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ON THE TOPOLOGICAL DIMENSION OF THE SOLUTION SET OF A CLASS OF NONLOCAL ELLIPTIC PROBLEMS

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Dedicated to Prof. Biagio Ricceri, 'il miglior fabbro'

Abstract. We study a Dirichlet problem for an elliptic equation of resonant type involving a general nonlocal term. Using a result of Ricceri, we prove that the solution set for such equation has a positive topological dimension, and contains a nondegenerate connected component. In particular, the solution set has the cardinality of the continuum.

1. Introduction

Nonlocal problems, of both elliptic and parabolic type, are characterized by the presence of a term $A(u)$ depending on all values of the unknown function u (*nonlocal term*). These problems can be studied with a variety of methods, leading to different results. We mention the works of Allegretto and Barabanova [1], Chang and Chipot [4], Chipot [5], Chipot, Valente and Vergara Caffarelli [6] and Gomes and Sanchez [9]. The existence of one solution for nonlocal problems is proved in [1] via a priori estimates and degree theory and in [4] via fixed points methods. Multiplicity results, under special assumptions on the nonlocal term, are obtained in [6] and in [9] via variational methods. In [5], among other results, the author proves that the solution set of a non-local elliptic Dirichlet problem may have the cardinality of the continuum (see Section 4 below for more details).

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In the present paper we deal with the following boundary value problem for an elliptic partial differential equation involving a general nonlocal term:

$$\begin{cases} -\Delta u - \lambda_1 u = A(u)f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded, smooth domain, $\lambda_1 > 0$ is the first positive eigenvalue of the operator $-\Delta$ in Ω with homogeneous Dirichlet boundary conditions, with corresponding eigenfunction $\varphi_1 \in C^1_0(\overline{\Omega})_+$. Moreover,

$$A(u) = a\bigg(\int_\Omega G(x, u, \nabla u)\, dx\bigg),$$

where $a \in C(\mathbb{R})$ is a bounded function and $G\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying

$$\sup_{|s|,|\xi| \leq M} |G(\,\cdot\,, s, \xi)| \in L^1(\Omega) \quad \text{for all } M > 0. \tag{1.2}$$

Finally $f \in C^{0,\alpha}(\overline{\Omega})$ ($0 < \alpha < 1$) satisfies the orthogonality condition:

$$\int_\Omega \varphi_1(x) f(x)\, dx = 0. \tag{1.3}$$

We denote by S the set of (classical) solutions of problem (1.1), that is:

$$S = \{u \in C^{2,\alpha}_0(\overline{\Omega}) : u \text{ is a solution of (1)}\}.$$

Our main result establishes a lower bound for the dimension of S:

Theorem 1.1. *Let Ω, A, f be as above. Then*:

(a) $\dim(S) \geq 1$;
(b) *S contains a nondegenerate connected component.*

By $\dim(S)$ we denote the *covering dimension* of S (for the definition and basic properties of dim, see Section 2 below). As a consequence, we will prove that the cardinality of S is $\mathfrak{c}$ (see Corollary 3.1). So, our result can be regarded as a strong multiplicity theorem for problem (1.1), which is of *resonant* type due to the presence of λ_1.

Our approach is based on a result of Ricceri [13] (see also Ricceri [12] and Saint Raymond [14]) which assures that, if L is a non-injective, bounded linear operator from a Banach space X onto a Banach space Y, and N is a completely continuous mapping from X into Y with bounded range, then the solution set of the equation

$$L(u) = N(u)$$

has a positive covering dimension. We will deal with problem (1.1) within this framework, proving (a). Moreover, (b) will follow from technical results in dimension theory.

The same method was already applied to other problems by Ricceri [12], Anello [3] and Dzedzej and Gel'man [8]. In [12], lower bounds are established for the dimension of the solution set of an elliptic equation involving a nonlocal term. In [3] and [8], differential inclusions are investigated.

The paper is organized as follows. In Section 2 we recall some preliminary notions and results. In Section 3 we prove Theorem 1.1 and a useful corollary. Finally, in Section 4 we present some examples and discuss possible developments of our results.

2. Mathematical background

First, we introduce some basic features of dimension theory (we will only recall what is needed for our purposes, referring the interested reader to the book of Engelking [7]). In what follows, S will be a metric space and $\mathbb{N} = \{0, 1, \dots\}$. We start with the definition of *covering dimension*:

Definition 2.1 ([7, p. 385]). The *covering dimension* of S, denoted $\dim(S) \in \mathbb{N} \cup \{\infty\}$, is defined as follows:

(a) $\dim(S) \leq n$, $n \in \mathbb{N}$, if for all finite open cover $\mathcal{A}$ of S, there exists a finite open refinement $\mathcal{B}$ of $\mathcal{A}$ such that $B_1 \cap \ldots \cap B_{n+2} = \emptyset$ for all $B_1, \ldots B_{n+2} \in \mathcal{B}$ pairwise distinct;

(b) $\dim(S) = n$, $n \in \mathbb{N}$ if $\dim(S) \leq n$ holds and $\dim(S) \leq n - 1$ does not;

(c) $\dim(S) = \infty$ if $\dim(S) \leq n$ does not hold for any $n \in \mathbb{N}$.

We will use some properties of the covering dimension. We recall that the covering dimension is a *topological invariant*, that is, if S and T are metric spaces and $\varphi: S \to T$ is a homeomorphism, then $\dim(S) = \dim(T)$. Moreover, in the case of Euclidean spaces, the covering dimension equals the usual one, that is,

$$\dim(\mathbb{R}^n) = n \quad \text{for all } n \in \mathbb{N}. \tag{2.1}$$

There is a strict relation between the covering dimension of spaces and their topological properties and cardinality. For instance, it is easily seen from Definition 2.1 that, if $\dim(S) > 0$, then S has at least a cluster point, in particular the cardinality of S, denoted $|S|$, is infinite. The question of 'how big' is $|S|$ is more delicate, as the following examples show: set $S_1 = \mathbb{Q}$ and $S_2 = \mathbb{R} \setminus \mathbb{Q}$, then

$$\dim(S_1) = \dim(S_2) = 0,$$

despite the fact that S_1 is countable while $|S_2| = \mathfrak{c}$.

Under special assumptions, the covering dimension of a metric space gives some information about disconnectedness issues:

Lemma 2.2 ([7, p. 388]). *If T is a locally compact metric space, then the following are equivalent*:

(a) $\dim(T) = 0$;
(b) *T is hereditarily disconnected (i.e. the connected components of T are singletons).*

We conclude this section by recalling the result of Ricceri [13] which will be our main tool (see also [12] for an alternative version and some applications):

Theorem 2.3 ([13], statement and proof of Theorem 1). *If X and Y are Banach spaces, $L\colon X \to Y$ is a surjective, bounded linear operator, $N\colon X \to Y$ is a completely continuous operator such that $N(X)$ is bounded, then there exists a compact set $T \subset X$ such that*:

(a) $T \subseteq \{u \in X : L(u) = N(u)\}$;
(b) $\dim(T) \geq \dim(\ker(L))$.

3. Main results

In this section we prove our main result, Theorem 1.1, and establish a consequence.

Proof of Theorem 1.1. We will apply Theorem 2.3. We set $X = C_0^{2,\alpha}(\overline{\Omega})$ and $Z = C^{0,\alpha}(\overline{\Omega})$ $(0 < \alpha < 1)$. X and Z, endowed with the usual norms, are Banach spaces.

From spectral theory (see Ambrosetti and Malchiodi [2, p. 8]) we know that the Laplacian operator $-\Delta$, considered as a linear operator between X and Z, admits an increasing sequence of positive eigenvalues $0 < \lambda_1 < \lambda_2 < \ldots$ and that the first eigenvalue λ_1 is isolated and simple. We denote $\varphi_1 \in C_0^1(\overline{\Omega})$ $(\varphi_1(x) > 0$ for all $x \in \Omega)$ the positive eigenfunction corresponding to λ_1. For all $h \in Z$, the linear problem

$$\begin{cases} -\Delta u - \lambda_1 u = h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

has a solution if and only if h satisfies the orthogonality condition:

$$\int_\Omega \varphi_1(x)h(x)\,dx = 0.$$

Moreover, if this is the case, the solution set of (3.1) is a closed, linear, one-dimensional submanifold of X. We define $L\colon X \to Z$ by

$$L(u) = -\Delta u - \lambda_1 u \quad \text{for all } u \in X.$$

Then, L is a bounded linear operator and we have

$$\ker(L) = \{\mu\varphi_1 : \mu \in \mathbb{R}\} \quad \text{and} \quad L(X) = \left\{h \in Z : \int_\Omega \varphi_1(x)h(x)\,dx = 0\right\}.$$

Since $\ker(L)$ is homeomorphic to $\mathbb{R}$, by (2.1) we get

$$\dim(\ker(L)) = 1. \tag{3.2}$$

We set $Y = L(X)$ (note that Y is a Banach space) and rephrase $L: X \to Y$, so that L is surjective. We define $N: X \to Y$ by putting

$$N(u) = A(u)f \quad \text{for all } u \in X.$$

The mapping N is well defined, as for all $u \in X$ we have by (1.2)

$$\int_\Omega G(x, u, \nabla u)\, dx \in \mathbb{R}$$

and $A(u)f \in Y$ by (1.3). We prove now that N is continuous. Assume that $u_n \to u$ in X, in particular $u_n \to u$ and $\nabla u_n \to \nabla u$ uniformly in $\overline{\Omega}$. By (1.2) and the dominated convergence theorem we have

$$\int_\Omega G(x, u_n, \nabla u_n)\, dx \to \int_\Omega G(x, u, \nabla u)\, dx$$

and so $N(u_n) \to N(u)$ in Y (recall that a is continuous). Since a is bounded, we have also that N has a bounded range, namely

$$N(X) \subseteq a(\mathbb{R})f.$$

Thus, the set $N(X)$ is relatively compact. In particular, N is completely continuous. Moreover, we observe that, for all $u \in X$, u is a solution of (1.1) if and only if

$$L(u) = N(u) \quad \text{in } Y.$$

Now we apply Theorem 2.3: there exists a compact subset T of X such that $T \subseteq S$ and

$$\dim(T) \geq 1 \quad (\text{see (3.2)}). \tag{3.3}$$

From (3.3) we have

$$\dim(S) \geq \dim(T) \geq 1 \quad (\text{see [7, p. 387]}),$$

so (a) is achieved.

Now we prove (b). Since T is a compact metric space, by Lemma 2.2 and (3.3) it contains a nondegenerate connected set, which is obviously a nondegenerate connected subset of S. □

We point out an interesting consequence of Theorem 1.1:

Corollary 3.1. *Let Ω, A, f be as above. Then, $|S| = \mathfrak{c}$.*

Proof. From (b) of Theorem 1.1 we know that S contains a nondegenerate connected subset. So, by classical results, $|S| \geq \mathfrak{c}$. On the other hand, an easy cardinality argument shows that $|C_0^{2,\alpha}(\overline{\Omega})| = \mathfrak{c}$, so we get $|S| = \mathfrak{c}$. □

4. Examples and further discussion

In this final section we wish to present some examples, compare our results with the ones found in the current literature and discuss possible further developments.

We rapidly resume the main ideas of the paper of Chipot [5]. The Author studies the problem:

$$\begin{cases} -\Delta u = a(\overline{u})f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $f \in W^{-1,2}(\Omega)$, $a \in C(\mathbb{R})$ is such that $\lambda \le a(t) \le \Lambda$ $(0 < \lambda < \Lambda)$ and

$$\overline{u} = \frac{1}{|\Omega|_N} \int_\Omega u(x)\,dx \quad (|\Omega|_N \text{ is the Lebesgue } N\text{-dimensional measure of } \Omega)$$

denotes the mean value of u. A one-to-one correspondence is established between the solutions of (4.1) in the space $H^1_0(\Omega)$ and the solutions of the equation

$$\frac{t}{a(t)} = \overline{\varphi}, \tag{4.2}$$

where φ is the unique solution of the Dirichlet problem

$$\begin{cases} -\Delta u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In particular, whenever the solution set of (4.2) has cardinality $\mathfrak{c}$, so does also the solution set of (4.1). The arguments of [5] are fully based on the homogeneity of the functional $u \mapsto \overline{u}$ and on the injectivity of the Laplace operator.

Our setting, of course, is different: we use a non-injective operator ($u \mapsto -\Delta u - \lambda_1 u$) and a very general nonlocal term. Nevertheless, some comparison can be made:

Example 4.1. We consider the domain $B_1 = \{x \in \mathbb{R}^2 : |x| < 1\}$. We recall that, in this case, we have

$$\lambda_1 = j^2_{0,1}, \ \varphi_1(x) = \frac{J_0(j_{0,1}|x|)}{\sqrt{\pi}|J_0'(j_{0,1})|},$$

where $j_{0,1}$ denotes the first zero of the Bessel function J_0 (see Henrot [10, p. 11]). We choose $f \in C^{0,\alpha}(\overline{B}_1)$ satisfying (1.3). We set

$$a(t) = \frac{1}{1+|t|} \quad \text{for all } t \in \mathbb{R}$$

and

$$G(x,s,\xi) = \frac{s}{\pi} \quad \text{for all } (x,s,\xi) \in B_1 \times \mathbb{R} \times \mathbb{R}^2,$$

so that

$$\int_{B_1} G(x,u,\nabla u)\,dx = \overline{u} \quad \text{for all } u \in C^{2,\alpha}_0(\overline{B}_1).$$

Then, by our results, the problem

$$\begin{cases} -\Delta u - \lambda_1 u = \dfrac{f(x)}{1+|\overline{u}|} & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases}$$

admits a set of solutions in $C_0^{2,\alpha}(\overline{B}_1)$ with positive dimension (in particular, with cardinality $\mathfrak{c}$). Note that a goes to 0 as $|t| \to \infty$, which places our nonlocal term out of the class considered in [5]. Note, also, that equation (4.2) in this case admits only a finite number of solutions.

We also present an example related to a two-point problem:

Example 4.2. We know that the first eigenvalue of the two-point problem

$$\begin{cases} -u'' = \lambda u & \text{in } [0,\pi], \\ u(0) = u(\pi) = 0, \end{cases}$$

is $\lambda_1 = 1$, corresponding to the eigenfunction $\varphi_1(x) = \sin(x)$. Set

$$f(x) = \cos(x) \quad \text{for all } x \in [0,\pi],$$

so that (1.3) holds. Moreover, set

$$G(x,s,\xi) = |\xi|^2 \quad \text{for all } (x,s,\xi) \in [0,\pi] \times \mathbb{R} \times \mathbb{R}$$

and let $a \in C(\mathbb{R})$ be a bounded function. Then, we have a nonlinear nonlocal term, namely

$$A(u) = a\biggl(\int_0^\pi (u')^2\,dx\biggr).$$

By our results, the nonlocal problem

$$\begin{cases} -u'' - u = A(u)\cos(x) & \text{in } [0,\pi], \\ u(0) = u(\pi) = 0, \end{cases}$$

admits a set of solutions in $C_0^{2,\alpha}([0,\pi])$ (for an arbitrary $\alpha \in \,]0,1[$) containing a nondegenerate connected subset and having cardinality $\mathfrak{c}$. A similar problem was studied by Ricceri [12, Proposition 4]. Besides, nonlocal terms of the same type have been considered by Chipot, Valente and Vergara Caffarelli [6].

Remark 4.3. Our Theorem 1.1 is not strictly related to the Laplace operator: we could replace, in problem (1.1), the term $-\Delta u - \lambda_1 u$ with any $\mathcal{L}u$, where $\mathcal{L}$ is a bounded linear differential operator such that $\dim(\ker(\mathcal{L})) > 0$, provided f belongs to the range of $\mathcal{L}$. Of course, the latter assumption (which in our case is equivalent to the orthogonality condition (1.3)) is not, in general, easy to satisfy. For instance, Nirenberg and Walker [11] deal with an operator $\mathcal{L}$ such that $\dim(\ker(\mathcal{L})) = \infty$, but we do not know how the range of such $\mathcal{L}$ can be characterized.

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