_0^{1,p}(0,1)$, we prepare two inequalities for $x, y \in \mathbb{R}$,

$$(|x|^{p-2}x - |y|^{p-2}y)(x-y) \geq 2^{-(p-2)}|x-y|^p \quad \text{if } p \geq 2, \tag{4.12}$$

$$(|x|^{p-2}x - |y|^{p-2}y)(x-y) \geq \frac{(p-1)|x-y|^2}{(|x|+|y|)^{2-p}} \quad \text{if } 1 < p < 2. \tag{4.13}$$

(4.12) follows from an easy calculation. (4.13) is obtained by the mean-value theorem

$$\frac{|x|^{p-2}x - |y|^{p-2}y}{x-y} = (p-1)|z|^{p-2},$$

with a certain z lying between x and y. Since $|z| < |x| + |y|$ and $p < 2$, we have

$$\frac{|x|^{p-2}x - |y|^{p-2}y}{x-y} \geq (p-1)(|x|+|y|)^{p-2}.$$

This proves (4.13). By (4.3), we make a relation

$$\begin{aligned} I'(u_n)(u_n - u_\infty) &- I'(u_\infty)(u_n - u_\infty) \\ &= \int_0^1 (|u_n'|^{p-2}u_n' - |u_\infty'|^{p-2}u_\infty')(u_n' - u_\infty')\,dt \\ &\quad - \lambda \int_0^1 h(t)(\widetilde{f}(t,u_n) - \widetilde{f}(t,u_\infty))(u_n - u_\infty)\,dt. \end{aligned} \tag{4.14}$$

Since $\|I'(u_n)\|$ converges to zero and u_n to u_∞ weakly in $W_0^{1,p}(0,1)$, the left-hand side converges to zero. We will show that the second integral on the right-hand side converges to zero. Since $\|u_n\|_{1,p}$ is bounded, we use (A3) with (4.2) to get a constant $C > 0$ such that

$$|h(t)(\widetilde{f}(t,u_n) - \widetilde{f}(t,u_\infty))(u_n - u_\infty)| \leq Ch(t)\rho(t) \in L^1(0,1).$$

By the Lebesgue convergence theorem, the second integral on the right-hand side in (4.14) converges to zero. Therefore we have

$$\lim_{n\to\infty} \int_0^1 (|u_n'|^{p-2}u_n' - |u_\infty'|^{p-2}u_\infty')(u_n' - u_\infty')\,dt = 0. \tag{4.15}$$

Let $p \geq 2$. By (4.12), we have

$$2^{-(p-2)} \int_0^1 |u_n' - u_\infty'|^p\,dt \leq \int_0^1 (|u_n'|^{p-2}u_n' - |u_\infty'|^{p-2}u_\infty')(u_n' - u_\infty')\,dt.$$

Hence $\|u_n - u_\infty\|_{1,p}$ converges to zero. Let $1 < p < 2$. Putting $w = |u_n'| + |u_\infty'|$, $q = (2-p)p/2$, $r = 2/(2-p)$ and using the Hölder inequality, we get

$$\begin{aligned} \int_0^1 |u_n' - u_\infty'|^p\,dt &\leq \left(\int_0^1 |u_n' - u_\infty'|^2 w^{-2q/p}\,dt\right)^{p/2} \left(\int_0^1 w^{qr}\,dt\right)^{1/r} \\ &\leq \left(\int_0^1 |u_n' - u_\infty'|^2(|u_n'| + |u_\infty'|)^{-(2-p)}\,dt\right)^{p/2} (\|u_n\|_{1,p} + \|u_\infty\|_{1,p})^{(2-p)p/2}. \end{aligned}$$

Thus from (4.13) and (4.15), we see that $\|u_n - u_\infty\|_{1,p}$ converges to zero. Consequently, $I(u)$ satisfies the Palais–Smale condition. □

Lemma 4.2. *If u is a critical point of I, then $u_0(t) \le u(t)$ for $t \in (0,1)$ and u becomes a solution of* (P_λ) *in the distribution sense.*

Proof. Suppose on the contrary that $D := \{t \in (0,1) : u(t) < u_0(t)\} \ne \emptyset$. Since $\widetilde{f}(t,u(t)) = f(u_0(t))$ in D, we have

$$-\varphi_p(u'(t))' = \lambda h(t) f(u_0(t)) = -\varphi_p(u_0'(t))' \quad \text{in } D.$$

Since $u_0 \equiv u$ on ∂D, from the comparison theorem it follows that $u \equiv u_0$ in D. This is a contradiction. Therefore $u_0(t) \le u(t)$ for all t, and $\widetilde{f}(t,u) = f(u)$. Consequently, u is a solution of (P_λ) in the distribution sense. □

In the next lemma, we show the C^1-regularity for a critical point of $I(u)$.

Lemma 4.3. *Let $u \in W_0^{1,p}(0,1)$ satisfy* (P_λ) *in the distribution sense and $u(t) > 0$ for $t \in (0,1)$. Then $u \in C_0^1[0,1]$, $\varphi_p(u') \in W^{1,1}(0,1)$ and u satisfies* (P_λ) *almost everywhere in* $(0,1)$.

Proof. Since $h \in L^1_{\mathrm{loc}}(0,1)$, it is easy to verify that $u \in C^1(0,1)$. We shall show the C^1-regularity at $t = 0$ and 1. Since $u > 0$ in $(0,1)$, it is concave, and so $u(t)/t$ is decreasing and moreover $u'(t) > 0$ for $t > 0$ small enough. To prove $u \in C^1[0,1)$, it is enough to show that $u(t)/t$ is bounded above as $t \to 0$. By (A3), there exists a constant $A > 0$ depending on $\|u\|_\infty$ such that $|f(u(t))| \le A|u(t)|^{p-1}$ in $[0,1]$. We remind the well-known inequality

$$(x+y)^{1/(p-1)} \le C_p(x^{1/(p-1)} + y^{1/(p-1)}) \quad \text{for } x, y \ge 0, \tag{4.16}$$

where $C_p = 1$ if $p \ge 2$ and $C_p = 2^{(2-p)/(p-1)}$ if $1 < p < 2$. Choose $\varepsilon > 0$ so small that $C_p\varepsilon < (p-1)/p$. Then we take a small $\delta > 0$ such that $u'(t) > 0$ in $(0,\delta)$ and

$$\lambda A \int_0^\delta h(\tau)\tau^{p-1}\,d\tau \le \varepsilon^{p-1}.$$

Integrating (P_λ) over (t,δ) and using the decrease of $u(t)/t$, we have

$$\begin{aligned} u'(t)^{p-1} &\le u'(\delta)^{p-1} + \lambda A \int_t^\delta h(\tau) u^{p-1}\,d\tau \\ &\le u'(\delta)^{p-1} + \lambda A \int_t^\delta h(\tau)\tau^{p-1}\,d\tau (u(t)/t)^{p-1} \\ &\le u'(\delta)^{p-1} + \varepsilon^{p-1}(u(t)/t)^{p-1}. \end{aligned}$$

We use (4.16) to get

$$u'(t) \le C_p u'(\delta) + C_p\varepsilon(u(t)/t).$$

Rewriting $C_p\varepsilon$ by a and $C_pu'(\delta)$ by C, we get $(t^{-a}u(t))' \le Ct^{-a}$. Integrating both sides over (s,t), we obtain

$$t^{-a}u(t) \le s^{-a}u(s) + (C/(1-a))(t^{1-a} - s^{1-a}) \quad \text{for } 0 < s < t \le \delta.$$

By (2.1) with $a < (p-1)/p$, $s^{-a}u(s)$ converges to zero as $s \to 0$. Thus we get

$$u(t) \le (C/(1-a))t \quad \text{for } 0 < t \le \delta.$$

This implies that $u(t)/t$ is bounded as $t \to 0$. Consequently, $u \in C^1[0,1)$. The argument above remains valid for the proof of the C^1-regularity at $t = 1$.

Since $u \in C_0^1[0,1]$, there is $C > 0$ such that $|u(t)| \le Ct(1-t)$. We use $|f(u(t))| \le A|u(t)|^{p-1}$ to get

$$|hf(u)| \le AC^{p-1}h(t)(t(1-t))^{p-1} \in L^1(0,1),$$

which implies that $\varphi_p(u')' = -\lambda h(t)f(u) \in L^1(0,1)$. Thus $\varphi_p(u') \in W^{1,1}(0,1)$.□

To prove Theorem 3.1, we use Brezis–Nirenberg's method [4]. For $p = 2$, see [1] and [6]. For $p \ne 2$, one may refer to [7] and [9]. They studied the N-dimensional problem, but we investigate the one-dimensional case. However, since we have a singular weight h and a different nonlinear term $f(u)$ from theirs, the proofs of lemmas below are more complicated.

LEMMA 4.4. *$J(u)$ has a global minimizer u_1 in $W_0^{1,p}(0,1)$. Moreover, u_1 becomes a positive solution of* (P_λ) *belonging to $C_0^1[0,1]$ which satisfies $0 < u_1'(0) < \overline{u}'(0)$, $\overline{u}'(1) < u_1'(1) < 0$ and*

$$u_0(t) \le u_1(t) < \overline{u}(t) \quad \text{in } (0,1). \tag{4.17}$$

PROOF. By the definition of $\widetilde{G}(t,s)$ with (A3), there is $C > 0$ such that

$$|\widetilde{G}(t,s)| \le C\overline{u}(t)^{p-1}|s| \quad \text{for } t \in [0,1],\ s \in \mathbb{R}. \tag{4.18}$$

By (4.18) with (4.2), $J(u)$ can be estimated as;

$$J(u) \ge \frac{1}{p}\|u\|_{1,p}^p - \lambda C\int_0^1 h\overline{u}^{p-1}|u|\,dt \ge \frac{1}{p}\|u\|_{1,p}^p - 2^{p-1}\lambda C\int_0^1 h\rho\,dt\|\overline{u}\|_{1,p}^{p-1}\|u\|_{1,p}.$$

Thus $J(u)$ is bounded from below, and it has a global minimizer u_1 because of the Palais-Smale condition (see [16, Theorem 2.7]). Therefore u_1 is a critical point of $J(u)$, i.e.

$$\begin{aligned} \varphi_p(u_1')' + \lambda h(t)\widetilde{g}(t,u_1) &= 0 \quad \text{in } (0,1), \\ u_1(0) = u_1(1) &= 0. \end{aligned} \tag{4.19}$$

By the same method as in Lemma 4.2, it follows that $u_0 \le u_1 \le \overline{u}$ in $[0,1]$. Thus $\widetilde{g}(t,u_1) = f(u_1)$ and u_1 becomes a solution of (P_λ). By Lemma 4.3, u_1 belongs to $C_0^1[0,1]$. By the same way as in Lemma 3.4, we have $0 < u_1'(0) < \overline{u}'(0)$, $\overline{u}'(1) < u_1'(1) < 0$ and we obtain (4.17) also. □

Lemma 4.5. *u_1 is a local minimizer of $I(u)$ in $C_0^1[0,1]$.*

Proof. Since u_1 is a global minimizer of $J(u)$ in $W_0^{1,p}(0,1)$, it is clear that $J(u_1)\le J(u)$ for all $u\in C_0^1[0,1]$. By Lemma 4.4, there is $\varepsilon>0$ such that

$$u(t)<\overline{u}(t)\quad \text{in } (0,1)\quad \text{if } u\in B_\varepsilon,$$

where $B_\varepsilon:=\{u\in C_0^1[0,1]:\|u-u_1\|_{C^1}<\varepsilon\}$.

Since $\widetilde{G}(t,s)=\widetilde{F}(t,s)$ when $s<\overline{u}(t)$, $I(u)\equiv J(u)$ in B_ε. Therefore u_1 is a local minimizer of $I(u)$ in $C_0^1[0,1]$. □

Lemma 4.6. *u_1 is a local minimizer of $I(u)$ in $W_0^{1,p}(0,1)$.*

The above lemma is not trivial since $C_0^1[0,1]$ is a proper subspace of $W_0^{1,p}(0,1)$. We need some subsidiary lemmas for the proof of Lemma 4.6.

Lemma 4.7. *Let $\{\nu_n\}$ be a nonnegative sequence and let $u_n,v,w_n\in C[0,1]$ satisfy*

$$\varphi_p(u_n(t))+\nu_n\varphi_p(u_n(t)-v(t))=w_n(t)\quad \text{on } [0,1]. \tag{4.20}$$

If $\{\nu_n\}$ converges and $\{w_n\}$ uniformly converges on $[0,1]$, then so does $\{u_n\}$.

Proof. Put $\Phi(s,r,\nu):=\varphi_p(s)+\nu\varphi_p(s-r)$ for $s,r\in\mathbb{R}$ and $\nu\ge 0$. For any $r\in\mathbb{R}$ and $\nu\ge0$ fixed, $\Phi(s,r,\nu)$ is strictly increasing with respect to s and $\Phi(\,\cdot\,,r,\nu)$ is surjective on $\mathbb{R}$. Hence $t=\Phi(s,r,\nu)$ has an inverse function $s=\Psi(t,r,\nu)$. Moreover, it is easy to verify that $\Psi(t,r,\nu)$ is continuous in three variables. Relation (4.20) is rewritten as $u_n(t)=\Psi(w_n(t),v(t),\nu_n)$. Therefore the convergence of $\{\nu_n\}$ and the uniform convergence of $\{w_n\}$ imply that of $\{u_n\}$. □

Lemma 4.8. *Let $u\in W_0^{1,p}(0,1)$ satisfy*

$$\varphi_p(u')'+\nu\varphi_p(u'-u_1')'+\lambda h\widetilde{f}(t,u)=0, \tag{4.21}$$

in the distribution sense with a certain $\nu\ge0$. Then u belongs to $C_0^1[0,1]$ and $|u'(t)|\le C$ on $[0,1]$, where C depends only on $\|u\|_\infty$ and not on ν.

Proof. Since $\{\varphi_p(u')+\nu\varphi_p(u'-u_1')\}'=-\lambda h\widetilde{f}(t,u)\in L^1_{\rm loc}(0,1)$, we see that $\varphi_p(u')+\nu\varphi_p(u'-u_1')\in W^{1,1}_{\rm loc}(0,1)$. Therefore (4.21) is satisfied almost everywhere in $(0,1)$. By definition, $\widetilde{f}(t,s)$ is nonnegative for all $s\in\mathbb{R}$. Thus (4.21) implies that $\varphi_p(u')+\nu\varphi_p(u'-u_1')$ is non-increasing. Since u_1 is concave, $-u_1'$ is non-decreasing. Therefore u' is non-increasing, i.e. u is concave in $(0,1)$. Since $u(t)=0$ at $t=0,1$, we have either $u\equiv0$ or $u>0$ in $(0,1)$. The conclusion is clear when $u\equiv0$ so that we suppose $u>0$. Then by (A3), there exists $A>0$ depending only on $\|u\|_\infty$ such that

$$|\widetilde{f}(t,u(t))|\le A(u(t)^{p-1}+u_0(t)^{p-1})\quad \text{on } [0,1].$$

Integrating (4.21) over (s,t), we have

$$(4.22)\quad \varphi_p(u'(s)) + \nu\varphi_p(u'(s) - u_1'(s)) = \varphi_p(u'(t)) + \nu\varphi_p(u'(t) - u_1'(t)) + \lambda \int_s^t h(\tau)\widetilde{f}(\tau,u)\,d\tau,$$

for $s,t \in [0,1]$. Fix t and denote the right-hand side by $w(s)$. Then the relation above is rewritten as $u'(s) = \Psi(w(s), u_1'(s), \nu)$, where Ψ is defined in the proof of Lemma 4.7. Since $w \in C(0,1)$, u belongs to $C^1(0,1)$, we shall show that $u \in C^1[0,1]$. Let ε and δ be as in the proof of Lemma 4.3 with $u'(t) > 0, u_1'(t) > 0$ in $(0,\delta)$. Then employing the inequality

$$x^{p-1} \le \varphi_p(x) + \nu\varphi_p(x-y) + \nu y^{p-1} \quad \text{for } x,y \ge 0,$$

and using the decrease of $u(t)/t$, we estimate (4.22) with $t = \delta$ as

$$\begin{aligned} u'(s)^{p-1} &\le C + \lambda A \int_s^\delta h(\tau)u^{p-1}\,d\tau \\ &\le C + \lambda A \int_s^\delta h(\tau)\tau^{p-1}\,d\tau (u(s)/s)^{p-1} \le C + \varepsilon^{p-1}(u(s)/s)^{p-1}. \end{aligned}$$

Here

$$C = |u'(\delta)|^{p-1} + \nu|u'(\delta) - u_1'(\delta)|^{p-1} + \nu\|u_1\|_{C^1}^{p-1} + \lambda A \int_0^1 h(\tau)u_0^{p-1}\,d\tau.$$

Following the proof of Lemma 4.3, we can show that $u(t)/t$ is bounded as $t \to 0$. Similarly, $u(t)/(1-t)$ is bounded as $t \to 1$. Thus $u \in C^1[0,1]$.

We shall show an *a priori* estimate of $|u'(t)|$. Choose $a \in (0,1/2)$ so small that

$$(4.23)\qquad \int_0^a h(\tau)\tau^{p-1}\,d\tau + \int_{1-a}^1 h(\tau)(1-\tau)^{p-1}\,d\tau \le \frac{1}{2\lambda A}.$$

Since u is concave, it has a unique critical point $t_0 \in (0,1)$. Putting $t = t_0$ in (4.22), we have

$$(4.24)\quad \varphi_p(u'(s)) + \nu\varphi_p(u'(s) - u_1'(s)) = -\nu\varphi_p(u_1'(t_0)) + \lambda \int_s^{t_0} h(\tau)\widetilde{f}(\tau,u)\,d\tau,$$

for $s \in [0,1]$. First, we deal with the case where $0 \le \nu \le 1$. Using the inequality

$$|x|^{p-1} \le |\varphi_p(x) + \nu\varphi_p(x-y)| + |y|^{p-1} \quad \text{for } x,y \in \mathbb{R},$$

we reduce (4.24) to

$$(4.25)\qquad |u'(s)|^{p-1} \le B + \lambda A \left| \int_s^{t_0} h(\tau)u(\tau)^{p-1}\,d\tau \right|,$$

for $s \in [0,1]$. Here we put

$$B = 2\|u_1'\|_\infty^{p-1} + \lambda A \int_0^1 h(t) u_0(t)^{p-1}\, dt.$$

Since u is concave, we have

$$0 \le u(\tau) \le \tau u'(0) \quad \text{for } \tau \in [0,1]. \tag{4.26}$$

We divide the proof into two cases.

Case 1. $0 < t_0 \le 1 - a$, where a is determined by (4.23).

Putting $s = 0$ in (4.25) and using (4.26) with $t_0 \le 1 - a$, we have

$$|u'(0)|^{p-1} \le B + \lambda A |u'(0)|^{p-1} \int_0^a h(\tau)\tau^{p-1}\, d\tau + \lambda A \int_a^{1-a} h(\tau) d\tau \|u\|_\infty^{p-1}.$$

Using (4.23), we have

$$(1/2)|u'(0)|^{p-1} \le B + \lambda A \int_a^{1-a} h(\tau)\, d\tau \|u\|_\infty^{p-1}.$$

Thus $|u'(0)|$ has an *a priori* bound depending only on $\|u\|_\infty$.

Case 2. $1 - a < t_0 < 1$.

By the concavity of u, we have $u(\tau) \le (1-\tau)|u'(1)|$. Putting $s = 1$ in (4.25), we have

$$|u'(1)|^{p-1} \le B + \lambda A |u'(1)|^{p-1} \int_{1-a}^1 h(\tau)(1-\tau)^{p-1}\, d\tau,$$

which with (4.23) implies $|u'(1)| \le (2B)^{1/(p-1)} \equiv C$. Thus we have $u(\tau) \le (1-\tau)|u'(1)| \le C(1-\tau)$. From (4.25) with $s = 0$, we estimate

$$\begin{aligned} |u'(0)|^{p-1} \le{}& B + \lambda A |u'(0)|^{p-1} \int_0^a h(\tau)\tau^{p-1}\, d\tau \\ &+ \lambda A \int_a^{1-a} h(\tau) d\tau \|u\|_\infty^{p-1} + \lambda A C^{p-1} \int_{1-a}^1 h(\tau)(1-\tau)^{p-1}\, d\tau. \end{aligned}$$

By (4.23), $|u'(0)|$ has an *a priori* bound depending on $\|u\|_\infty$. In both Cases 1 and 2, $|u'(0)|$ has an *a priori* bound C. Similarly, we have $|u'(1)| \le C$ also. Since u is concave, it follows that

$$|u'(t)| \le \max(|u'(0)|, |u'(1)|) \le C \quad \text{for } \ t \in [0,1].$$

Let us consider the case where $\nu \ge 1$. Dividing (4.24) by ν, we have

$$\begin{aligned} &\varphi_p(u'(s) - u_1'(s)) + (1/\nu)\varphi_p(u'(s)) \\ &\qquad = -\varphi_p(u_1'(t_0)) + (\lambda/\nu) \int_s^{t_0} h(\tau)\widetilde{f}(\tau, u)\, d\tau. \end{aligned} \tag{4.27}$$

Using the inequality

$$A_p |x|^{p-1} \le |\varphi_p(x-y) + (1/\nu)\varphi_p(x)| + |y|^{p-1} \quad \text{for } x, y \in \mathbb{R},$$

where $A_p > 0$ depends only on p, we estimate (4.27) as

$$A_p|u'(s)|^{p-1} \le B + \lambda A\left|\int_s^{t_0} hu(\tau)^{p-1}\,d\tau\right|.$$

In the similar argument as in Cases 1 and 2, we can obtain the boundedness of $|u'(t)|$. □

LEMMA 4.9. *Let $\{u_n\} \subset W_0^{1,p}(0,1)$ satisfy (4.21) with a nonnegative sequence $\{\nu_n\}$ and suppose that $\{u_n\}$ is bounded in $L^\infty(0,1)$. Then a subsequence of $\{u_n\}$ converges strongly in $C_0^1[0,1]$.*

PROOF. Suppose that $u_n \not\equiv 0$ for large n. Then $\{u_n\}$ is positive and concave in $(0,1)$, which has been shown in the proof of Lemma 4.8. By assumption with Lemma 4.8, $\|u_n\|_{C^1}$ is bounded. Let t_n be a critical point of $u_n(t)$. We divide the proof into two cases.

Case 1. $\{\nu_n\}$ is bounded.

After taking a subsequence if needed, we may assume that $\{t_n\}$ and $\{\nu_n\}$ converge to limits t_∞ and ν_∞, respectively, and $\{u_n\}$ converges to a certain limit u_∞ strongly in $C[0,1]$ by the Ascoli-Arzelà theorem. Putting $u = u_n$, $\nu = \nu_n$ and $t_0 = t_n$ in (4.24), we get

$$\text{(4.28)}\quad \varphi_p(u_n'(s)) + \nu_n\varphi_p(u_n'(s) - u_1'(s)) = -\nu_n\varphi_p(u_1'(t_n)) + \lambda\int_s^{t_n} h(\tau)\widetilde{f}(\tau,u_n)\,d\tau.$$

Since $u_n \ge 0$ and $|u_n'(s)| \le C$ on $[0,1]$, we see $0 \le u_n(s) \le 2Cs(1-s)$, for $s \in (0,1)$. By (A3) with the boundedness of $\|u_n\|_\infty$, there is a constant $A > 0$ such that

$$|\widetilde{f}(s,u_n(s))| \le A(u_n(s)^{p-1} + u_0(s)^{p-1}) \quad \text{on } [0,1].$$

Thus we have a constant $B > 0$ independent of n such that

$$|h(s)\widetilde{f}(s,u_n)| \le Bh(s)(s(1-s))^{p-1} \in L^1(0,1).$$

By the Lebesgue convergence theorem, the right-hand side of (4.28) converges uniformly on $[0,1]$. By Lemma 4.7, u_n converges to u_∞ strongly in $C_0^1[0,1]$.

Case 2. $\{\nu_n\}$ is unbounded.

Taking a subsequence, we assume that $\{\nu_n\}$ diverges to infinity. Dividing (4.28) by ν_n, we get

$$\varphi_p(u_n'(s) - u_1'(s)) = -\varphi_p(u_1'(t_n)) - (1/\nu_n)\varphi_p(u_n'(s)) + (\lambda/\nu_n)\int_s^{t_n} h(\tau)\widetilde{f}(\tau,u_n)\,d\tau.$$

The right-hand side converges to $-\varphi_p(u_1'(t_\infty))$ uniformly on $[0,1]$, which implies the uniform convergence of $\{u_n'\}$. Thus $\{u_n\}$ strongly converges in $C_0^1[0,1]$. □

We prove the existence of a minimizer of $I(u)$. The next lemma seems well-known but for the sake of completeness, we give a proof.

Lemma 4.10. *$I(u)$ achieves a minimum on any bounded closed convex subset of $W_0^{1,p}(0,1)$.*

Proof. Let M be a bounded closed convex subset of $W_0^{1,p}(0,1)$. Then $I(u)$ is bounded on M. Indeed, we choose $C > 0$ such that $\|u\|_{1,p} \le C$ for $u \in M$. We note that M is bounded in $L^\infty(0,1)$. Thus there exist constants A, $C_1 > 0$ by (4.2) such that

$$|\widetilde{F}(t,u(t))| \le A(|u(t)|^p + |u_0(t)|^p) \le C_1(t(1-t))^{p-1} \quad \text{for } u \in M, \tag{4.29}$$

which shows the boundedness of $I(u)$ on M. We choose a minimizing sequence $\{u_k\}$, i.e. $u_k \in M$ and

$$\inf\{I(u) : u \in M\} = \lim_{k\to\infty} I(u_k).$$

Since $W_0^{1,p}(0,1)$ is reflexive and M is bounded closed convex, M is weakly sequentially compact. Thus a subsequence of $\{u_k\}$ (denoted by $\{u_k\}$ again) converges weakly to a limit $u_\infty \in M$ and $\|u_\infty\|_{1,p} \le \liminf_{k\to\infty} \|u_k\|_{1,p}$. By the compact imbedding, u_k converges to u_∞ strongly in $C[0,1]$. By the Lebesgue convergence theorem with (4.29),

$$\int_0^1 h(t)\widetilde{F}(t,u_k(t))\,dt \longrightarrow \int_0^1 h(t)\widetilde{F}(t,u_\infty(t))\,dt.$$

Accordingly, we have

$$I(u_\infty) = (1/p)\|u_\infty\|_{1,p}^p - \lambda \int_0^1 h(t)\widetilde{F}(t,u_\infty)\,dt \le \liminf_{k\to\infty} I(u_k).$$

Thus u_∞ is a minimizer of $I(u)$ on M. □

We denote the closed ball of center u_1 with radius r in $W_0^{1,p}(0,1)$ by

$$B_r = B(u_1,r) := \{u \in W_0^{1,p}(0,1) : \|u - u_1\|_{1,p} \le r\}.$$

We are now in a position to prove Lemma 4.6.

Proof of Lemma 4.6. It is enough to prove the existence of $r > 0$ such that $I(u_1) \le I(u)$ if $\|u - u_1\|_{1,p} \le r$. Suppose on the contrary that for any $r \in (0,1)$ there exists $v_r \in B_r = B(u_1,r)$ such that $I(v_r) < I(u_1)$. Then by Lemma 4.10, there exists a minimizer $u_r \in B_r$ such that $I(u_r) \le I(v_r) < I(u_1)$. Define $K(u)$ by

$$K(u) := \frac{1}{p}\|u - u_1\|_{1,p}^p = \frac{1}{p}\int_0^1 |u' - u_1'|^p\,dt.$$

We shall show that there is a sequence $\{r_n\}$ converging to zero such that $\{u_{r_n}\} \subset C_0^1[0,1]$ and it converges strongly in $C_0^1[0,1]$. Since $u_r \neq u_1$, we have

$$K'(u_r)(u_r - u_1) = \|u_r - u_1\|_{1,p}^p > 0. \tag{4.30}$$

Hence $K'(u_r)$ is surjective from $W_0^{1,p}(0,1)$ onto $\mathbb{R}$. Put $\delta = \|u_r - u_1\|_{1,p}$. Since u_r is a minimizer of $I(u)$ on the restriction $K(u) = \delta^p/p$, there is a Lagrange multiplier $\nu_r \in \mathbb{R}$ such that

$$I'(u_r) + \nu_r K'(u_r) = 0, \tag{4.31}$$

or equivalently

$$\varphi_p(u_r')' + \lambda h\widetilde{f}(t, u_r) + \nu_r \varphi_p(u_r' - u_1')' = 0.$$

Since $I(su_r + (1-s)u_1)$ has a minimum at $s = 1$ on $[0,1]$, its derivative at $s = 1$ is nonpositive, that is, $I'(u_r)(u_r - u_1) \leq 0$. Thus by (4.30) and (4.31), ν_r is nonnegative. Since $\|u_r - u_1\|_{1,p} \leq r \leq 1$, Lemmas 4.8 and 4.9 assure that $u_r \in C_0^1[0,1]$ and there is a subsequence $\{r_n\}$ converging to zero such that $\{u_{r_n}\}$ converges to u_1 strongly in $C_0^1[0,1]$. However the fact $I(u_{r_n}) < I(u_1)$ contradicts Lemma 4.5 and this completes the proof. □

By Lemma 4.6, u_1 is a critical point of $I(u)$ and hence it becomes a solution of (P_λ) which belongs to $C_0^1[0,1]$ and satisfies $u_0 \leq u_1$.

5. Distinctness of u_0 and u_1

In this section, we show u_0, u_1 known to exist in the previous section are distinct. Suppose on the contrary that $u_0(t) = u_1(t)$, for all $t \in (0,1)$. Then u_0 is a local minimizer of $I(u)$ in $W_0^{1,p}(0,1)$. We fix $\delta > 0$ so small that

$$I(u_0) \leq I(u) \quad \text{for all } u \in B(u_0, 2\delta).$$

To get a second positive solution, we divide our discussion into two cases:

Case 1. $I(u_0) = d$,

Case 2. $I(u_0) < d$,

where d is defined by

$$d := \inf\{I(u) : u \in \partial B_\delta\}, \qquad \partial B_\delta := \{u \in W_0^{1,p}(0,1) : \|u - u_0\|_{1,p} = \delta\}.$$

In Case 1, note that the infimum of I on ∂B_δ is equal to that on $B_{2\delta}$. Using this fact, we shall prove that I achieves a minimum on ∂B_δ although ∂B_δ is not convex.

Lemma 5.1. *In Case* 1, *the functional I achieves a minimum on ∂B_δ.*

Proof. Let $\{u_k\}$ be a minimizing sequence of I on ∂B_δ. After choosing a subsequence, we assume that $\{u_k\}$ converges to a limit u_∞ weakly in $W_0^{1,p}(0,1)$. Then

$$\|u_\infty - u_0\|_{1,p} \le \liminf_{k\to\infty} \|u_k - u_0\|_{1,p} = \delta.$$

Thus $u_\infty \in B_\delta$ and $d = I(u_0) \le I(u_\infty)$. By the compact imbedding, $\{u_k\}$ converges to u_∞ strongly in $C[0,1]$. By the Lebesgue convergence theorem with the boundedness of $\|u_k\|_{1,p}$, we have the convergence

$$(1/p)\|u_k\|_{1,p}^p = I(u_k) + \lambda \int_0^1 h\widetilde{F}(t,u_k)\,dt \longrightarrow d + \lambda \int_0^1 h\widetilde{F}(t,u_\infty)\,dt.$$

The right-hand side is bounded above by

$$I(u_\infty) + \lambda \int_0^1 h\widetilde{F}(t,u_\infty)\,dt = (1/p)\|u_\infty\|_{1,p}^p.$$

Consequently, $\limsup_{k\to\infty} \|u_k\|_{1,p} \le \|u_\infty\|_{1,p}$. From the weak convergence it follows that $\liminf_{k\to\infty} \|u_k\|_{1,p} \ge \|u_\infty\|_{1,p}$. Accordingly, $\|u_k\|_{1,p}$ converges to $\|u_\infty\|_{1,p}$. Since $W_0^{1,p}(0,1)$ is uniformly convex, $\{u_k\}$ strongly converges to u_∞ in $W_0^{1,p}(0,1)$. Hence $u_\infty \in \partial B_\delta$ and $I(u_\infty) = d$. This completes the proof. □

In Case 1, the Ghoussoub–Preiss version of the mountain pass lemma (see [10] or p. 140 in [5]) is applicable, however we use Lemma 5.1 directly to get the second positive solution.

Lemma 5.2. *In Case* 1, *there exists a positive solution u_1 different from u_0 satisfying $u_0(t) \le u_1(t)$ for $t \in [0.1]$.*

Proof. By Lemma 5.1, there exists a minimizer u_1 of $I(u)$ on ∂B_δ. Thus we have $I(u_1) = d = I(u_0) \le I(u)$ for $u \in B(u_1,\delta)$. This implies that u_1 is a critical point of $I(u)$. By Lemmas 4.2 and 4.3, u_1 is a solution of (P_λ) satisfying $u_0 \le u_1$. Moreover, it is different from u_0 because $\|u_0 - u_1\|_{1,p} = \delta$. □

To deal with Case 2, we apply the mountain pass lemma by Ambrosetti–Rabinowitz [2]. To this end, we need the following lemma.

Lemma 5.3. *There exists $\phi_1 \in W_0^{1,p}(0,1)$ such that $\|\phi_1 - u_0\|_{1,p} > \delta$ and $I(\phi_1) < I(u_0)$.*

Proof. Choose $\phi \in W_0^{1,p}(0,1)$ such that $\phi > 0$ in $(0,1)$ and $\max(u_0(t_0),1) < \phi(t_0)$ at some point t_0. It is enough to show that $I(R\phi) \to -\infty$ as $R \to \infty$. Put $D := \{t \in (0,1) : \phi(t) > \max\{u_0(t),1\}\}$. By the definition of $\widetilde{F}(t,s)$, we have

$$\widetilde{F}(t,s) = F(s) + f(u_0(t))u_0(t) - F(u_0(t)) \quad \text{if } s > u_0(t).$$

Since $\widetilde{F}(t,s) \geq 0$ for all $s \geq 0$, we have for $R > 1$,

$$I(R\phi) \leq (R^p/p)\|\phi\|_{1,p}^p - \lambda \int_D h\widetilde{F}(t, R\phi)\,dt = (R^p/p)\|\phi\|_{1,p}^p - \lambda \int_D hF(R\phi)\,dt - C, \tag{5.1}$$

where

$$C = \lambda \int_D h(t)(f(u_0)u_0 - F(u_0))\,dt.$$

Put $M(R) := \inf\{F(s)/s^p : s \geq R\}$. Then (5.1) can be estimated as

$$I(R\phi) \leq R^p \bigg[(1/p)\|\phi\|_{1,p}^p - \lambda M(R) \int_D h\phi^p\,dt\bigg] - C.$$

Since $M(R)$ diverges to ∞ as $R \to \infty$ by (A4), we conclude that $I(R\phi) \to -\infty$. Putting $\phi_1 = R\phi$, for large R, we completes the proof. □

PROOF OF THEOREM 3.1. It remains to prove Theorem 3.1 for Case 2, that is, $I(u_0) < d = \inf\{I(u) : \|u - u_0\|_{1,p} = \delta\}$. Let ϕ_1 be as in Lemma 5.3. We define

$$\Gamma := \{\gamma \in C([0,1], W_0^{1,p}(0,1)) : \gamma(0) = u_0, \gamma(1) = \phi_1\},$$
$$c := \inf_{\gamma \in \Gamma} \max_{0 \leq \theta \leq 1} I(\gamma(\theta)).$$

By the definition of c, we see $I(u_0) < d \leq c$. By the mountain pass lemma [2] (see [16] also), $I(u)$ has a critical point u_1 such that $I(u_1) = c$. Consequently, u_1 is a positive solution of (P_λ) different from u_0. □

PROOF OF THEOREM 1.4. Let λ_1 be the supremum of λ for which (P_λ) has a positive solution. Then λ_1 is finite by Proposition 1.3(c). By Theorem 1.2, (P_λ) has at least one positive solution for $\lambda \in (0, \mu_1]$. By Theorem 3.1, (P_λ) for $\lambda \in (\mu_1, \lambda_1)$ has at least two distinct positive solutions u_0 and u_1 such that $u_0(t) \leq u_1(t)$ for $t \in [0,1]$. Let $\{(\lambda_n, u_n)\}$ be a sequence of positive solutions such that $\{\lambda_n\}$ converges increasingly to λ_1. Then by Lemmas 2.3–2.5, a subsequence of $\{u_n\}$ converges to a limit u_∞ strongly in $C_0^1[0,1]$. Hence $u_\infty \geq 0$. Since $\lambda_1 > \mu_1$ and $u_n > 0$, Lemma 2.4 assures that $u_\infty \not\equiv 0$. By the strong maximum principle, u_∞ is strictly positive. Thus for $\lambda = \lambda_1$, there is at least one positive solution. By the definition of λ_1, there are no positive solutions for $\lambda > \lambda_1$. □

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