Topological **M**ethods in **N**onlinear **A**nalysis
Volume 42, No. 2, 2013, 233–256

RESOLVENT CONVERGENCE FOR LAPLACE OPERATORS ON UNBOUNDED CURVED SQUEEZED DOMAINS

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Abstract. We establish a resolvent convergence result for the Laplace operator on certain classes of unbounded curved squeezed domains Ω_ε as $\varepsilon \to 0$. As a consequence, we obtain Trotter–Kato-type convergence results for the corresponding family of C^0-semigroups. This extends previous results obtained by Antoci and Prizzi in [1] in the flat squeezing case.

1. Introduction

Let ω be an arbitrary domain in $\mathbb{R}^\ell$, bounded or not, with Lipschitz boundary. Define the bilinear forms $\widetilde{a}_\omega$ and $\widetilde{b}_\omega$ by

$$\begin{aligned}\widetilde{a}_\omega\colon H^1(\omega)\times H^1(\omega) &\to \mathbb{R}, \quad (\widetilde{u},\widetilde{v}) \mapsto \int_\omega \nabla\widetilde{u}(x)\cdot\nabla\widetilde{v}(x)dx;\\ \widetilde{b}_\omega\colon L^2(\omega)\times L^2(\omega) &\to \mathbb{R}, \quad (\widetilde{u},\widetilde{v}) \mapsto \int_\omega \widetilde{u}(x)\widetilde{v}(x)dx.\end{aligned}$$

Then the pair $(\widetilde{a}_\omega,\widetilde{b}_\omega)$ generates a densely defined selfadjoint operator B_ω on $L^2(\omega)$, which is commonly interpreted as the operator $-\Delta$ on ω with Neumann boundary condition.

2010 *Mathematics Subject Classification.* Primary 35B25; Secondary 35B40,35K57.

Key words and phrases. singular perturbations, resolvent convergence, Trotter-Kato-type convergence.

The research of the first author was partially supported by the Brazilian grant CNPq 300656/2009-2.

In this paper we are interested in the case $\omega = \Omega_\varepsilon$, where Ω_ε, for $\varepsilon > 0$ small, is 'thin' of order ε. As $\varepsilon \to 0^+$, the domain Ω_ε 'degenerates' to some limit set, which may no longer be a domain in $\mathbb{R}^\ell$. Our purpose is then to determine the asymptotic behavior of the corresponding family $(B_{\Omega_\varepsilon})_\varepsilon$ as $\varepsilon \to 0$.

More specifically, let $\mathcal{M} \subset \mathbb{R}^\ell$ be a smooth k-dimensional submanifold of $\mathbb{R}^\ell$ and $\mathcal{U} \supset \mathcal{M}$ be a normal (tubular) neighbourhood of $\mathcal{M}$ with normal projection ϕ. For $\varepsilon \in [0,1]$ define the *squeezing operator* $\Gamma_\varepsilon: \mathcal{U} \to \mathcal{U}$ by $x \mapsto \varepsilon x + (1-\varepsilon)s\phi(x)$. For any domain Ω in $\mathbb{R}^\ell$ with Lipschitz boundary and $\mathrm{Cl}\,\Omega \subset \mathcal{U}$ we set $\Omega_\varepsilon = \Gamma_\varepsilon(\Omega)$ and $\mathbf{B}_\varepsilon = B_{\Omega_\varepsilon}$ for $\varepsilon \in \,]0,1]$. A particular case is the *flat squeezing* case in which, writing $\mathbb{R}^\ell = \mathbb{R}^k \times \mathbb{R}^{\ell-k}$, $x = (x_1, x_2)$, we set $\mathcal{M} = \mathbb{R}^k \times \{0\}$, $\mathcal{U} = \mathbb{R}^\ell$ and $\phi(x) = (x_1, 0)$.

Now using the change of variables defined by Γ_ε we may pull $\mathbf{B}_\varepsilon$ back to $L^2(\Omega)$ and thus obtain the densely defined selfadjoint operator $\mathbf{A}_\varepsilon$ in $L^2(\Omega)$ given by:

(a) $u \in D(\mathbf{B}_\varepsilon)$ if and only if $u \circ \Gamma_\varepsilon \in D(\mathbf{A}_\varepsilon)$;

(b) $\mathbf{A}_\varepsilon(u \circ \Gamma_\varepsilon) = (\mathbf{B}_\varepsilon u) \circ \Gamma_\varepsilon$ for $u \in D(\mathbf{B}_\varepsilon)$.

If Ω is bounded, then there is a closed linear subspace $L^2_s(\Omega)$ of $L^2(\Omega)$ and a densely defined selfadjoint operator $\mathbf{A}_0$ on $L^2_s(\Omega)$ such that, as $\varepsilon \to 0^+$, the eigenvalues and eigenfunctions of $\mathbf{A}_\varepsilon$ converge, in a certain strong sense, to the eigenvalues and eigenfunctions of $\mathbf{A}_0$. This *spectral convergence result* was first proved in [6] (cf. also [7]) in the flat squeezing case, and later in [9] in the general curved squeezing case.

The spectral convergence theorem implies various Trotter–Kato-type linear convergence theorems of the C^0-semigroups $e^{-t\mathbf{A}_\varepsilon}$ to $e^{-t\mathbf{A}_0}$, cf. [6], [2], [9], which are used to prove attractor semicontinuity and Conley index continuation results for reaction-diffusion equations with nonlinearities satisfying certain growth assumptions.

It is shown in [3] that certain abstract singular spectral convergence properties of families of selfadjoint operators imply fairly general linear convergence results. These abstract results are then applied in [3] to reaction-diffusion equations with localized singularities, while in [10] they are applied to general curved squeezed domains, relaxing the growth assumptions from [6], [2], [9].

If Ω is unbounded, then, in general, the operators $\mathbf{A}_\varepsilon$ do not have compact resolvents and so spectral convergence results in the above form are not expected to hold. However, as shown in [1] in the flat squeezing case, the resolvents of $\mathbf{A}_\varepsilon$ converge in some sense to the resolvents of $\mathbf{A}_0$ and this is sufficient for the validity of a corresponding linear convergence result.

It is the purpose of this paper to extend the above results from [1] to the curved squeezing case. To this end we impose a geometric condition on $\mathcal{M}$ requiring that $\mathcal{M}$ have bounded normal curvature, see Subsection 2.2. This condition is trivially satisfied in the flat squeezing case (the normal curvature of $\mathcal{M} = \mathbb{R}^k \times \{0\}$ being zero) and for compact manifolds $\mathcal{M}$. A simple example of a noncompact manifold with bounded normal curvature is provided by the graph of the exponential function $\exp\colon \mathbb{R} \to \mathbb{R}$, while the graph of the function $g\colon]0,\infty[\to \mathbb{R}$, $x \mapsto \sin(1/x)$, is a manifold with unbounded normal curvature.

Given a manifold with bounded normal curvature there is normal neighbourhood $\mathcal{U}$ of $\mathcal{M}$ such that for every open set Ω with $\mathrm{Cl}\,\Omega \subset \mathcal{U}$ the analogue of the above mentioned resolvent convergence holds, see Theorem 2.11. We also obtain the corresponding Trotter-Kato-type convergence results, see Corollary 4.5. This latter result is actually obtained, in the spirit of [3], as a consequence of an abstract resolvent assumption, cf. condition (Res) and Theorem 3.4.

The paper is organized as follows. In Section 2 we introduce some notation and discuss the concept of bounded normal curvature for a smooth k-dimensional submanifold $\mathcal{M}$ of $\mathbb{R}^\ell$. We prove that $\mathcal{M}$ has bounded normal curvature if and only if $\mathcal{M}$ has bounded second fundamental form (cf. Proposition 2.2). We also state the main result of this paper, a resolvent convergence for the Laplace operator in the curved squeezing case (cf. Theorem 2.11). The abstract condition (Res) is introduced in Section 3, where we show that this condition implies two linear convergence results (cf. Theorem 3.4). The proof of Theorem 2.11 is given in Section 4.

2. Curved squeezing and resolvent convergence

In this section we introduce some notation and establish preliminary results required for the statement of the resolvent convergence result for the Laplace operator on curved squeezed domains.

In this paper all linear spaces are over the reals.

2.1. Let H be a linear space and V be a linear subspace of H. Let $a\colon V \times V \to \mathbb{R}$ be a bilinear form on V and $b\colon H \times H \to \mathbb{R}$ be a bilinear form on H. Define $R = R(a,b)$ to be the set of all pairs $(u,w) \in V \times H$ such that $a(u,v) = b(w,v)$ for all $v \in V$. We call R the *operator relation generated by the pair* (a,b). If R is the graph of a mapping $B\colon D(B) \to H$, then this map is called the *operator generated by the pair* (a,b).

The following result is well-known (cf. also [8, Lemma 4.4 and its proof]).

Proposition 2.1. *Let V, H be two Hilbert spaces. Suppose V is a dense linear subspace of H and the inclusion map from V to H is continuous. Let $b = \langle\,\cdot\,,\,\cdot\,\rangle$ be the inner product of H and $\|\cdot\|$ and $|\cdot|$ denote the Euclidean norms of V and H. Let $a\colon V \times V \to \mathbb{R}$ be a symmetric bilinear form on V. Assume* *that there are constants d, C, $\alpha \in \mathbb{R}$, $\alpha > 0$, such that, for all u, $v \in V$,*

$$|a(u,v)| \le C\|u\|\|v\|,$$
$$a(u,u) \ge \alpha\|u\|^2 - d|u|^2.$$

Then the operator relation generated by (a,b) is the graph of a densely defined selfadjoint operator B in (H,b). If $d = 0$, then B is positive, $D(B^{1/2}) = V$ and

$$a(u,v) = b(B^{1/2}u, B^{1/2}v), \quad \text{for } u,\ v \in V.$$

Given a Hilbert space $(H, \langle\,\cdot\,,\,\cdot\,\rangle_H)$ and a densely defined selfadjoint operator A on $(H, \langle\,\cdot\,,\,\cdot\,\rangle_H)$ which is nonnegative, i.e. $\langle Au, u\rangle_H \ge 0$ for $u \in D(A)$, the operator $A + I$, where $I = \mathrm{Id}_H$, generates a family $(A+I)^\alpha: D((A+I)^\alpha) \to H$, $\alpha \in [0,\infty[$ of fractional power spaces. The space $D((A+I)^\alpha)$, equipped with the scalar product

$$(u,v) \mapsto \langle u,v\rangle_{(A+I)^\alpha} := \langle (A+I)^\alpha u, (A+I)^\alpha v\rangle_H \tag{2.1}$$

is a Hilbert space.

2.2. We assume throughout this paper that ℓ, k and r are positive integers with $r \ge 2$, $\ell \ge 2$ and $k < \ell$. Let $\langle\,\cdot\,,\,\cdot\,\rangle$ be an inner product on $\mathbb{R}^\ell$ and $\|\cdot\|$ be the corresponding Euclidean norm.

Let $\mathcal{M} \subset \mathbb{R}^\ell$ be a k-dimensional submanifold of $\mathbb{R}^\ell$ of class C^r. Given $m \in [1\,.\,.\,r]$ and $p \in \mathcal{M}$, a *C^m-curve in $\mathcal{M}$ at p* is a C^m-map $\gamma:]-\delta,\delta[\to \mathbb{R}^\ell$ for some $\delta \in]0,\infty[$ with $\gamma(]-\delta,\delta[) \subset \mathcal{M}$ and $\gamma(0) = p$. We identify, in the usual way, the tangent space $T_p\mathcal{M}$ to $\mathcal{M}$ at p with the linear subspace of $\mathbb{R}^\ell$ consisting of all points $\gamma'(0) \in \mathbb{R}^\ell$ where γ is a C^1-curve in $\mathcal{M}$ at p.

If V is open in $\mathcal{M}$, $p \in V$, E is a normed space and $f: V \to E$ is differentiable at p, then there is an open set $U = U_p$ in $\mathbb{R}^\ell$ with $p \in U$ and a map $\widetilde{f} = \widetilde{f}_p: U \to E$ with $\widetilde{f}_{|(U\cap V)} = f_{|(U\cap V)}$ and $\widetilde{f}$ is differentiable at p. The *derivative* $Df(p): T_p(\mathcal{M}) \to E$ *of f at p* is defined as the restriction $D\widetilde{f}_{|T_p(\mathcal{M})}$. $Df(p)$ is independent of the choice of U or $\widetilde{f}$.

For $p \in \mathcal{M}$ let $Q(p): \mathbb{R}^\ell \to \mathbb{R}^\ell$ (resp. $P(p): \mathbb{R}^\ell \to \mathbb{R}^\ell$) be the orthogonal projection of $\mathbb{R}^\ell$ onto $T_p(\mathcal{M})$ (resp. $T_p^\perp(\mathcal{M})$). Thus $P(p) = \mathrm{Id}_{\mathbb{R}^\ell} - Q(p)$.

It is well known that $Q: \mathcal{M} \to \mathcal{L}(\mathbb{R}^\ell, \mathbb{R}^\ell)$ is of class C^{r-1} (cf. Remark 4.2).

It is also well known that $(DQ(p)a)b \in T_p^\perp(\mathcal{M})$ for each $p \in \mathcal{M}$ and all a, $b \in T_p(\mathcal{M})$ (cf. Remark 4.2) and the map

$$\mathrm{II}_p: T_p(\mathcal{M}) \times T_p(\mathcal{M}) \to T_p^\perp(\mathcal{M}), \quad (a,b) \mapsto (DQ(p)a)b$$

is bilinear and symmetric. The map II_p is called the *second fundamental form of $\mathcal{M}$ at p*. Let $s_p: T_p(\mathcal{M}) \to T_p^\perp(\mathcal{M})$ be the quadratic form of II_p, i.e. $s_p(a) = \mathrm{II}_p(a,a)$ for $a \in T_p(\mathcal{M})$. Let

$$\|\mathrm{II}_p\| = \sup\{\, \|\mathrm{II}_p(a,b)\| \mid (a,b) \in T_p(\mathcal{M}) \times T_p(\mathcal{M}),\ \|a\| \le 1,\ \|b\| \le 1\,\}$$

and

$$\|s_p\| = \sup\{\, \|s_p(a)\| \mid a \in T_p(\mathcal{M}),\ \|a\| \le 1 \,\}.$$

It follows from the polarization identity that

$$\|s_p\| \le \|\mathrm{II}_p\| \le 2\|s_p\|. \tag{2.2}$$

Given a C^2 curve γ in $\mathcal{M}$ at p with $\gamma'(0) \ne 0$ its *normal curvature* $c_\gamma(p)$ is defined as $c_\gamma(p) = \|P(p)\gamma''(0)\|/\|\gamma'(0)\|^2$.

We say that $\mathcal{M}$ *has bounded normal curvature* if

$$\sup\{c_\gamma(p) \mid\ p \in \mathcal{M} \text{ and } \gamma \text{ is a } C^2 \text{ curve in } \mathcal{M} \text{ at } p \text{ with } \gamma'(0) \ne 0\ \} < \infty.$$

We say that $\mathcal{M}$ *has bounded second fundamental form* if

$$\sup\{\, \|\mathrm{II}_p(a,b)\| \mid p \in \mathcal{M},\ (a,b) \in T_p(\mathcal{M}) \times T_p(\mathcal{M}),\ \|a\| \le 1,\ \|b\| \le 1 \,\} < \infty.$$

Since $s_p(\gamma'(0)) = (DQ(p)\gamma'(0))\gamma'(0) = P(p)\gamma''(0)$ for every C^2 curve γ in $\mathcal{M}$ at p and since for each $a \in T_p(\mathcal{M})$ there is a C^2 curve γ in $\mathcal{M}$ at p with $a = \gamma'(0)$, the following result follows from (2.2):

Proposition 2.2. *$\mathcal{M}$ has bounded normal curvature if and only if $\mathcal{M}$ has bounded fundamental form.*

Moreover, the following result holds.

Proposition 2.3. *The following conditions are equivalent*:

(a) $M_0 := \sup_{(p,a,c) \in A} \|(DQ(p)a)c\| < \infty$, *where*

$$A = \{\, (p,a,c) \in \mathcal{M} \times T_p(\mathcal{M}) \times T_p^{\perp}(\mathcal{M}) \mid \|a\| \le 1,\ \|c\| \le 1 \,\}.$$

(b) *$\mathcal{M}$ has bounded second fundamental form, i.e.*

$$M_0' := \sup_{(p,a,b) \in A'} \|(DQ(p)a)b\| < \infty,$$

where $A' = \{\, (p,a,b) \in \mathcal{M} \times T_p(\mathcal{M}) \times T_p(\mathcal{M}) \mid \|a\| \le 1,\ \|b\| \le 1 \,\}$.

The proof of Proposition 2.3 is given in Section 4.

Definition 2.4. An open set $\mathcal{U}$ in $\mathbb{R}^\ell$ with $\mathcal{M} \subset \mathcal{U}$ is called a *normal neighbourhood* (or *normal strip*) *of* $\mathcal{M}$ if there is a map $\phi\colon \mathcal{U} \to \mathcal{M}$ of class C^{r-1}, called an *orthogonal projection of* $\mathcal{U}$ *onto* $\mathcal{M}$ and a continuous function $\delta\colon \mathcal{M} \to\,]0,\infty]$, called *the thickness of* $\mathcal{U}$ such that:

(a) whenever $x \in \mathcal{U}$ and $p \in \mathcal{M}$ then $\phi(x) = p$ if and only if the vector $x - p$ is orthogonal to $T_p\mathcal{M}$ (in $\mathbb{R}^\ell$) and $\|x - p\| < \delta(p)$;

(b) $\varepsilon x + (1-\varepsilon)\phi(x) \in \mathcal{U}$ for all $x \in \mathcal{U}$ and all $\varepsilon \in [0,1]$.

Remark. Although in the papers [9], [10] the condition $\mathcal{M} \subset \mathcal{U}$ is erroneously missing, it is implied and satisfied by all normal neighbourhoods constructed and considered there.

The following result follows immediately from Definition 2.4:

Lemma 2.5. *Let $\mathcal{U}$, ϕ and δ be as in Definition 2.4 and $\delta_0\colon \mathcal{M} \to]0,\infty]$ be a continuous function with $\delta_0 \le \delta$. Let $\mathcal{U}_0$ be the set of all $x \in \mathcal{U}$ with $\|x - \phi(x)\| < \delta_0(\phi(x))$. Then $\mathcal{U}_0$ is a normal neighbourhood of $\mathcal{M}$ (relative to the orthogonal projection $\phi_0 = \phi_{|\mathcal{U}_0}$ and the thickness δ_0).*

For the rest of this paper assume the following hypothesis:

(2.3) *$\mathcal{M}$ has bounded normal curvature, i.e. bounded second fundamental form.*

Choose $M \in \mathbb{R}$ arbitrarily with

$$\sup\{\, \|(DQ(p)a)c\| \mid (p,a,c) \in \mathcal{M} \times T_p(\mathcal{M}) \times T_p^{\perp}(\mathcal{M}),\ \|a\| \le 1,\ \|c\| \le 1\,\} \le M < \infty. \tag{2.4}$$

This is possible by assumption (2.3) and Proposition 2.3.

The following proposition is proved in Section 4.

Proposition 2.6. *Let $q_0 \in]0,1[$ be arbitrary and M be as in (2.4). There is normal neighbourhood $\mathcal{U}$ of $\mathcal{M}$ with normal projection ϕ and thickness δ such that $M\delta(p) \le q_0$ for all $p \in \mathcal{M}$.*

For the rest of this paper we fix a $q_0 \in]0,1[$ and a normal neighbourhood $\mathcal{U}$ with normal projection ϕ and thickness δ such that the assertions of Proposition 2.6 are satisfied.

2.3. For $\varepsilon \in [0,1]$ define the maps $\Gamma_\varepsilon\colon \mathcal{U} \to \mathcal{U}$ by

$$x \mapsto \phi(x) + \varepsilon(x - \phi(x)),$$

$J_\varepsilon\colon \mathcal{U} \to \mathbb{R}$ by $J_\varepsilon(x) = |\det(D\Gamma_\varepsilon(x)_{|T_{\phi(x)}(\mathcal{M})})|$, $x \in \mathcal{U}$, and $S_\varepsilon\colon \mathcal{U} \to \mathcal{L}(\mathbb{R}^\ell, \mathbb{R}^\ell)$ by

$$S_\varepsilon(x)h = D\phi(\Gamma_\varepsilon(x))h - (DQ(\phi(x))(D\phi(\Gamma_\varepsilon(x))h))(x - \phi(x)) \tag{2.5}$$

for $x \in \mathcal{U}$ and $h \in \mathbb{R}^\ell$.

In the sequel, given a linear map $B\colon \mathbb{R}^\ell \to \mathbb{R}^\ell$ we denote by B^T the adjoint of B relative to the scalar product $\langle\cdot,\cdot\rangle$.

For the rest of this paper we will assume that

(2.6) Ω is open in $\mathbb{R}^\ell$ with $\mathrm{Cl}(\Omega) \subset \mathcal{U}$. For $\varepsilon \in]0,1]$, we write $\Omega_\varepsilon = \Gamma_\varepsilon(\Omega)$.

For $\varepsilon \in]0,1]$ define the following bilinear forms:

$$\widetilde{a}_\varepsilon\colon H^1(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon) \to \mathbb{R}, \qquad (\widetilde{u},\widetilde{v}) \mapsto \int_{\Omega_\varepsilon} \nabla\widetilde{u}(x)\cdot\nabla\widetilde{v}(x)\,dx;$$

$$\begin{aligned}
\check{a}_\varepsilon &: H^1(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon) \to \mathbb{R}, & (\widetilde{u}, \widetilde{v}) &\mapsto \varepsilon^{-(\ell-k)} \int_{\Omega_\varepsilon} \nabla \widetilde{u}(x) \cdot \nabla \widetilde{v}(x)\, dx;\\
\widetilde{b}_\varepsilon &: L^2(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon) \to \mathbb{R}, & (\widetilde{u}, \widetilde{v}) &\mapsto \int_{\Omega_\varepsilon} \widetilde{u}(x)\widetilde{v}(x)\, dx;\\
\check{b}_\varepsilon &: L^2(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon) \to \mathbb{R}, & (\widetilde{u}, \widetilde{v}) &\mapsto \varepsilon^{-(\ell-k)} \int_{\Omega_\varepsilon} \widetilde{u}(x)\widetilde{v}(x)\, dx,
\end{aligned}$$

and finally let $a_\varepsilon : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ be defined by

$$\begin{aligned}
a_\varepsilon(u,v) = &\int_\Omega J_\varepsilon(x) \langle S_\varepsilon(x)^T \nabla u(x), S_\varepsilon(x)^T \nabla v(x)\rangle\, dx\\
&+ \frac{1}{\varepsilon^2} \int_\Omega J_\varepsilon(x) \langle P(x)\nabla u(x), P(x)\nabla v(x)\rangle\, dx, \quad u, v \in H^1(\Omega).
\end{aligned}$$

For $\varepsilon \in [0,1]$ define the bilinear form $b_\varepsilon : L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}$ by

$$b_\varepsilon(u,v) = \int_\Omega J_\varepsilon(x) u(x) v(x)\, dx, \quad u, v \in L^2(\Omega).$$

We have

$$\widetilde{a}_\varepsilon(u,u) + \widetilde{b}_\varepsilon(u,u) = |u|^2_{H^1(\Omega_\varepsilon)}, \quad \varepsilon \in \,]0,1]\,,\ u \in H^1(\Omega_\varepsilon). \tag{2.7}$$

Let $\varepsilon \in \,]0,1]$ be arbitrary. Then, Proposition 2.1 and (2.7) imply that the pair $(\widetilde{a}_\varepsilon, \widetilde{b}_\varepsilon)$ generates a densely defined selfadjoint operator $\mathbf{B}_\varepsilon$ in $(L^2(\Omega_\varepsilon), \widetilde{b}_\varepsilon)$, which we interpret, as usual, as the operator $-\Delta$ on Ω_ε with Neumann boundary condition on $\partial\Omega_\varepsilon$. Since $\check{a}_\varepsilon = \varepsilon^{-(\ell-k)} \widetilde{a}_\varepsilon$ and $\check{b}_\varepsilon = \varepsilon^{-(\ell-k)} \widetilde{b}_\varepsilon$, we see that

(2.8) the pair $(\check{a}_\varepsilon, \check{b}_\varepsilon)$ generates $\mathbf{B}_\varepsilon$ and both $\mathbf{B}_\varepsilon$ and $\mathbf{B}_\varepsilon + \mathrm{Id}_{L^2(\Omega_\varepsilon)}$ are densely defined selfadjoint linear operators in $(L^2(\Omega_\varepsilon), \check{b}_\varepsilon)$ with $\mathbf{B}_\varepsilon + \mathrm{Id}_{L^2(\Omega_\varepsilon)}$ positive.

Furthermore the following estimates hold:

PROPOSITION 2.7. *There are constants k_1, $k_2 \in \,]0,\infty[$ such that*

$$k_1 b_\varepsilon(u,u) \le |u|^2_{L^2(\Omega)} \le k_2 b_\varepsilon(u,u), \quad \textit{for } \varepsilon \in [0,1] \textit{ and } u \in L^2(\Omega). \tag{2.9}$$

The proof of Proposition 2.7 is given in Section 4. Let us now define the space

$$H^1_s(\Omega) := \{\, u \in H^1(\Omega) \mid P(x)\nabla u(x) = 0 \text{ a.e.} \,\}.$$

Note that

(2.10) $H^1_s(\Omega)$ is a closed linear subspace of the Hilbert space $H^1(\Omega)$.

Now define the 'limit' bilinear form

$$a_0 : H^1_s(\Omega) \times H^1_s(\Omega) \to \mathbb{R}, \qquad (u,v) \mapsto \int_\Omega J_0(x) \langle S_0(x)^T \nabla u(x), S_0(x)^T \nabla v(x)\rangle\, dx.$$

Finally, let $L^2_s(\Omega)$ be the closure of $H^1_s(\Omega)$ in $L^2(\Omega)$. Note that

(2.11) $L^2_s(\Omega)$ is a closed linear subspace of the Hilbert space $L^2(\Omega)$.

For $\varepsilon \in \,]0,1]$ and u, $v \in L^2(\Omega)$ set

$$\langle u,v\rangle_\varepsilon := b_\varepsilon(u,v).$$

For $\varepsilon \in \,]0,1]$ and u, $v \in H^1(\Omega)$ set

$$\langle\!\langle u,v\rangle\!\rangle_\varepsilon := a_\varepsilon(u,v) + b_\varepsilon(u,v).$$

By (2.9), $\langle\,\cdot\,,\,\cdot\,\rangle_\varepsilon$ (resp. $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle_\varepsilon$) is a scalar product on $L^2(\Omega)$ (resp. $H^1(\Omega)$). Let $|\cdot|_\varepsilon$ (resp. $\|\cdot\|_\varepsilon$) be the Euclidean norm on $L^2(\Omega)$ (resp. $H^1(\Omega)$) induced by $\langle\,\cdot\,,\,\cdot\,\rangle_\varepsilon$ (resp. $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle_\varepsilon$). Furthermore, for $u,v \in L^2_s(\Omega)$ set

$$\langle u,v\rangle_0 := b_0(u,v).$$

Finally, for $u,v \in H^1_s(\Omega)$ set

$$\langle\!\langle u,v\rangle\!\rangle_0 := a_0(u,v) + b_0(u,v).$$

Again by (2.9), $\langle\,\cdot\,,\,\cdot\,\rangle_0$ (resp. $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle_0$) is a scalar product on $L^2_s(\Omega)$ (resp. $H^1_s(\Omega)$).

Let $|\cdot|_0$ (resp. $\|\cdot\|_0$) be the Euclidean norm on $L^2_s(\Omega)$ (resp. $H^1_s(\Omega)$) induced by $\langle\,\cdot\,,\,\cdot\,\rangle_0$ (resp. $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle_0$). By (2.9) the norms $|\cdot|_\varepsilon$, $\varepsilon \in [0,1]$, are all equivalent to the usual norm on $L^2(\Omega)$, with equivalence constants independent of ε. Writing $H^\varepsilon = L^2(\Omega)$ for $\varepsilon \in \,]0,1]$ and $H^0 = L^2_s(\Omega)$ we thus see that

(2.12) for $\varepsilon \in [0,1]$, $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_\varepsilon)$ is a Hilbert space.

We have the following result which is also proved in Section 4.

Proposition 2.8. *The following properties hold:*

(2.13) *There exist constants* γ_1, $\gamma_2 \in \,]0,\infty[$ *such that*
$\gamma_1|u|_{H^1(\Omega)} \le \|u\|_\varepsilon \le (1/\varepsilon^2)\gamma_2|u|_{H^1(\Omega)}$ *for all* $\varepsilon \in \,]0,1]$ *and* $u \in H^1(\Omega)$
and
$\gamma_1|u|_{H^1(\Omega)} \le \|u\|_0 \le \gamma_2|u|_{H^1(\Omega)}$ *for all* $u \in H^1_s(\Omega)$

and

(2.14) *there exists a constant* $C \in \,]1,\infty[$ *such that, for* $\varepsilon \in \,]0,1]$,
$\|u\|_\varepsilon \le C\|u\|_0$ *and* $\|u\|_0 \le C\|u\|_\varepsilon$, *whenever* $u \in H^1_s(\Omega)$
and
$|u|_\varepsilon \le C|u|_0$ *and* $|u|_0 \le C|u|_\varepsilon$, *whenever* $u \in H^0$.

We obtain from (2.13) that the norm $\|\cdot\|_\varepsilon$ is, for each $\varepsilon \in [0,1]$, equivalent to the usual norm $|\cdot|_{H^1(\Omega)}$.

Now let $(u_k)_k$ in $H^1_s(\Omega)$ be a $\|\cdot\|_0$-Cauchy sequence. By (2.14) we have that, for any given $\varepsilon \in \,]0,1]$, $(u_k)_k$ is a $\|\cdot\|_\varepsilon$-Cauchy sequence and consequently also a $|\cdot|_{H^1(\Omega)}$-Cauchy sequence. Thus, for some $u \in H^1(\Omega)$, the sequence $(u_k)_k$

converges to u in the $|\cdot|_{H^1(\Omega)}$-norm. Hence $u \in H^1_s(\Omega)$ as $H^1_s(\Omega)$ is closed in $H^1(\Omega)$ in the $|\cdot|_{H^1(\Omega)}$-norm. It follows that $(u_k)_k$ converges to u in the $\|\cdot\|_\varepsilon$-norm and thus, since $u_k - u \in H^1_s(\Omega)$ for $k \in \mathbb{N}$, $(u_k)_k$ converges to u in the $\|\cdot\|_0$-norm. It follows that $(H^1_s(\Omega), \langle\!\langle\cdot\,,\,\cdot\rangle\!\rangle_0)$ is a Hilbert space. By definition, $H^1_s(\Omega)$ is dense in $(H^0, \langle\cdot\,,\,\cdot\rangle_0)$.

Altogether we obtain that

(2.15) for $\varepsilon \in \,]0,1]$, $(H^1(\Omega), \langle\!\langle\cdot\,,\,\cdot\rangle\!\rangle_\varepsilon)$ is a Hilbert space which is continuously and densely embedded in $(H^\varepsilon, \langle\cdot,\cdot\rangle_\varepsilon)$. $(H^1_s(\Omega), \langle\!\langle\cdot\,,\,\cdot\rangle\!\rangle_0)$ is a Hilbert space which is continuously and densely embedded in $(H^0, \langle\cdot\,,\,\cdot\rangle_0)$.

Now, using (2.13) we obtain the estimates

$$a_\varepsilon(u,u) \geq \gamma_1^2 |u|^2_{H^1(\Omega)} - |u|^2_\varepsilon, \quad \varepsilon \in \,]0,1]\,,\ u \in H^1(\Omega), \tag{2.16}$$

$$a_0(u,u) \geq \gamma_1^2 |u|^2_{H^1(\Omega)} - |u|^2_0, \quad u \in H^1_s(\Omega). \tag{2.17}$$

Proposition 2.1 together with (2.16) and (2.17) implies that,

(2.18) for $\varepsilon \in [0,1]$, the pair $(a_\varepsilon, \langle\cdot\,,\,\cdot\rangle_\varepsilon)$ generates a densely defined selfadjoint operator $\mathbf{A}_\varepsilon$ on $(H^\varepsilon, \langle\cdot\,,\,\cdot\rangle_\varepsilon)$.

The definition of a_ε implies that

(2.19) $\mathbf{A}_\varepsilon$ is nonnegative, i.e. $\langle \mathbf{A}_\varepsilon u, u\rangle_\varepsilon \geq 0$ for $\varepsilon \in [0,1]$ and $u \in D(\mathbf{A}_\varepsilon)$.

Let us relate the operators $\mathbf{A}_\varepsilon$ and $\mathbf{B}_\varepsilon$, $\varepsilon \in \,]0,1]$, to each other. Note that, for $\varepsilon \in \,]0,1]$, the inverse $\Gamma_\varepsilon^{-1}: \mathcal{U} \to \mathcal{U}$ of Γ_ε exists and is given by $y \mapsto \phi(y) + \varepsilon^{-1}(y - \phi(y))$ so Γ_ε^{-1} is of class C^{r-1} and

$$D(\Gamma_\varepsilon^{-1})(y)w = D\phi(y)w + \varepsilon^{-1}(w - D\phi(y)w), \quad \varepsilon \in \,]0,1]\,,\ y \in \mathcal{U},\ w \in \mathbb{R}^\ell. \tag{2.20}$$

Define

$$L_\varepsilon: \mathcal{U} \to \mathcal{L}(\mathbb{R}^\ell, \mathbb{R}^\ell), \qquad x \mapsto D(\Gamma_\varepsilon^{-1})(\Gamma_\varepsilon(x)).$$

PROPOSITION 2.9. *For $\varepsilon \in \,]0,1]$, the assignment*

$$u \mapsto \widetilde{u} = u \circ (\Gamma_\varepsilon)_{|\Omega}$$

restricts to linear isomorphisms $L^2(\Omega_\varepsilon) \to L^2(\Omega)$ and $H^1(\Omega_\varepsilon) \to H^1(\Omega)$. Moreover,

$$\nabla u(\Gamma_\varepsilon(x)) = (L_\varepsilon(x))^T(\nabla \widetilde{u}(x)), \quad \textit{for all } u \in H^1(\Omega_\varepsilon) \textit{ and almost all } x \in \Omega.$$

For $\varepsilon \in \,]0,1]$,

$$a_\varepsilon(u \circ (\Gamma_\varepsilon)_{|\Omega}, v \circ (\Gamma_\varepsilon)_{|\Omega}) = \check{a}_\varepsilon(u,v), \quad \textit{for all } u,v \in H^1(\Omega_\varepsilon). \tag{2.21}$$

Moreover,

$$b_\varepsilon(u \circ (\Gamma_\varepsilon)_{|\Omega}, v \circ (\Gamma_\varepsilon)_{|\Omega}) = \check{b}_\varepsilon(u,v), \quad \textit{for all } u,v \in L^2(\Omega_\varepsilon). \tag{2.22}$$

For the proof of Proposition 2.9 we refer to Section 4. Using formulas (2.21) and (2.22) we obtain the following

Proposition 2.10. *The (linear) operators* $\mathbf{B}_\varepsilon$ *(resp.* $\mathbf{A}_\varepsilon$*) defined by* $(\breve{a}_\varepsilon, \breve{b}_\varepsilon)$ *(resp.* $(a_\varepsilon, b_\varepsilon)$*) satisfy the following properties:*

(a) $u \in D(\mathbf{B}_\varepsilon)$ *if and only if* $u \circ (\Gamma_\varepsilon)_{|\Omega} \in D(\mathbf{A}_\varepsilon)$;
(b) $\mathbf{A}_\varepsilon(u \circ (\Gamma_\varepsilon)_{|\Omega}) = (\mathbf{B}_\varepsilon u) \circ (\Gamma_\varepsilon)_{|\Omega}$ *for* $u \in D(\mathbf{B}_\varepsilon)$.

Given $\varepsilon \in [0,1]$ and letting I_ε be the identity operator on H^ε we see that the operator $\mathbf{A}_\varepsilon + I_\varepsilon$ is the operator generated by the pair $(\langle\!\langle \cdot\,,\,\cdot \rangle\!\rangle_\varepsilon, \langle \cdot\,,\,\cdot \rangle_\varepsilon)$. Applying Proposition 2.1 we thus obtain

$$
\begin{aligned}
&\text{For } \varepsilon \in [0,1],\ \mathbf{A}_\varepsilon + I_\varepsilon\colon D(\mathbf{A}_\varepsilon + I_\varepsilon) = D(\mathbf{A}_\varepsilon) \subset H^\varepsilon \to H^\varepsilon \text{ is a densely}\\
&\text{defined positive selfadjoint operator in } (H^\varepsilon, \langle \cdot\,,\,\cdot \rangle_{H^\varepsilon}).\\
&\text{For } \varepsilon \in\,]0,1],\ D((\mathbf{A}_\varepsilon + I_\varepsilon)^{1/2}) = H^1(\Omega) \text{ and}\\
&\langle\!\langle u, v \rangle\!\rangle_\varepsilon = \langle (\mathbf{A}_\varepsilon + I_\varepsilon)^{1/2} u, (\mathbf{A}_\varepsilon + I_\varepsilon)^{1/2} v \rangle_\varepsilon \text{ for } u,\ v \in H^1(\Omega).\\
&D((\mathbf{A}_0 + I_0)^{1/2}) = H^1_s(\Omega) \text{ and}\\
&\langle\!\langle u, v \rangle\!\rangle_0 = \langle (\mathbf{A}_0 + I_0)^{1/2} u, (\mathbf{A}_0 + I_0)^{1/2} v \rangle_\varepsilon \text{ for } u,\ v \in H^1_s(\Omega).
\end{aligned}
\tag{2.23}
$$

We can now state the first main result of this paper:

Theorem 2.11. *Whenever* $(\varepsilon_n)_n$ *is a sequence in* $]0,1]$ *converging to zero,* $w \in H^0$ *and* $(w_n)_n$ *is a sequence in* H^0 *with* $|w_n - w|_0 \to 0$ *as* $n \to \infty$*, then* $\|(\mathbf{A}_{\varepsilon_n} + I_{\varepsilon_n})^{-1} w_n - (\mathbf{A}_0 + I_0)^{-1} w\|_{\varepsilon_n} \to 0$ *as* $n \to \infty$.

The proof is presented in Section 4.

3. An abstract convergence result for linear semiflows

In this section we introduce an abstract hypothesis, condition (Res), and we show that this condition implies two Trotter–Kato-type convergence results for C^0-semigroups of linear operators.

Definition 3.1. We say that the family $(H^\varepsilon, \langle \cdot\,,\,\cdot \rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon \in [0,\varepsilon_0]}$ *satisfies condition* (Res) if the following properties are satisfied:

(a) $\varepsilon_0 \in\,]0,\infty[$ and for every $\varepsilon \in [0,\varepsilon_0]$, $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon})$ is a Hilbert space and $A_\varepsilon\colon D(A_\varepsilon) \subset H^\varepsilon \to H^\varepsilon$ is a densely defined nonnegative self-adjoint operator on $(H^\varepsilon, \langle \cdot\,,\,\cdot \rangle_{H^\varepsilon})$. For $\alpha \in [0,\infty[$ write $H^\varepsilon_\alpha := D((A_\varepsilon + I_\varepsilon)^{\alpha/2})$, where $I_\varepsilon = \mathrm{Id}_{H^\varepsilon}$, and $\langle \cdot\,,\,\cdot \rangle_{H^\varepsilon_\alpha} := \langle \cdot\,,\,\cdot \rangle_{(A_\varepsilon + I_\varepsilon)^{\alpha/2}}$ with the corresponding norm $|\cdot|_{H^\varepsilon_\alpha}$. In particular, $H^\varepsilon_0 = H^\varepsilon$;
(b) for each $\varepsilon \in\,]0,\varepsilon_0]$, H^0 is a linear subspace of H^ε and H^0_1 is a linear subspace of H^ε_1;
(c) there exists a constant $C \in\,]1,\infty[$ such that, for $\varepsilon \in\,]0,\varepsilon_0]$,

$$
\begin{aligned}
&|u|_{H^\varepsilon_1} \le C|u|_{H^0_1} \quad \text{and} \quad |u|_{H^0_1} \le C|u|_{H^\varepsilon_1}, \quad \text{whenever } u \in H^0_1,\\
&|u|_{H^\varepsilon} \le C|u|_{H^0} \quad \text{and} \quad |u|_{H^0} \le C|u|_{H^\varepsilon}, \quad \text{whenever } u \in H^0;
\end{aligned}
$$

(d) whenever $(\varepsilon_n)_n$ is a sequence in $]0,\varepsilon_0]$ converging to zero, $w \in H^0$ and $(w_n)_n$ is a sequence in H^0 with $|w_n - w|_{H^0} \to 0$ as $n \to \infty$, then

$$|(A_{\varepsilon_n} + I_{\varepsilon_n})^{-1} w_n - (A_0 + I_0)^{-1} w|_{H_1^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty.$$

Proposition 3.2. *If $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy condition* (Res)*, then for every $\varepsilon \in\,]0,\varepsilon_0]$, the subspace H_1^0 is closed in $(H_1^\varepsilon, |\cdot|_{H_1^\varepsilon})$.*

Proof. Let $\varepsilon \in\,]0,\varepsilon_0]$ and suppose $(u_n)_n$ is a sequence in H_1^0 with $|u_n - u|_{H_1^\varepsilon} \to 0$ as $n \to \infty$ for some $u \in H_1^\varepsilon$. Part (c) of condition (Res) implies that

$$|u_n - u_m|_{H_1^0} \le C|u_n - u_m|_{H_1^\varepsilon},$$

so $(u_n)_n$ is a Cauchy sequence in the Banach space $(H_1^0, |\cdot|_{H_1^0})$. Therefore $(u_n)_n$ converges in H_1^0 to some v in H_1^0. But part (c) of condition (Res) implies that

$$|u_n - v|_{H_1^\varepsilon} \le C|u_n - v|_{H_1^0}.$$

Hence $u = v$ and thus $u \in H_1^0$. This proves the proposition. □

Remark 3.3. Note that, for α, $t \in\,]0,\infty[$ and $\lambda \in [0,\infty[$

$$\lambda^\alpha e^{-\lambda t} \le C(\alpha) t^{-\alpha} \text{ with } C(\alpha) = (\alpha/e)^\alpha.$$

Let $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy condition (Res). Let $\varepsilon \in [0,\varepsilon_0]$ and $r \in\,]0,\infty[$. Using the Stone–Neumann operational calculus together with the above estimate with $\alpha = 1/2$ we obtain the estimates:

$$|e^{-A_\varepsilon r} u|_{H^\varepsilon} \le |u|_{H^\varepsilon}, \qquad u \in H^\varepsilon, \tag{3.1}$$

$$|e^{-A_\varepsilon r} u|_{H_1^\varepsilon} \le C_0 r^{-1/2} e^r |u|_{H^\varepsilon}, \quad u \in H^\varepsilon, \tag{3.2}$$

where $C_0 = C(1/2)$.

We now state and prove the second main result of this paper.

Theorem 3.4. *Let $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy condition* (Res)*. Suppose $(\varepsilon_n)_n$ is a sequence in $]0,\varepsilon_0]$ with $\varepsilon_n \to 0$. Then the following properties hold:*

(a) *If $u_0 \in H^0$ and $(u_n)_n$ is a sequence such that $u_n \in H^{\varepsilon_n}$ for each $n \in \mathbb{N}$ and $|u_n - u_0|_{H^{\varepsilon_n}} \to 0$ as $n \to \infty$, then*

$$|e^{-tA_{\varepsilon_n}} u_n - e^{-tA_0} u_0|_{H_1^{\varepsilon_n}} \to 0 \quad \textit{as } n \to \infty,$$

uniformly on compact intervals in $]0,\infty[$.

(b) *If $u_0 \in H_1^0$ and $(u_n)_n$ is a sequence such that $u_n \in H_1^{\varepsilon_n}$ for each $n \in \mathbb{N}$ and $|u_n - u_0|_{H_1^{\varepsilon_n}} \to 0$ as $n \to \infty$, then*

$$|e^{-tA_{\varepsilon_n}} u_n - e^{-tA_0} u_0|_{H_1^{\varepsilon_n}} \to 0 \quad \textit{as } n \to \infty,$$

uniformly on compact intervals in $[0,\infty[$.

Proof. We follow, with the appropriate modifications, the arguments in the proof of [1, Proposition 3.2]. Let $u \in H^0$ be arbitrary. Given $n \in \mathbb{N}$ and $t \in [0,\infty[$ consider the function $w_n = w_{n,t}\colon [0,t] \to H^{\varepsilon_n}$ defined by

$$\begin{aligned} w_n(s) &= (A_{\varepsilon_n} + I_{\varepsilon_n})^{-1/2} e^{-A_{\varepsilon_n}(t-s)} (e^{-A_0 s}(A_0 + I_0)^{-1} u - e^{-A_{\varepsilon_n} s}(A_0 + I_0)^{-1} u) \\ &= (A_{\varepsilon_n} + I_{\varepsilon_n})^{-1/2} e^{-A_{\varepsilon_n}(t-s)} e^{-A_0 s}(A_0 + I_0)^{-1} u \\ &\quad - (A_{\varepsilon_n} + I_{\varepsilon_n})^{-1/2} e^{-A_{\varepsilon_n} t}(A_0 + I_0)^{-1} u. \end{aligned}$$

It is an easy exercise, using the estimate (3.1) and part (c) of condition (Res), to prove that w_n is continuous on $[0,t]$, differentiable on $]0,t[$ and, for all $s \in]0,t[$,

$$\begin{aligned} w_n'(s) &= (A_{\varepsilon_n} + I_{\varepsilon_n})^{-1/2} A_{\varepsilon_n} e^{-A_{\varepsilon_n}(t-s)} e^{-A_0 s}(A_0 + I_0)^{-1} u \\ &\quad - (A_{\varepsilon_n} + I_{\varepsilon_n})^{-1/2} e^{-A_{\varepsilon_n}(t-s)} A_0 e^{-A_0 s}(A_0 + I_0)^{-1} u. \end{aligned}$$

Since

$$\begin{aligned} &(A_{\varepsilon_n} + I_{\varepsilon_n})^{-1} A_{\varepsilon_n} e^{-A_{\varepsilon_n}(t-s)} e^{-A_0 s}(A_0 + I_0)^{-1} u \\ &\qquad - (A_{\varepsilon_n} + I_{\varepsilon_n})^{-1} e^{-A_{\varepsilon_n}(t-s)} A_0 e^{-A_0 s}(A_0 + I_0)^{-1} u \\ &\quad = (A_{\varepsilon_n} + I_{\varepsilon_n})^{-1} (A_{\varepsilon_n} + I_{\varepsilon_n}) e^{-A_{\varepsilon_n}(t-s)} e^{-A_0 s}(A_0 + I_0)^{-1} u \\ &\qquad - (A_{\varepsilon_n} + I_{\varepsilon_n})^{-1} e^{-A_{\varepsilon_n}(t-s)} e^{-A_0 s}(A_0 + I_0)^{-1} u \\ &\qquad - (A_{\varepsilon_n} + I_{\varepsilon_n})^{-1} e^{-A_{\varepsilon_n}(t-s)} (A_0 + I_0) e^{-A_0 s}(A_0 + I_0)^{-1} u \\ &\qquad + (A_{\varepsilon_n} + I_{\varepsilon_n})^{-1} e^{-A_{\varepsilon_n}(t-s)} e^{-A_0 s}(A_0 + I_0)^{-1} u \\ &\quad = e^{-A_{\varepsilon_n}(t-s)} (A_0 + I_0)^{-1} e^{-A_0 s} u - e^{-A_{\varepsilon_n}(t-s)} (A_{\varepsilon_n} + I_{\varepsilon_n})^{-1} e^{-A_0 s} u, \end{aligned}$$

it follows that

$$w_n'(s) = (A_{\varepsilon_n} + I_{\varepsilon_n})^{1/2} e^{-A_{\varepsilon_n}(t-s)} ((A_0 + I_0)^{-1} e^{-A_0 s} u - (A_{\varepsilon_n} + I_{\varepsilon_n})^{-1} e^{-A_0 s} u).$$

Let $\tau \in]0,\infty[$ be arbitrary. Since the set of all $e^{-A_0 s} u$ with $s \in [0,\tau]$ is compact in $(H^0, |\cdot|_{H^0})$, part (d) of condition (Res) implies that

$$\rho_n(\tau) := \sup_{s \in [0,\tau]} |(A_{\varepsilon_n} + I_{\varepsilon_n})^{1/2} ((A_0 + I_0)^{-1} e^{-A_0 s} u - (A_{\varepsilon_n} + I_{\varepsilon_n})^{-1} e^{-A_0 s} u)|_{H^{\varepsilon_n}} \to 0$$

as $n \to \infty$. The mean-value theorem now implies that, if $t \le \tau$,

$$\begin{aligned} &|(A_{\varepsilon_n} + I_{\varepsilon_n})^{-1/2} (e^{-A_0 t}(A_0 + I_0)^{-1} u - e^{-A_{\varepsilon_n} t}(A_0 + I_0)^{-1} u)|_{H^{\varepsilon_n}} \\ &\qquad = |w_n(t) - w_n(0)|_{H^{\varepsilon_n}} \le t \cdot \sup_{s \in]0,t[} |w_n'(s)|_{H^{\varepsilon_n}} \le \tau \rho_n(\tau). \end{aligned}$$

Therefore we obtain

$$\sup_{t \in [0,\tau]} |(A_{\varepsilon_n} + I_{\varepsilon_n})^{-1/2} (e^{-A_0 t}(A_0 + I_0)^{-1} u - e^{-A_{\varepsilon_n} t}(A_0 + I_0)^{-1} u)|_{H^{\varepsilon_n}} \to 0$$

as $n \to \infty$. Thus, for each $v \in D(A_0)$,

$$\sup_{t \in [0,\tau]} |(A_{\varepsilon_n} + I_{\varepsilon_n})^{-1/2} (e^{-A_0 t} v - e^{-A_{\varepsilon_n} t} v)|_{H^{\varepsilon_n}} \to 0, \quad \text{as } n \to \infty. \tag{3.3}$$

Since $|(A_{\varepsilon_n}+I_{\varepsilon_n})^{-1/2}|_{\mathcal{L}(H^{\varepsilon_n},H^{\varepsilon_n})} \le 1$, $|e^{-A_0 t}w|_{H^{\varepsilon_n}} \le C|e^{-A_0 t}w|_{H^0} \le C|w|_{H^0}$ and $|e^{-A_{\varepsilon_n} t}w|_{H^{\varepsilon_n}} \le |w|_{H^{\varepsilon_n}} \le C|w|_{H^0}$ for $w \in H^0$ and $n \in \mathbb{N}$, we see that

$$\sup_{t\in[0,\tau]} |(A_{\varepsilon_n}+I_{\varepsilon_n})^{-1/2}(e^{-A_0 t}w - e^{-A_{\varepsilon_n} t}w)|_{H^{\varepsilon_n}} \le 2C|w|_{H^0}.$$

This, together with the validity of (3.3) for $v \in D(A_0)$ and the density of $D(A_0)$ in H^0 implies that (3.3) holds for all $v \in H^0$.

Now for all $t \in [0,\tau]$ we obtain, using (3.3), (3.1) and part (d) of condition (Res), that

$$\begin{aligned}
&|(A_{\varepsilon_n}+I_{\varepsilon_n})^{1/2}(e^{-A_0 t}(A_0+I_0)^{-1}u - e^{-A_{\varepsilon_n} t}(A_0+I_0)^{-1}u)|_{H^{\varepsilon_n}}\\
&\quad\le |(A_{\varepsilon_n}+I_{\varepsilon_n})^{1/2}(e^{-A_0 t}(A_0+I_0)^{-1}u - (A_{\varepsilon_n}+I_{\varepsilon_n})^{-1}e^{-A_0 t}u)|_{H^{\varepsilon_n}}\\
&\qquad + |(A_{\varepsilon_n}+I_{\varepsilon_n})^{1/2}((A_{\varepsilon_n}+I_{\varepsilon_n})^{-1}e^{-A_0 t}u - e^{-A_{\varepsilon_n} t}(A_{\varepsilon_n}+I_{\varepsilon_n})^{-1}u)|_{H^{\varepsilon_n}}\\
&\qquad + |(A_{\varepsilon_n}+I_{\varepsilon_n})^{1/2}(e^{-A_{\varepsilon_n} t}(A_{\varepsilon_n}+I_{\varepsilon_n})^{-1}u - e^{-A_{\varepsilon_n} t}(A_0+I_0)^{-1}u)|_{H^{\varepsilon_n}}\\
&\quad\le |(A_{\varepsilon_n}+I_{\varepsilon_n})^{1/2}((A_0+I_0)^{-1}e^{-A_0 t}u - (A_{\varepsilon_n}+I_{\varepsilon_n})^{-1}e^{-A_0 t}u)|_{H^{\varepsilon_n}}\\
&\qquad + |(A_{\varepsilon_n}+I_{\varepsilon_n})^{-1/2}(e^{-A_0 t}u - e^{-A_{\varepsilon_n} t}u)|_{H^{\varepsilon_n}}\\
&\qquad + |(A_{\varepsilon_n}+I_{\varepsilon_n})^{1/2}((A_{\varepsilon_n}+I_{\varepsilon_n})^{-1}u - (A_0+I_0)^{-1}u)|_{H^{\varepsilon_n}}\\
&\quad\le 2\rho_n(\tau) + \sup_{t\in[0,\tau]} |(A_{\varepsilon_n}+I_{\varepsilon_n})^{-1/2}(e^{-A_0 t}u - e^{-A_{\varepsilon_n} t}u)|_{H^{\varepsilon_n}}.
\end{aligned}$$

Thus, for all $v \in D(A_0)$,

$$\sup_{t\in[0,\tau]} |(A_{\varepsilon_n}+I_{\varepsilon_n})^{1/2}(e^{-A_0 t}v - e^{-A_{\varepsilon_n} t}v)|_{H^{\varepsilon_n}} \to 0 \quad \text{as } n\to\infty. \tag{3.4}$$

Now let $t \in [0,\tau]$ and $w \in H^0$ be arbitrary.

$$\begin{aligned}
&|(A_{\varepsilon_n}+I_{\varepsilon_n})^{1/2}(e^{-A_0 t}w - e^{-A_{\varepsilon_n} t}w)|_{H^{\varepsilon_n}}\\
&\quad\le |(A_{\varepsilon_n}+I_{\varepsilon_n})^{1/2}e^{-A_0 t}w|_{H^{\varepsilon_n}} + |(A_{\varepsilon_n}+I_{\varepsilon_n})^{1/2}e^{-A_{\varepsilon_n} t}w|_{H^{\varepsilon_n}}\\
&\quad\le C|(A_0+I_0)^{1/2}e^{-A_0 t}w|_{H^0} + |(A_{\varepsilon_n}+I_{\varepsilon_n})^{1/2}e^{-A_{\varepsilon_n} t}w|_{H^{\varepsilon_n}} =: T,
\end{aligned}$$

where, by (3.1), (3.2) and part (c) of condition (Res),

$$T \le \begin{cases} 2C_0Ct^{-1/2}e^t|w|_{H^0}, & \text{if } t>0,\\ 2C|w|_{H^0_1}, & \text{if } t\ge 0 \text{ and } w\in H^0_1. \end{cases}$$

Hence, if $\beta \in]0,\tau]$ we obtain

$$\sup_{t\in[\beta,\tau]} |(A_{\varepsilon_n}+I_{\varepsilon_n})^{1/2}(e^{-A_0 t}w - e^{-A_{\varepsilon_n} t}w)|_{H^{\varepsilon_n}} \le 2C_0C\beta^{-1/2}e^{\tau}|w|_{H^0}.$$

Moreover, if $w \in H^0_1$ we have

$$\sup_{t\in[0,\tau]} |(A_{\varepsilon_n}+I_{\varepsilon_n})^{1/2}(e^{-A_0 t}w - e^{-A_{\varepsilon_n} t}w)|_{H^{\varepsilon_n}} \le 2C|w|_{H^0_1}.$$

This implies, together with the validity of (3.4) for $v \in D(A_0)$ and the density of $D(A_0)$ both in $(H^0, |\cdot|_{H^0})$ and in $(H^0_1, |\cdot|_{H^0_1})$, that

$$\sup_{t\in[\beta,\tau]} |e^{-A_0 t}u - e^{-A_{\varepsilon_n} t}u|_{H_1^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty, \text{ if } u \in H^0 \text{ and } \beta \in\,]0,\tau] \tag{3.5}$$

and

$$\sup_{t\in[0,\tau]} |e^{-A_0 t}u - e^{-A_{\varepsilon_n} t}u|_{H_1^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty, \text{ if } u \in H^0_1. \tag{3.6}$$

Now, if u_0 and $(u_n)_n$ are as in part (a), then for $\beta \in\,]0,\tau]$, $t \in [\beta,\tau]$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} |e^{-A_0 t}u_0 - e^{-A_{\varepsilon_n} t}u_n|_{H_1^{\varepsilon_n}} &\le |e^{-A_0 t}u_0 - e^{-A_{\varepsilon_n} t}u_0|_{H_1^{\varepsilon_n}} + |e^{-A_{\varepsilon_n} t}(u_0 - u_n)|_{H_1^{\varepsilon_n}} \\ &\le |e^{-A_0 t}u_0 - e^{-A_{\varepsilon_n} t}u_0|_{H_1^{\varepsilon_n}} + C_0\beta^{-1/2}e^{\tau}|u_n - u_0|_{H^{\varepsilon_n}}. \end{aligned}$$

Together with (3.5) this implies part (a) of the theorem.

If u_0 and $(u_n)_n$ are as in part (b), then for $t \in [0,\tau]$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} |e^{-A_0 t}u_0 - e^{-A_{\varepsilon_n} t}u_n|_{H_1^{\varepsilon_n}} &\le |e^{-A_0 t}u_0 - e^{-A_{\varepsilon_n} t}u_0|_{H_1^{\varepsilon_n}} + |e^{-A_{\varepsilon_n} t}(u_0 - u_n)|_{H_1^{\varepsilon_n}} \\ &\le |e^{-A_0 t}u_0 - e^{-A_{\varepsilon_n} t}u_0|_{H_1^{\varepsilon_n}} + |u_n - u_0|_{H_1^{\varepsilon_n}}. \end{aligned}$$

This together with (3.6) implies part (b) of the theorem. The theorem is proved. □

4. The proof of Theorem 2.11

In this section we prove Propositions 2.3, 2.6–2.9 and Theorem 2.11.

We will need the following easy local result:

Lemma 4.1. *For every $q \in \mathcal{M}$ there is an open set V_q in $\mathcal{M}$ and C^{r-1}-maps $h_i = h_{q,i}\colon V_q \to \mathbb{R}^\ell$, $i \in [1..k]$ and $\nu_j = \nu_{q,j}\colon V_q \to \mathbb{R}^\ell$, $j \in [1..\ell-k]$ such that for every $p \in V_q$ the vectors $h_i(p)$, $i \in [1..k]$, form an orthonormal basis of $T_p\mathcal{M}$, and the vectors $\nu_j(p)$, $j \in [1..\ell-k]$, form an orthonormal basis of the orthogonal complement $T_p^\perp(\mathcal{M})$ of $T_p\mathcal{M}$ in $\mathbb{R}^\ell$.*

It follows that

$$Q(p)h = \sum_{i=1}^{k} \langle h, h_i(p)\rangle h_i(p), \quad q \in \mathcal{M},\ p \in V_q,\ h \in \mathbb{R}^\ell \tag{4.1}$$

and

$$(\mathrm{Id}_{\mathbb{R}^\ell} - Q(p))h = P(p)h = \sum_{j=1}^{\ell-k} \langle h, \nu_j(p)\rangle \nu_j(p), \quad q \in \mathcal{M},\ p \in V_q,\ h \in \mathbb{R}^\ell. \tag{4.2}$$

Therefore we obtain

$$(4.3)\qquad (DQ(p)a)b = \sum_{i=1}^{k}(\langle b, Dh_i(p)a\rangle h_i(p) + \langle b, h_i(p)\rangle Dh_i(p)a), \qquad q \in \mathcal{M},\ p \in V_q,\ a \in T_p(\mathcal{M}),\ b \in \mathbb{R}^\ell,$$

$$(4.4)\qquad -(DQ(p)a)b = \sum_{j=1}^{\ell-k}(\langle b, D\nu_j(p)a\rangle \nu_j(p) + \langle b, \nu_j(p)\rangle D\nu_j(p)a), \qquad q \in \mathcal{M},\ p \in V_q,\ a \in T_p(\mathcal{M}),\ b \in \mathbb{R}^\ell.$$

Remark 4.2. It follows from (4.1), (4.2) and Lemma 4.1 that $Q\colon \mathcal{M} \to \mathcal{L}(\mathbb{R}^\ell, \mathbb{R}^\ell)$ and $P\colon \mathcal{M} \to \mathcal{L}(\mathbb{R}^\ell, \mathbb{R}^\ell)$ are C^{r-1}-maps. Moreover, (4.3) and (4.4) imply that, for $p \in \mathcal{M}$ and $a \in T_p(\mathcal{M})$, $(DQ(p)a)b \in T_p^\perp(\mathcal{M})$ for $b \in T_p(\mathcal{M})$ and $(DQ(p)a)c \in T_p(\mathcal{M})$ for $c \in T_p^\perp(\mathcal{M})$.

Lemma 4.3. *For all $q \in \mathcal{M}$, all $p \in V_q$ and all $a \in T_p(\mathcal{M})$*

$$\sum_{i=1}^{k} \|((DQ(p)a)h_i(p))\|^2 = \sum_{j=1}^{\ell-k} \|((DQ(p)a)\nu_j(p))\|^2.$$

Proof. It follows from (4.3) and (4.4) that for $p \in V_q$ and $a \in T_p(\mathcal{M})$

$$\begin{aligned} (DQ(p)a)\nu_j(p) &= \sum_{i=1}^{k}\langle \nu_j(p), Dh_i(p)a\rangle h_i(p), \quad j \in [1\,..\,\ell-k]\,, \\ -(DQ(p)a)h_i(p) &= \sum_{j=1}^{\ell-k}\langle h_i(p), D\nu_j(p)a\rangle \nu_j(p), \quad i \in [1\,..\,k]\,. \end{aligned}$$

Given $i \in [1\,..\,k]$ and $j \in [1\,..\,\ell-k]$ we have $\langle h_i(p), \nu_j(p)\rangle = 0$ for $p \in V_q$ and we obtain

$$\langle Dh_i(p)a, \nu_j(p)\rangle + \langle h_i(p), D\nu_j(p)a\rangle = 0,$$

$$(DQ(p)a)h_i(p) = \sum_{j=1}^{\ell-k}\langle Dh_i(p)a, \nu_j(p)\rangle \nu_j(p), \quad i \in [1\,..\,k]\,.$$

Hence

$$\begin{aligned} \|(DQ(p)a)\nu_j(p)\|^2 &= \sum_{i=1}^{k} |\langle \nu_j(p), Dh_i(p)a\rangle|^2, \quad j \in [1\,..\,\ell-k]\,, \\ \|(DQ(p)a)h_i(p)\|^2 &= \sum_{j=1}^{\ell-k} |\langle Dh_i(p)a, \nu_j(p)\rangle|^2, \quad i \in [1\,..\,k]\,. \end{aligned}$$

Finally we conclude

$$\sum_{i=1}^{k} \|((DQ(p)a)h_i(p))\|^2 = \sum_{i=1}^{k}\sum_{j=1}^{\ell-k} |\langle \nu_j(p), Dh_i(p)a\rangle|^2$$

and

$$\sum_{j=1}^{\ell-k}\|((DQ(p)a)\nu_j(p))\|^2=\sum_{j=1}^{\ell-k}\sum_{i=1}^{k}|\langle\nu_j(p),Dh_i(p)a\rangle|^2. \qquad \square$$

Proof of Proposition 2.3. Let $p\in\mathcal{M}$ be arbitrary. Lemma 4.1 implies that there is a $q\in\mathcal{M}$ with $p\in V_q$. Let $h_i=h_{q,i}$, $i\in[1\,.\,.\,k]$, and $\nu_j=\nu_{q,j}$, $j\in[1\,.\,.\,\ell-k]$, be as in that lemma.

Given $(a,b,c)\in T_p(\mathcal{M})\times T_p(\mathcal{M})\times T_p^{\perp}(\mathcal{M})$ with $\|a\|\le 1$, $\|b\|\le 1$ and $\|c\|\le 1$, then

$$\|(DQ(p)a)b\|^2=\|\sum_{i=1}^{k}\langle b,h_i(p)\rangle(DQ(p)a)h_i(p)\|^2\le\|b\|^2\|\sum_{i=1}^{k}(DQ(p)a)h_i(p)\|^2.$$

Lemma 4.3 implies

$$\|(DQ(p)a)b\|^2\le\sum_{j=1}^{\ell-k}\|((DQ(p)a)\nu_j(p))\|^2\le M_0^2(\ell-k). \tag{4.5}$$

Analogously,

$$\|(DQ(p)a)c\|^2=\|\sum_{j=1}^{\ell-k}\langle c,\nu_j(p)\rangle(DQ(p)a)\nu_j(p)\|^2\le\|c\|^2\|\sum_{j=1}^{\ell-k}(DQ(p)a)\nu_j(p)\|^2.$$

It follows from Lemma 4.3 that

$$\|(DQ(p)a)c\|^2\le\sum_{i=1}^{k}\|((DQ(p)a)h_i(p))\|^2\le(M_0')^2k. \tag{4.6}$$

Now (4.5) and (4.6) imply that $M_0^2\le(M_0')^2k$ and $(M_0')^2\le M_0^2(\ell-k)$, so $M_0<\infty$ if and only if $M_0'<\infty$. This shows that (a) is equivalent to (b). $\square$

Proof of Proposition 2.6. By [10, Proposition 3.4] there is a normal neighbourhood $\mathcal{U}$ of $\mathcal{M}$ with normal projection ϕ and thickness δ. If $M=0$, then there is nothing to prove. Let $M>0$ and define $\delta_0\colon\mathcal{M}\to\,]0,\infty]$ by $p\mapsto\min(\delta(p),q_0/M)$. An application of Lemma 2.5 yields a normal neighbourhood $\mathcal{U}_0\subset\mathcal{U}$ of $\mathcal{M}$ with normal projection $\phi_0=\phi_{|\mathcal{U}_0}$ and thickness δ_0 satisfying $M\delta_0(p)\le q_0$ for all $p\in\mathcal{M}$. Dropping the index '0' in '$\mathcal{U}_0$', 'δ_0' and 'ϕ_0' we obtain the assertion of the proposition. $\square$

Recall that $q_0\in\,]0,1[$ and $\mathcal{U}$ is a normal neighbourhood with normal projection ϕ and thickness δ such that the assertions of Proposition 2.6 are satisfied. Since $x-\phi(x)\in T_p^{\perp}(\mathcal{M})$ for $x\in\mathcal{U}$, it thus follows that

$$\|(DQ(\phi(x))h)(x-\phi(x))\|\le q_0\|h\| \quad \text{for } x\in\mathcal{U} \text{ and } h\in T_{\phi(x)}(\mathcal{M}). \tag{4.7}$$

We also have $Q(\phi(x))(x-\phi(x))=0$ for $x\in\mathcal{U}$, and so

$$(DQ(\phi(x))D\phi(x)h)(x-\phi(x))+Q(\phi(x))(h-D\phi(x)h)=0 \text{ for } x\in\mathcal{U} \text{ and } h\in\mathbb{R}^{\ell}. \tag{4.8}$$

It follows from (4.8) and the fact that $D\phi(x)h \in T_{\phi(x)}(\mathcal{M})$ for $x \in \mathcal{U}$ and $h \in \mathbb{R}^\ell$ that, for $x \in \mathcal{U}$ and $h \in \mathbb{R}^\ell$

$$D\phi(x)h = Q(\phi(x))h + (DQ(\phi(x))(D\phi(x)h))(x - \phi(x)). \tag{4.9}$$

For $\varepsilon \in [0,1]$ recall that the map $\Gamma_\varepsilon : \mathcal{U} \to \mathcal{U}$ is defined by $x \mapsto \phi(x) + \varepsilon(x - \phi(x))$. It follows that

(4.10) For $\varepsilon \in [0,1]$, Γ_ε is C^{r-1}-map and, for $x \in \mathcal{U}$, $D\Gamma_\varepsilon(x)(T_{\phi(x)}(\mathcal{M})) \subset T_{\phi(x)}(\mathcal{M})$ and $D\Gamma_\varepsilon(x)(T^\perp_{\phi(x)}(\mathcal{M})) \subset T^\perp_{\phi(x)}(\mathcal{M})$ with $D\Gamma_\varepsilon(x)\nu = \varepsilon\nu$ for $\nu \in T^\perp_{\phi(x)}(\mathcal{M})$.

Now (4.10) implies that

$$\det D\Gamma_\varepsilon(x) = \det(D\Gamma_\varepsilon(x)_{|T_{\phi(x)}(\mathcal{M})}) \cdot \det(D\Gamma_\varepsilon(x)_{|T^\perp_{\phi(x)}(\mathcal{M})}).$$

Hence

$$\det D\Gamma_\varepsilon(x) = \varepsilon^{\ell-k} \det(D\Gamma_\varepsilon(x)_{|T_{\phi(x)}(\mathcal{M})}) \quad \text{for } \varepsilon \in [0,1] \text{ and } x \in \mathcal{U}. \tag{4.11}$$

Moreover, since

$$\phi(\Gamma_\varepsilon(x)) = \phi(x), \quad \text{for } \varepsilon \in [0,1] \text{ and } x \in \mathcal{U} \tag{4.12}$$

it follows that

$$D\phi(\Gamma_\varepsilon(x)) \circ D\Gamma_\varepsilon(x) = D\phi(x), \quad \varepsilon \in [0,1]\,,\, x \in \mathcal{U}. \tag{4.13}$$

Using (4.12) we also obtain from (4.9)

$$\begin{aligned} D\phi(\Gamma_\varepsilon(x))h &= Q(\phi(x))h + (DQ(\phi(x))(D\phi(\Gamma_\varepsilon(x))h))(\Gamma_\varepsilon(x) - \phi(x)) \\ &= Q(\phi(x))h + (DQ(\phi(x))(\varepsilon D\phi(\Gamma_\varepsilon(x))h))(x - \phi(x)). \end{aligned} \tag{4.14}$$

It follows from (2.5) that

$$S_\varepsilon(x)(\mathbb{R}^\ell) \subset T_{\phi(x)}(\mathcal{M}). \tag{4.15}$$

Moreover, by (2.5) and (4.13) we have

$$\begin{aligned} S_\varepsilon(x)(D\Gamma_\varepsilon(x)h) &= D\phi(\Gamma_\varepsilon(x))(D\Gamma_\varepsilon(x)h) \\ &\qquad - (DQ(\phi(x))(D\phi(\Gamma_\varepsilon(x))(D\Gamma_\varepsilon(x)h)))(x - \phi(x)) \\ &= D\phi(x)h - (DQ(\phi(x))(D\phi(x)h))(x - \phi(x)). \end{aligned}$$

Hence (4.9) implies

$$S_\varepsilon(x)(D\Gamma_\varepsilon(x)h) = Q(\phi(x))h, \quad \varepsilon \in [0,1]\,,\; x \in \mathcal{U},\; h \in \mathbb{R}^\ell. \tag{4.16}$$

In particular,

$$S_\varepsilon(x)(D\Gamma_\varepsilon(x)h) = h, \quad \varepsilon \in [0,1]\,,\; x \in \mathcal{U},\; h \in T_{\phi(x)}(\mathcal{M})$$

so $D\Gamma_\varepsilon(x)_{|T_{\phi(x)}(\mathcal{M})}\colon T_{\phi(x)}(\mathcal{M}) \to T_{\phi(x)}(\mathcal{M})$ is linear and injective, i.e. bijective as $T_{\phi(x)}(\mathcal{M})$ has finite dimension. We thus obtain

$$D\Gamma_\varepsilon(x)_{|T_{\phi(x)}(\mathcal{M})} = (S_\varepsilon(x)_{|T_{\phi(x)}(\mathcal{M})})^{-1}, \quad \varepsilon \in [0,1]\,,\ x \in \mathcal{U}. \tag{4.17}$$

Since $Q(p)^T = Q(p)$ for all $p \in \mathcal{M}$, we obtain from (4.16) that

$$D\Gamma_\varepsilon(x)^T \circ S_\varepsilon(x)^T = Q(\phi(x)), \quad \varepsilon \in [0,1]\,,\ x \in \mathcal{U}. \tag{4.18}$$

We also have, in view of (4.15),

$$\langle S_\varepsilon(x)^T\nu, h\rangle = \langle \nu, S_\varepsilon(x)h\rangle = 0$$

for $\varepsilon \in [0,1]$, $x \in \mathcal{U}$, $h \in T_{\phi(x)}(\mathcal{M})$, $\nu \in T_{\phi(x)}{}^{\perp}(\mathcal{M})$. Thus

$$S_\varepsilon(x)^T\nu = 0, \quad \varepsilon \in [0,1]\,,\ x \in \mathcal{U},\ \nu \in T_{\phi(x)}{}^{\perp}(\mathcal{M}). \tag{4.19}$$

We have the following technical result:

Lemma 4.4. *For all ε_0, $\varepsilon \in [0,1]$, $x \in \mathcal{U}$ and $h \in \mathbb{R}^\ell$:*

(a) $\|D\phi(x)\|_{\mathcal{L}(\mathbb{R}^\ell,\mathbb{R}^\ell)} \le C_1$ *with* $C_1 = (1-q_0)^{-1}$.
(b) $\|D\Gamma_\varepsilon(x)\|_{\mathcal{L}(\mathbb{R}^\ell,\mathbb{R}^\ell)} \le C_2$ *with* $C_2 = (2(1-q_0)^{-1}+1)$.
(c) $\|S_\varepsilon(x)\|_{\mathcal{L}(\mathbb{R}^\ell,\mathbb{R}^\ell)} \le C_3$ *with* $C_3 = (1+q_0)(1-q_0)^{-1}$.
(d) $C_4 \le J_\varepsilon(x) \le C_5$ *with* $C_4 = (k!C_3^k)^{-1}$ *and* $C_5 = k!C_2^k$.
(e) $C_6\|Q(\phi(x))h\| \le \|S_\varepsilon(x)^Th\| \le C_3\|Q(\phi(x))h\|$ *with* $C_6 = C_2^{-1}$.
(f) $\|D\Gamma_\varepsilon(x) - D\Gamma_{\varepsilon_0}(x)\|_{\mathcal{L}(\mathbb{R}^\ell,\mathbb{R}^\ell)} \le C_7|\varepsilon-\varepsilon_0|$ *with* $C_7 = (1+(1-q_0)^{-1})$.
(g) $\|S_\varepsilon(x) - S_{\varepsilon_0}(x)\|_{\mathcal{L}(\mathbb{R}^\ell,\mathbb{R}^\ell)} \le C_8|\varepsilon-\varepsilon_0|$ *with* $C_8 = (1+q_0)q_0(1-q_0)^{-2}$.
(h) $|\det(D\Gamma_\varepsilon(x)_{|T_{\phi(x)}(\mathcal{M})}) - \det(D\Gamma_{\varepsilon_0}(x)_{|T_{\phi(x)}(\mathcal{M})})| \le C_9|\varepsilon-\varepsilon_0|$ *with* $C_9 = k!kC_2^{k-1}C_7$.

Proof. It follows from (4.9) and (4.7) that

$$\|D\phi(x)h\| \le \|h\| + M\|D\phi(x)h\|\cdot\|x-\phi(x)\| \le \|h\| + q_0\|D\phi(x)h\|,$$

for $x \in \mathcal{U}$, $h \in \mathbb{R}^\ell$. This proves part (a).

Now

$$D\Gamma_\varepsilon(x)h = D\phi(x)h + \varepsilon(h - D\phi(x)h), \quad \varepsilon \in [0,1]\,,\ x \in \mathcal{U},\ h \in \mathbb{R}^\ell \tag{4.20}$$

hence, by part (a), we obtain

$$\|D\Gamma_\varepsilon(x)h\| \le \|D\phi(x)h\| + \|h\| + \|D\phi(x)h\| \le (2(1-q_0)^{-1}+1)\|h\|$$

for $x \in \mathcal{U}$ and $h \in \mathbb{R}^\ell$. This implies part (b).

In view of (2.5), (4.7) and part (a) we have

$$\begin{aligned}\|S_\varepsilon(x)h\| &\le \|D\phi(\Gamma_\varepsilon(x))h\| + \|(DQ(\phi(x))(D\phi(\Gamma_\varepsilon(x))h))(x-\phi(x))\| \\ &\le \|D\phi(\Gamma_\varepsilon(x))h\| + q_0\|D\phi(\Gamma_\varepsilon(x))h\| \le (1+q_0)(1-q_0)^{-1}\|h\|\end{aligned}$$

for $x \in \mathcal{U}$ and $h \in \mathbb{R}^\ell$. This implies part (c).

Let $x \in \mathcal{U}$. There is a $q \in \mathcal{M}$ with $\phi(x) \in V_q$. Let $h_i = h_{q,i}$ for $i \in [1..k]$. For i, $j \in [1..k]$ let $a_{i,j} = \langle h_i(\phi(x)), D\Gamma_\varepsilon(x)h_j(\phi(x))\rangle$ and $b_{i,j} = \langle h_i(\phi(x)), DS_\varepsilon(x)h_j(\phi(x))\rangle$. By parts (b) and (c), $|a_{i,j}| \le C_2$ and $|b_{i,j}| \le C_3$.

Denoting by Perm_k the set of all permutations σ of the set $[1..k]$ and writing $(-1)^\sigma$ for the sign of σ, we have

$$\det(D\Gamma_\varepsilon(x)_{|T_{\phi(x)}(\mathcal{M})}) = \det(a_{i,j})_{i,j} = \sum_{\sigma\in\mathrm{Perm}_k} (-1)^\sigma \prod_{i=1}^{k} a_{i,\sigma(i)}.$$

Thus

$$|\det(D\Gamma_\varepsilon(x)_{|T_{\phi(x)}(\mathcal{M})})| \le k!C_2^k.$$

Analogously,

$$|\det(DS_\varepsilon(x)_{|T_{\phi(x)}(\mathcal{M})})| \le k!C_3^k.$$

These estimates together with (4.17) prove part (d).

Now (4.18) and part (b) yield

$$\begin{aligned}\|Q(\phi(x))h\| &\le \|D\Gamma_\varepsilon(x)^T\|_{\mathcal{L}(\mathbb{R}^\ell,\mathbb{R}^\ell)}\|S_\varepsilon(x)^T h\|\\ &= \|D\Gamma_\varepsilon(x)\|_{\mathcal{L}(\mathbb{R}^\ell,\mathbb{R}^\ell)}\|S_\varepsilon(x)^T h\| \le C_2\|S_\varepsilon(x)^T h\|.\end{aligned}$$

Therefore we obtain

$$C_6\|Q(\phi(x))h\| \le \|S_\varepsilon(x)^T h\|$$

which proves the first estimate in part (e).

To prove the second estimate in part (e) notice that it follows from (4.19) that $S_\varepsilon(x)^T(P(\phi(x))h) = 0$. This implies that $S_\varepsilon(x)^T h = S_\varepsilon(x)^T(Q(\phi(x))h)$. Therefore, by part (c),

$$\begin{aligned}\|S_\varepsilon(x)^T h\| &= \|S_\varepsilon(x)^T(Q(\phi(x))h)\| \le \|DS_\varepsilon(x)^T\|_{\mathcal{L}(\mathbb{R}^\ell,\mathbb{R}^\ell)}\|Q(\phi(x))h\|\\ &= \|DS_\varepsilon(x)\|_{\mathcal{L}(\mathbb{R}^\ell,\mathbb{R}^\ell)}\|Q(\phi(x))h\| \le C_3\|Q(\phi(x))h\|.\end{aligned}$$

This completes the proof of the second estimate in part (e).

An application of (4.20) yields

$$D\Gamma_\varepsilon(x)h - D\Gamma_{\varepsilon_0}(x)h = (\varepsilon-\varepsilon_0)(h - D\phi(x)h), \quad \varepsilon,\varepsilon_0 \in [0,1],\ x\in\mathcal{U},\ h\in\mathbb{R}^\ell.$$

Hence

$$\|D\Gamma_\varepsilon(x) - D\Gamma_{\varepsilon_0}(x)\|_{\mathcal{L}(\mathbb{R}^\ell,\mathbb{R}^\ell)} \le (1+(1-q_0)^{-1})|\varepsilon-\varepsilon_0|, \quad \varepsilon,\varepsilon_0\in[0,1],\ x\in\mathcal{U}.$$

This proves part (f).

We turn to the proof of part (g). Using formula (4.14) we obtain

$$\begin{aligned}D\phi(\Gamma_\varepsilon(x))h - D\phi(\Gamma_{\varepsilon_0}(x))h = {}&(DQ(\phi(x))((\varepsilon-\varepsilon_0)D\phi(\Gamma_\varepsilon(x))h\\ &+ \varepsilon_0(D\phi(\Gamma_\varepsilon(x))h - D\phi(\Gamma_{\varepsilon_0}(x))h))(x-\phi(x)),\end{aligned}$$

and so

$$\begin{aligned}\|D\phi(\Gamma_\varepsilon(x))h - D\phi(\Gamma_{\varepsilon_0}(x))h\| \\ \le q_0\big(|\varepsilon-\varepsilon_0|\|D\phi(\Gamma_\varepsilon(x))h\| + \varepsilon_0\|D\phi(\Gamma_\varepsilon(x))h - D\phi(\Gamma_{\varepsilon_0}(x))h\|\big)\end{aligned}$$

which implies that

$$\|D\phi(\Gamma_\varepsilon(x))h - D\phi(\Gamma_{\varepsilon_0}(x))h\| \le (|\varepsilon-\varepsilon_0|q_0(1-q_0)^{-2})\|h\| \tag{4.21}$$

for ε, $\varepsilon_0 \in [0,1]$, $x \in \mathcal{U}$ and $h \in \mathbb{R}^\ell$. Moreover, it follows from (2.5) that, for ε, $\varepsilon_0 \in [0,1]$, $x \in \mathcal{U}$ and $h \in \mathbb{R}^\ell$,

$$\begin{aligned}S_\varepsilon(x)h - S_{\varepsilon_0}(x)h &= (D\phi(\Gamma_\varepsilon(x))h - D\phi(\Gamma_{\varepsilon_0}(x))h) \\ &\quad - (DQ(\phi(x))(D\phi(\Gamma_\varepsilon(x))h - D\phi(\Gamma_{\varepsilon_0}(x))h))(x-\phi(x))\end{aligned}$$

so (4.21) implies that

$$\begin{aligned}\|S_\varepsilon(x)h - S_{\varepsilon_0}(x)h\| &\le (1+q_0)\|(D\phi(\Gamma_\varepsilon(x))h - D\phi(\Gamma_{\varepsilon_0}(x))h)\| \\ &\le (1+q_0)\big(|\varepsilon-\varepsilon_0|q_0(1-q_0)^{-2}\big)\|h\|\end{aligned}$$

for ε, $\varepsilon_0 \in [0,1]$, $x \in \mathcal{U}$ and $h \in \mathbb{R}^\ell$. We finally obtain

$$\|S_\varepsilon(x)h - S_{\varepsilon_0}(x)h\|_{\mathcal{L}(\mathbb{R}^\ell,\mathbb{R}^\ell)} \le (1+q_0)\big(|\varepsilon-\varepsilon_0|q_0(1-q_0)^{-2}\big)$$

for ε, $\varepsilon_0 \in [0,1]$ and $x \in \mathcal{U}$. This implies part (g).

To complete the proof of the lemma note that if a_i, $\widetilde{a}_i$, $i \in [1\,..\,k]$ are real numbers then

$$\bigg|\prod_{i=1}^k a_i - \prod_{i=1}^k \widetilde{a}_i\bigg| \le k\Big(\max_{i\in[1..k]} \max(|a_i|,|\widetilde{a}_i|)\Big)^{k-1} \max_{i\in[1..k]} |a_i - \widetilde{a}_i|. \tag{4.22}$$

This is easily proved by induction on k.

Let $x \in \mathcal{U}$. There is a $q \in \mathcal{M}$ with $\phi(x) \in V_q$. Let $h_i = h_{q,i}$ for $i \in [1\,..\,k]$. For i, $j \in [1\,..\,k]$ let $a_{i,j} = \langle h_i(\phi(x)), D\Gamma_\varepsilon(x)h_j(\phi(x))\rangle$ and $\widetilde{a}_{i,j} = \langle h_i(\phi(x)), D\Gamma_{\varepsilon_0}(x)h_j(\phi(x))\rangle$. By part (b) and part (f), $|a_{i,j}| \le (2(1-q_0)^{-1}+1)$, $|\widetilde{a}_{i,j}| \le (2(1-q_0)^{-1}+1)$ and $|a_{i,j} - \widetilde{a}_{i,j}| \le (1+(1-q_0)^{-1})|\varepsilon-\varepsilon_0|$. Now

$$\begin{aligned}&\det(D\Gamma_\varepsilon(x)_{|T_{\phi(x)}(\mathcal{M})}) - \det(D\Gamma_{\varepsilon_0}(x)_{|T_{\phi(x)}(\mathcal{M})}) \\ &\qquad = \det(a_{i,j})_{i,j} - \det(\widetilde{a}_{i,j})_{i,j} = \sum_{\sigma\in\mathrm{Perm}_k} (-1)^\sigma \bigg(\prod_{i=1}^k a_{i,\sigma(i)} - \prod_{i=1}^k \widetilde{a}_{i,\sigma(i)}\bigg),\end{aligned}$$

hence, an application of (4.22) and the above estimates yield

$$\begin{aligned}&|\det(D\Gamma_\varepsilon(x)_{|T_{\phi(x)}(\mathcal{M})}) - \det(D\Gamma_{\varepsilon_0}(x)_{|T_{\phi(x)}(\mathcal{M})})| \\ &\qquad \le \sum_{\sigma\in\mathrm{Perm}_k} \bigg|\prod_{i=1}^k a_{i,\sigma(i)} - \prod_{i=1}^k \widetilde{a}_{i,\sigma(i)}\bigg| \\ &\qquad \le k!k(2(1-q_0)^{-1}+1)^{k-1}(1+(1-q_0)^{-1})|\varepsilon-\varepsilon_0|.\end{aligned}$$

This proves part (h). □

Recall that for $\varepsilon \in \,]0,1]$, the inverse $\Gamma_\varepsilon^{-1}\colon \mathcal{U} \to \mathcal{U}$ of Γ_ε is given by $y \mapsto \phi(y) + \varepsilon^{-1}(y - \phi(y))$. Therefore, it follows from (2.20) that

$$\|D(\Gamma_\varepsilon^{-1})(y)\| \le C_1 + \varepsilon^{-1} + \varepsilon^{-1}C_1, \quad \varepsilon \in \,]0,1]\,,\ y \in \mathcal{U}, \tag{4.23}$$

where the constant $C_1 \in \,]0,\infty[$ is as in Lemma 4.4 part (a). Since $L_\varepsilon(x) = D(\Gamma_\varepsilon^{-1})(\Gamma_\varepsilon(x)) = (D\Gamma_\varepsilon(x))^{-1}$, $x \in \mathcal{U}$, we see from (4.16) that

$$Q(\phi(x))(L_\varepsilon(x)h) = S_\varepsilon(x)h, \quad \varepsilon \in \,]0,1]\,,\ x \in \mathcal{U},\ h \in \mathbb{R}^\ell. \tag{4.24}$$

Moreover, (2.20) shows that

$$P(\phi(x))(L_\varepsilon(x)h) = \varepsilon^{-1}P(\phi(x))h, \quad \varepsilon \in \,]0,1]\,,\ x \in \mathcal{U},\ h \in \mathbb{R}^\ell. \tag{4.25}$$

Therefore, (4.24) and (4.25) imply that

$$L_\varepsilon(x)h = S_\varepsilon(x)h + \varepsilon^{-1}P(\phi(x))h, \quad \varepsilon \in \,]0,1]\,,\ x \in \mathcal{U},\ h \in \mathbb{R}^\ell. \tag{4.26}$$

Proof of Proposition 2.7. This follows from Lemma 4.4. □

Proof of Proposition 2.8. This follows from Lemma 4.4. □

Proof of Proposition 2.9. It follows from Lemma 4.4, the estimate (4.23) and well known results from Sobolew space theory that, for $\varepsilon \in \,]0,1]$, the assignment $u \mapsto \widetilde{u} = u \circ ((\Gamma_\varepsilon)_{|\Omega})$ restricts to linear isomorphisms $L^2(\Omega_\varepsilon) \to L^2(\Omega)$ and $H^1(\Omega_\varepsilon) \to H^1(\Omega)$. It is easily checked that $\nabla u(\Gamma_\varepsilon(x)) = (L_\varepsilon(x))^T(\nabla\widetilde{u}(x))$ for all $u \in H^1(\Omega_\varepsilon)$ and almost all $x \in \Omega$. Therefore, using Lemma 4.4 and the change-of-variables formula we obtain formulas (2.21) and (2.22). □

Proof of Theorem 2.11. If the theorem is not true, then there exist a $\beta \in \,]0,\infty[$, a sequence $(\varepsilon_n)_n$ in $]0,1]$ converging to zero, $w \in H^0$ and a sequence $(w_n)_n$ in H^0 with $|w_n - w|_{H^0} \to 0$ as $n \to \infty$ and such that

$$\|u_n - u\|_{\varepsilon_n} \ge \beta, \quad n \in \mathbb{N}, \tag{4.27}$$

where $u = (\mathbf{A}_0 + I_0)^{-1}w$ and $u_n = (\mathbf{A}_{\varepsilon_n} + I_{\varepsilon_n})^{-1}w_n$ for $n \in \mathbb{N}$.

Now, for $n \in \mathbb{N}$, (2.14) implies

$$\begin{aligned}\|u_n\|_{\varepsilon_n}^2 &= \langle\!\langle u_n, u_n\rangle\!\rangle_{\varepsilon_n} = \langle w_n, u_n\rangle_{\varepsilon_n} \\ &\le |w_n|_{\varepsilon_n}|u_n|_{\varepsilon_n} \le |w_n|_{\varepsilon_n}\|u_n\|_{\varepsilon_n} \le C|w_n|_0\|u_n\|_{\varepsilon_n}\end{aligned}$$

so the boundedness of $(|w_n|_0)_n$ implies that

$$(\|u_n\|_{\varepsilon_n})_n \text{ is a bounded sequence.} \tag{4.28}$$

Therefore, in view of (2.13) we see that $(|u_n|_{H^1(\Omega)})_n$ bounded. Taking subsequences if necessary, we may therefore assume that $(u_n)_n$ is weakly convergent in $(H^1(\Omega), |\cdot|_{H^1(\Omega)})$ to some $\widetilde{u} \in H^1(\Omega)$. Now the linear operator $\Phi\colon H^1(\Omega) \mapsto L^2(\Omega,\mathbb{R}^\ell)$, $h \mapsto \widetilde{h}$, where $\widetilde{h}(x) = P(\phi(x))\nabla h(x)$, $x \in \Omega$, is strongly, hence weakly continuous, so $(\Phi(u_n))_n$ weakly converges to $\Phi(\widetilde{u})$ in $L^2(\Omega,\mathbb{R}^\ell)$. In particular,

$$|\Phi(\widetilde{u})|_{L^2(\Omega,\mathbb{R}^\ell)} \leq \liminf_{n\to\infty} |\Phi(u_n)|_{L^2(\Omega,\mathbb{R}^\ell)}.$$

Now (4.28) and the definition of a_ε together with Lemma 4.4 implies that $|\Phi(u_n)|_{L^2(\Omega,\mathbb{R}^\ell)} \to 0$ as $n\to\infty$, so that $\Phi(\widetilde{u}) = 0$ in $L^2(\Omega,\mathbb{R}^\ell)$, i.e. $\widetilde{u}\in H^1_s(\Omega)$. We show that $\widetilde{u} = u$. To this end it is sufficient to show that

$$\langle\!\langle \widetilde{u}, v\rangle\!\rangle_0 = \langle\!\langle u, v\rangle\!\rangle_0, \quad \text{for every } v\in H^1_s(\Omega). \tag{4.29}$$

Let $v\in H^1_s(\Omega)$ be arbitrary. Then

$$\begin{aligned}
&\langle\!\langle u_n, v\rangle\!\rangle_{\varepsilon_n} - \langle\!\langle \widetilde{u}, v\rangle\!\rangle_0 \\
&\quad = \int_\Omega \big(\langle \nabla u_n(x), J_{\varepsilon_n}(x) S_{\varepsilon_n}(x) S_{\varepsilon_n}(x)^T \nabla v(x)\rangle + u_n(x) J_{\varepsilon_n}(x) v(x)\big)\, dx \\
&\qquad - \int_\Omega \big(\langle \nabla \widetilde{u}(x), J_0(x) S_0(x) S_0(x)^T \nabla v(x)\rangle + \widetilde{u}(x) J_0(x) v(x)\big)\, dx
\end{aligned}$$

so

$$\begin{aligned}
|\langle\!\langle u_n, v\rangle\!\rangle_{\varepsilon_n} - \langle\!\langle \widetilde{u}, v\rangle\!\rangle_0| \leq{}& |\nabla u_n|_{L^2(\Omega,\mathbb{R}^\ell)} |J_{\varepsilon_n} S_{\varepsilon_n}(\,\cdot\,)\circ S_{\varepsilon_n}(\,\cdot\,)^T \\
&- J_0 S_0(\,\cdot\,)\circ S_0(\,\cdot\,)^T|_{L^\infty(\Omega,\mathcal{L}(\mathbb{R}^\ell,\mathbb{R}^\ell))} |\nabla v|_{L^2(\Omega)} \\
&+ |u_n|_{L^2(\Omega)} |J_{\varepsilon_n} - J_0|_{L^\infty(\Omega)} |v|_{L^2(\Omega)} \\
&+ \bigg| \int_\Omega (\langle \nabla u_n(x), J_0(x) S_0(x) S_0(x)^T \nabla v(x)\rangle + u_n(x) J_0(x) v(x))\, dx \\
&- \int_\Omega (\langle \nabla \widetilde{u}(x), J_0(x) S_0(x) S_0(x)^T \nabla v(x)\rangle + \widetilde{u}(x) J_0(x) v(x))\, dx \bigg|.
\end{aligned}$$

By Lemma 4.4,

$$\begin{aligned}
&|J_{\varepsilon_n} S_{\varepsilon_n}(\,\cdot\,)\circ S_{\varepsilon_n}(\,\cdot\,)^T - J_0 S_0(\cdot)\circ S_0(\,\cdot\,)^T|_{L^\infty(\Omega,\mathcal{L}(\mathbb{R}^\ell,\mathbb{R}^\ell))} \to 0, \text{ as } n\to\infty \\
&\text{and } |J_{\varepsilon_n} - J_0|_{L^\infty(\Omega)} \to 0, \text{ as } n\to\infty.
\end{aligned} \tag{4.30}$$

Since the linear maps $H^1(\Omega)\to L^2(\Omega,\mathbb{R}^\ell)$, $u\mapsto \nabla u$ and $H^1(\Omega)\to L^2(\Omega)$, $u\mapsto u$ are strongly, hence weakly continuous, the above estimate shows that

$$\text{for every } v\in H^1_s(\Omega),\ \langle\!\langle u_n, v\rangle\!\rangle_{\varepsilon_n} \to \langle\!\langle \widetilde{u}, v\rangle\!\rangle_0 \text{ as } n\to\infty. \tag{4.31}$$

The same argument, replacing 'u_n' by '$\widetilde{u}$', shows that

$$\text{for every } v\in H^1_s(\Omega),\ \langle\!\langle \widetilde{u}, v\rangle\!\rangle_{\varepsilon_n} \to \langle\!\langle \widetilde{u}, v\rangle\!\rangle_0 \text{ as } n\to\infty. \tag{4.32}$$

Now, again for $v\in H^1_s(\Omega)$,

$$\langle\!\langle u_n, v\rangle\!\rangle_{\varepsilon_n} - \langle\!\langle u, v\rangle\!\rangle_0 = \langle w_n, v\rangle_{\varepsilon_n} - \langle w, v\rangle_0$$

and

$$\begin{aligned}|\langle w_n,v\rangle_{\varepsilon_n} - \langle w,v\rangle_0| \\ &\leq |J_{\varepsilon_n} - J_0|_{L^\infty(\Omega)}|w_n|_{L^2(\Omega)}|v|_{L^2(\Omega)} + |J_0|_{L^\infty(\Omega)}|w_n - w|_{L^2(\Omega)}|v|_{L^2(\Omega)} \\ &\leq |J_{\varepsilon_n} - J_0|_{L^\infty(\Omega)}\sqrt{k_2}|w_n|_0|v|_{L^2(\Omega)} + |J_0|_{L^\infty(\Omega)}\sqrt{k_2}|w_n - w|_0|v|_{L^2(\Omega)} \to 0\end{aligned}$$

as $n \to \infty$, so

$$\text{for every } v \in H^1_s(\Omega),\ \langle\langle u_n, v\rangle\rangle_{\varepsilon_n} \to \langle\langle u,v\rangle\rangle_0 \text{ as } n \to \infty. \tag{4.33}$$

Note that (4.31) and (4.33) prove (4.29) and we obtain that $\widetilde{u} = u$. Now, using (4.31) and (4.32) we have

$$\|u_n - u\|^2_{\varepsilon_n} - \langle\langle u_n, u_n\rangle\rangle_{\varepsilon_n} = -2\langle\langle u_n, u\rangle\rangle_{\varepsilon_n} + \langle\langle u,u\rangle\rangle_{\varepsilon_n} \to -\langle\langle u,u\rangle\rangle_0 \tag{4.34}$$

as $n \to \infty$. Now

$$\langle\langle u_n, u_n\rangle\rangle_{\varepsilon_n} - \langle\langle u,u\rangle\rangle_0 = \langle w_n, u_n\rangle_{\varepsilon_n} - \langle w,u\rangle_0$$

and, using the fact that $(u_n)_n$ weakly converges in $L^2(\Omega)$ to u, we obtain

$$\begin{aligned}|\langle w_n, u_n\rangle_{\varepsilon_n} - \langle w,u\rangle_0| &\leq |w_n - w|_{\varepsilon_n}|u_n|_{\varepsilon_n} + |\langle w, u_n\rangle_{\varepsilon_n} - \langle w,u\rangle_0| \\ &\leq C|w_n - w|_0\|u_n\|_{\varepsilon_n} + |J_{\varepsilon_n} - J_0|_{L^\infty(\Omega)}|w|_{L^2(\Omega)}|u_n|_{L^2(\Omega)} \\ &\quad + \left|\int_\Omega u_n(x)J_0 w(x)\,dx - \int_\Omega u(x)J_0 w(x)\,dx\right| \to 0\end{aligned}$$

as $n \to \infty$. Altogether we obtain that $\|u_n - u\|_{\varepsilon_n} \to 0$ as $n \to \infty$, a contradiction to (4.27). The theorem is proved. □

The statements (2.12), (2.23), (2.18), (2.19), (2.10), (2.11), (2.13), (2.14) and Theorem 2.11 now imply the following result.

COROLLARY 4.5. *The family* $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, \mathbf{A}_\varepsilon)_{\varepsilon\in[0,1]}$ *defined in Section* 2 *satisfies assumption* (Res). *Consequently, the assertions of Theorem* 3.4 *hold in this case.*

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Manuscript received November 30, 2012

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