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WEAK LOCAL NASH EQUILIBRIUM

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Abstract. In this paper, we consider a concept of local Nash equilibrium for non-cooperative games - the so-called weak local Nash equilibrium. We prove its existence for a significantly more general class of sets of strategies than compact convex sets. The theorems on existence of the weak local equilibrium presented here are applications of Brouwer and Lefschetz fixed point theorems.

1. Introduction

Let $p_1, \ldots, p_n: S_1 \times \ldots \times S_n \to \mathbb{R}$, $n \geq 2$, be real functions defined in a cartesian product, $S = S_1 \times \ldots \times S_n$. A Nash equilibrium for the functions $p_1, \ldots, p_n$ is a point $\widetilde{s} = (\widetilde{s}_1, \ldots, \widetilde{s}_n) \in S_1 \times \ldots \times S_n$ such that

$$p_i(\widetilde{s}_1, \ldots, \widetilde{s}_{i-1}, s_i, \widetilde{s}_{i+1}, \ldots, \widetilde{s}_n) \leq p_i(\widetilde{s}), \quad \text{for any } s_i \in S_i,\ 1 \leq i \leq n.$$

This concept was established by John F. Nash in his PhD thesis. The interpretation is that in a non-cooperative game with n players, numbered from 1 to n, in which, for each i between 1 and n, the ith player has a set S_i of strategies and a payoff function $p_i: S_1 \times \ldots \times S_n \to \mathbb{R}$, the Nash equilibrium is a solution where there is no motivation to any player changes his strategy if the other do not.

In [1], the authors define a concept of local Nash equilibrium when the sets of strategies $S_1, \ldots, S_n$ are metric spaces. It is the following: Let $S_1, \ldots, S_n$ be metric spaces and $p_1, \ldots, p_n: S_1 \times \ldots \times S_n \to \mathbb{R}$ real functions. A point

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$\widetilde{s} = (\widetilde{s}_1, \dots, \widetilde{s}_n) \in S_1 \times \dots \times S_n$ is a local Nash equilibrium for the functions $p_1, \dots, p_n$ if there exists $\varepsilon > 0$ such that

$$p_i(\widetilde{s}_1, \dots, \widetilde{s}_{i-1}, s_i, \widetilde{s}_{i+1}, \dots, \widetilde{s}_n) \leq p_i(\widetilde{s}), \quad \text{for any } s_i \in B(\widetilde{s}_i, \varepsilon),\ 1 \leq i \leq n,$$

where $B(\widetilde{s}_i, \varepsilon)$ is the open ball in S_i with center in $\widetilde{s}_i$ and radius ε. Thus, in a competition situation, the interpretation is that in a local Nash equilibrium neither player has incentive to chance its strategy to a close strategy if the other players kept fixed in its strategies. In this sense, we can say that local Nash equilibrium is resistent to small unilateral changes.

In [2], we presented the following concept of local equilibrium for non-cooperative games:

Definition 1.1. Let $(S_1, d_1), \dots, (S_n, d_n)$ be metric spaces and $p_1, \dots, p_n \colon S_1 \times \dots \times S_n \to \mathbb{R}$ real functions. We say that $\widetilde{s} = (\widetilde{s}_1, \dots, \widetilde{s}_n) \in S$ is a *weak local equilibrium* (*w.l.e.*) for $p_1, \dots, p_n$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$p_i(\widetilde{s}_1, \dots, \widetilde{s}_{i-1}, s_i, \widetilde{s}_{i+1}, \dots, \widetilde{s}_n) \leq p_i(\widetilde{s}) + \varepsilon d_i(s_i, \widetilde{s}_i),$$

for every $s_i \in B(\widetilde{s}_i, \delta)$, $1 \leq i \leq n$.

We can say that local equilibrium and weak local equilibrium have the same interpretation in the study of competitions. In both cases, there is no motivation to small unilateral changes of strategy. Moreover, every weak local equilibrium is a candidate to local equilibrium.

In [6], J.F. Nash proved the existence of equilibrium points using the famous Brouwer fixed point theorem. In this paper, Section 2, we prove the existence of the weak local equilibrium for games such that each space of strategies is a compact and convex subset of an Euclidean space and each payoff function has a certain differentiability condition. This result is presented as an application of Brouwer fixed point theorem. In Section 3, this result is extended to a larger class of spaces of strategies: the compact Euclidean neighbourhood retracts with non-null Euler characteristic and with a special property of retraction. The proof depends on applying Lefschetz fixed point theorem.

2. Theorems on existence of the w.l.e.

Given an n-person game, $\mathbf{G} = \{p_1, \dots, p_n \colon S_1 \times \dots \times S_n \to \mathbb{R}\}$, where the player $i \in \{1, \dots, n\}$ is characterized by the space of strategies S_i and by the payoff function p_i, we denote by d_i the metric over the space S_i. As customary, we write $S = S_1 \times \dots \times S_n$.

The classical theorem on existence of equilibrium points ([6, Theorem 1]) asserts that if $\mathbf{G} = \{p_1, \dots, p_n \colon S_1 \times \dots \times S_n \to \mathbb{R}\}$ is an n-person non-cooperative game such that each space of strategies $S_i \subset \mathbb{R}^{m_i}$ is compact and convex and each payoff function $p_i: S \to \mathbb{R}$ is continuous as a function of n variables, and $p_i(s_1, \dots, s_n)$ is linear as a function of s_i when the other variables are kept fixed, then there exists at least one Nash equilibrium point to $\mathbf{G}$. In order to prove that, Nash used the Brouwer fixed point theorem.

Our next theorem shows the existence of the w.l.e. when we change the assumption of linearity of p_i by a condition of differentiability.

Theorem 2.1. *Let $\mathbf{G} = \{p_1, \dots, p_n : S_1 \times \ldots \times S_n \to \mathbb{R}\}$ be an n-person game such that S_i is a compact and convex subset of $\mathbb{R}^{m_i}$ and $p_i: S \to \mathbb{R}$ is continuous as a function of n variables and $p_i(s_1, \dots, s_n)$ is continuously differentiable in s_i, $1 \le i \le n$. Then the game has at least one w.l.e.*

Proof. For each $i \in \{1, \dots, n\}$, define the function $v_i: S \to \mathbb{R}^{m_i}$ by

$$v_i(s_1, \dots, s_n) = \overrightarrow{\nabla}_{s_i} p_i(s) = \left(\frac{\partial p_i}{\partial s_{i1}}(s), \dots, \frac{\partial p_i}{\partial s_{im_i}}(s) \right),$$

where $s_i = (s_{i1}, \dots, s_{im_i}) \in \mathbb{R}^{m_i}$. Consider the vector field $v: S \to \mathbb{R}^m$, $m = m_1 + \ldots + m_n$, given by $v(s) = (v_1(s), \dots, v_n(s))$, for all $s \in S$.

In this context, we have:

Lemma 2.2 (Variational Inequality Formulation of the w.l.e.). *Under the assumptions of Theorem* 2.1, *$\widetilde{s} \in S$ is a w.l.e. for $\mathbf{G}$ if it is a solution for the variational inequality*

$$\langle v(\widetilde{s}), s - \widetilde{s} \rangle \le 0, \quad \textit{for all } s \in S, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^m$.

Proof. From definition of the vector field v, given $s = (s_1, \dots, s_n) \in S$ and $s_i' \in S_i$, we have

$$p_i(s_1, \dots, s_{i-1}, s_i', s_{i+1}, \dots, s_n) = p_i(s) + \langle v_i(s), s_i' - s_i \rangle + r_i(s_i' - s_i) \|s_i' - s_i\|$$

with $\lim\limits_{h \to 0} r_i(h) = 0$. Suppose that $\widetilde{s} = (\widetilde{s}_1, \dots, \widetilde{s}_n) \in S$ is a solution for (2.1). Given $s_i \in S_i$, let $s = (\widetilde{s}_1, \dots, \widetilde{s}_{i-1}, s_i, \widetilde{s}_{i+1}, \dots, \widetilde{s}_n) \in S$. Then

$$\langle v(\widetilde{s}), s - \widetilde{s} \rangle = \langle v_i(\widetilde{s}), s_i - \widetilde{s}_i \rangle \le 0.$$

Thus $p_i(\widetilde{s}_1, \dots, \widetilde{s}_{i-1}, s_i, \widetilde{s}_{i+1}, \dots, \widetilde{s}_n) \le p_i(\widetilde{s}) + r_i(s_i - \widetilde{s}_i) \|s_i - \widetilde{s}_i\|$.

Since $\lim\limits_{h \to 0} r_i(h) = 0$, given $\varepsilon > 0$ there exists $\delta_i > 0$ such that

$$p_i(\widetilde{s}_1, \dots, \widetilde{s}_{i-1}, s_i, \widetilde{s}_{i+1}, \dots, \widetilde{s}_n) \le p_i(\widetilde{s}) + \varepsilon \|s_i - \widetilde{s}_i\|$$

if $\|s_i - \widetilde{s}_i\| < \delta_i$. Let $\delta = \min\limits_{1 \le i \le n} \{\delta_i\}$. Then

$$p_i(\widetilde{s}_1, \dots, \widetilde{s}_{i-1}, s_i, \widetilde{s}_{i+1}, \dots, \widetilde{s}_n) \le p_i(\widetilde{s}) + \varepsilon d_i(s_i, \widetilde{s}_i),$$

for all $s_i \in B(\widetilde{s}_i, \delta)$, $1 \le i \le n$. Hence $\widetilde{s}$ is a w.l.e. for $\mathbf{G}$. $\square$

Proof of Theorem 2.1 (continued). Since S is a compact and convex subset of $\mathbb{R}^m$, we can consider the natural retraction $r\colon \mathbb{R}^m \to S$ which $r(x) \in S$ is the point that realizes the distance of x at S, for every $x \in \mathbb{R}^m$. This natural retraction is characterized by the following variational inequality:

$$\langle x - r(x), y - r(x)\rangle \leq 0, \quad \text{for all } x \in \mathbb{R}^m \text{ and all } y \in S. \tag{2.2}$$

Let $f\colon S \to S$ be the function defined by $f(s) = r(s + v(s))$, for all $s \in S$. Since $f\colon S \to S$ is a continuous function and S is a compact and convex subset of $\mathbb{R}^m$, from Brouwer fixed point theorem, there exists $\widetilde{s} \in S$ such that $f(\widetilde{s}) = \widetilde{s}$. Thus, $\widetilde{s} = r(\widetilde{s} + v(\widetilde{s}))$. We verify that $\widetilde{s}$ is a w.l.e. for **G**. In fact, in order to show that, by Lemma 2.2, it is sufficient to show that the point $\widetilde{s}$ satisfies the variational inequality (2.1). From (2.2), we have that

$$\langle \widetilde{s} + v(\widetilde{s}) - r(\widetilde{s} + v(\widetilde{s})), s - r(\widetilde{s} + v(\widetilde{s}))\rangle \leq 0, \quad \text{for every } s \in S.$$

Since $r(\widetilde{s} + v(\widetilde{s})) = \widetilde{s}$, it follows that $\langle v(\widetilde{s}), s - \widetilde{s}\rangle \leq 0$, for every $s \in S$. Thus, by Lemma 2.2, $\widetilde{s}$ is a w.l.e. for **G**. □

Let $S \subset \mathbb{R}^m$ be a compact and convex subset of $\mathbb{R}^m$. If $s \in S$ is not in the boundary of S we say that s is in the relative interior of S and we denote by $\mathrm{ri}(S)$ the set consisting of such points.

Let $\mathbf{G} = \{p_1, \ldots, p_n\colon S_1 \times \ldots \times S_n \to \mathbb{R}\}$ be a game as in Theorem 2.1 and let $\widetilde{s} \in \mathrm{ri}(S)$. Then, $v(\widetilde{s}) = (\overrightarrow{\nabla}_{s_1} p_1(\widetilde{s}), \ldots, \overrightarrow{\nabla}_{s_n} p_n(\widetilde{s})) = (\overrightarrow{0}, \ldots, \overrightarrow{0})$ if $\widetilde{s}$ is a w.l.e. for **G**. This fact is easily proved using the following:

Lemma 2.3. *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function, differentiable in $x_0 \in (a,b)$ and with the property that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $f(x) \leq f(x_0) + \varepsilon|x - x_0|$ when $|x - x_0| < \delta$. Then $f'(x_0) = 0$.*

Proof. Let $\varepsilon > 0$. By assumption, there exists $\delta > 0$ such that if $0 < h < \delta$ then $f(x_0 + h) \leq f(x_0) + \varepsilon h$. Thus, for $0 < h < \delta$, we have

$$\frac{f(x_0 + h) - f(x_0)}{h} \leq \varepsilon$$

and, therefore,

$$\lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} \leq \varepsilon.$$

By the arbitrariness of ε, we conclude that

$$\lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} \leq 0.$$

Analogous, for $-\delta < h < 0$ we have $f(x_0 + h) \leq f(x_0) - \varepsilon h$ and, therefore,

$$\frac{f(x_0 + h) - f(x_0)}{h} \geq -\varepsilon.$$

Hence

$$\lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h} \geq -\varepsilon.$$

Again, by the arbitrariness of ε, we conclude that

$$\lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h} \geq 0.$$

Hence

$$f'(x_0) = \lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h} = 0. \qquad \square$$

On the other hand, if $v(\widetilde{s}) = 0$ then $\widetilde{s}$ is a fixed point to the map $f\colon S \to S$ defined in the proof of Theorem 2.1. Moreover, by proof of Theorem 2.1, it follows that $\widetilde{s}$ is a w.l.e. for $\mathbf{G}$. Thus, if we write

$$\mathrm{WLE}(\mathbf{G}) = \{\widetilde{s} \in S \mid \widetilde{s} \text{ is a w.l.e. for } \mathbf{G}\},$$

we have that:

Proposition 2.4. $\mathrm{WLE}(\mathbf{G}) \cap \mathrm{ir}(S) = \{\widetilde{s} \in \mathrm{ir}(S) \mid v(\widetilde{s}) = 0\}$.

Example 2.5. Consider the classical Cournot oligopoly model: there are $n \geq 2$ firms, $\{1, \ldots, n\}$, producing a homogeneous commodity. Let s_i be the production of firm i and let f_i be its production cost function. Let q be the demand price function. Suppose that the market consumes the total amount of production, $\sum s_j$. Thus, the profit of firm i is expressed by

$$p_i(s_1, \ldots, s_m) = q\biggl(\sum_{j=1}^{n} s_j\biggr) \cdot s_i - f_i(s_i).$$

Let $I_1, \ldots, I_n$ be compact intervals in $\mathbb{R}$. If q and f_i, $1 \leq i \leq n$, are continuously differentiable, by Theorem 2.1, there exists at least one solution $\widetilde{s} = (\widetilde{s}_1, \ldots, \widetilde{s}_m) \in I_1 \times \ldots \times I_n$ resistent to small unilateral changes in $I_1 \times \ldots \times I_n$. This solutions are exactly the solutions to the variational problem

$$\sum_{i=1}^{n} \biggl[q'\biggl(\sum_{j=1}^{n} \widetilde{s}_j\biggr) \cdot \widetilde{s}_i + q\biggl(\sum_{j=1}^{n} \widetilde{s}_j\biggr) - f_i'(\widetilde{s}_i)\biggr] \cdot (s_i - \widetilde{s}_i) \leq 0, \tag{2.3}$$

for every $s = (s_1, \ldots, s_n) \in S$, $S = I_1 \times \ldots \times I_n$.

It is well known that if the functions p_i are differentiable and concave in s_i then the solutions to (2.3) are exactly the Nash equilibria to $p_1, \ldots, p_n$ (see [5, p. 212]). Without the assumption of concavity of p_i, we know at least that the solutions of (2.3) are resistant to small unilateral deviations.

Example 2.6. Let $p_1, p_2\colon [-1,1]\times[-1,1] \to \mathbb{R}$ be the functions given by

$$p_1(x,y) = -xy \quad \text{and} \quad p_2(x,y) = (2y+x)^2.$$

The functions p_1, p_2 haven't Nash equilibria. However, by Theorem 2.1, they have at least one w.l.e. To find them, we take the natural retraction $r\colon \mathbb{R}^2 \to [-1,1]\times[-1,1]$, which is given by

$$r(x,y) = \begin{cases} (x,y) & \text{if } x, y \in [-1,1], \\ (x,1) & \text{if } x \in [-1,1] \text{ and } y \geq 1, \\ (x,-1) & \text{if } x \in [-1,1] \text{ and } y \leq -1, \\ (1,y) & \text{if } x \geq 1 \text{ and } y \in [-1,1], \\ (-1,y) & \text{if } x \leq -1 \text{ and } y \in [-1,1], \\ (1,1) & \text{if } x \geq 1 \text{ and } y \geq 1, \\ (1,-1) & \text{if } x \geq 1 \text{ and } y \leq -1, \\ (-1,-1) & \text{if } x \leq -1 \text{ and } y \leq -1, \\ (-1,1) & \text{if } x \leq -1 \text{ and } y \geq 1. \end{cases}$$

We still consider the vector field

$$v(x,y) = \left(\frac{\partial p_1}{\partial x}(x,y), \frac{\partial p_2}{\partial y}(x,y)\right) = (-y, 8y+4x),$$

and we solve the equation $r((x,y)+v(x,y)) = (x,y)$, whose solutions are $(0,0)$, $(-1,1/2)$, $(1,-1/2)$, $(-1,1)$. Hence, $(0,0)$, $(-1,1/2)$, $(1,-1/2)$ and $(-1,1)$ are all w.l.e. for p_1, p_2. Moreover, $(-1,1)$ is a local equilibrium for p_1, p_2 in the sense of [1]. In fact, note that

$$\begin{aligned} p_1(x,1) &= -x \leq 1 = p_1(-1,1), && \text{for all } x \in [-1,1], \\ p_2(-1,y) &= (2y-1)^2 \leq 1 = p_2(-1,1), && \text{for all } y \in [0,1]. \end{aligned}$$

3. The w.l.e. for a more general class of spaces of strategies

In this section, as a consequence of Lefschetz fixed point theorem, we prove that the w.l.e. occurs to a larger class of spaces of strategies.

3.1. Definitions. Throughout this section, we consider the singular homology with coefficients in $\mathbb{Q}$ – the field of the rational numbers.

Let $X \subset \mathbb{R}^m$. If there exists a retraction $r\colon V \to X$, where V is a neighbourhood of X in $\mathbb{R}^m$, the set X is said to be an *Euclidean neighbourhood retract* or an ENR *space*.

Let $X \subset \mathbb{R}^m$ be a compact ENR. Then, the space $\mathrm{H}_i(X)$ is a finite dimensional vector space and $\mathrm{H}_j(X) = 0$ for sufficiently large j ([2, Proposition 4.11]).

Thus, it is well defined the number

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{Q}} \mathrm{H}_i(X),$$

called the *Euler characteristic of* X. Moreover, if $f\colon X \to X$ is a continuous function, it is well defined the number

$$\Lambda(f) = \sum_{i=0}^{\infty} (-1)^i \operatorname{trace}(f_{*i}),$$

called the *Lefschetz number of* f. It is clear that if f is homotopic to identity map then $\Lambda(f) = \chi(X)$. The Lefschetz fixed point theorem asserts that if $\Lambda(f) \neq 0$ then f has at least one fixed point.

Definition 3.1. We say that a subset X of $\mathbb{R}^m$ has the *property of convenient retraction* (abbrev. p.c.r.) if there exists a retraction $r\colon V \to X$, where V is an open neighbourhood of X in $\mathbb{R}^m$, satisfying: given $x_0 \in V$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle x_0 - r(x_0), x - r(x_0)\rangle \le \varepsilon \|x - r(x_0)\|,$$

for all $x \in X$ with $\|x - r(x_0)\| < \delta$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the inner product in $\mathbb{R}^m$ and $\|\cdot\|$ is the norm induced by it. In this case, we say that $r\colon V \to X$ is a convenient retraction.

Example 3.2. Every closed convex subset K of $\mathbb{R}^m$ has the p.c.r. In fact, there is a natural retraction $r\colon \mathbb{R}^m \to K$ such that for each $x \in \mathbb{R}^m$ assigns the point $r(x) \in K$ which realizes the distance of x to K. This retraction satisfies $\langle x_0 - r(x_0), x - r(x_0)\rangle \le 0$ for every $x_0 \in \mathbb{R}^m$ and $x \in K$.

The next proposition gives us another example of spaces with the p.c.r.

Proposition 3.3 ([2, Proposition 4.3]). *Every submanifold M of $\mathbb{R}^n$, of class C^2, with or without boundary, has the p.c.r.*

Let X be a closed subset of the Euclidean space $\mathbb{R}^n$ and let V be an open neighbourhood of X in $\mathbb{R}^n$. A map $r\colon V \to X$ is called a proximative retraction (or metric projection) if

$$\|r(y) - y\| = \operatorname{dist}(y, X), \quad \text{for every } y \in V,$$

where $\operatorname{dist}(y, X) = \inf\{\|x - y\| \mid x \in X\}$ is the distance of y to X.

Evidently, every proximative retraction is a retraction map but not conversely.

A compact subset $K \subset \mathbb{R}^n$ is called a proximative neighbourhood retract (written $K \in \mathrm{PANR}$) if there exists an open neighbourhood V of K in $\mathbb{R}^n$ and a proximative retraction $r\colon V \to K$.

We have the following:

Proposition 3.4. *Let K be a compact subset of $\mathbb{R}^n$. If $K \in$ PANR then K is an* ENR *with the p.c.r.*

Proof. Suppose $K \in$ PANR and let $r: V \to K$ be a proximative retraction. Then, r is a convenient retraction. In fact, let $x_0 \in V$ and $x \in K$ be arbitraries. Since r is a proximative retraction, $\|x_0 - x\| \geq \|x_0 - r(x_0)\|$. Given $\varepsilon > 0$, if $\varepsilon \geq \|x_0 - r(x_0)\|$ then

$$\langle x_0 - r(x_0), x - r(x_0)\rangle \leq \varepsilon\|x - r(x_0)\|, \quad \text{for every } x \in K.$$

Suppose $0 < \varepsilon < \|x_0 - r(x_0)\|$. Let $0 < \theta < \pi/2$ be such that

$$\cos\theta = \frac{\varepsilon}{\|x_0 - r(x_0)\|}.$$

Now, if we take $\delta = 2\varepsilon$, it is easy to see that, for every $x \in K$ such that $\|x - r(x_0)\| < \delta$, the angle α between $(x_0 - r(x_0))$ and $(x - r(x_0))$ is in $(\theta, \pi]$. Thus, for every $x \in K$ such that $\|x - r(x_0)\| < \delta$, we have

$$\langle x_0 - r(x_0), x - r(x_0)\rangle \leq \frac{\varepsilon}{\|x_0 - r(x_0)\|}\|x_0 - r(x_0)\|\|x - r(x_0)\| = \varepsilon\|x - r(x_0)\|.$$

Hence, r is a convenient retraction and, therefore, K is an ENR with the p.c.r. $\square$

Remark 3.5. In Proposition 3.4, it was proved that every proximative retraction is a convenient retraction. However, the converse is not true. That is, a convenient retraction need not necessarily be a proximative retraction.

Example 3.6. Let

$$K = \{(2\cos t, 2\sin t) \mid 0 \leq t \leq 3\pi/2\} \cup \{(x, -2) \mid 0 \leq x \leq 5\}.$$

Let V be the open neighbourhood of K in $\mathbb{R}^2$ defined by

$$\begin{aligned} V = \{(\rho\cos t, \rho\sin t) \mid 0 \leq t \leq 3\pi/2,\ 1/2 < \rho < 3\} \\ \cup \{(x, y) \mid 0 \leq x \leq 5,\ -3 < y < -1/2\} \cup V_1 \cup V_2, \end{aligned}$$

where V_1 is an open semi-disc around $(2, 0)$ and V_2 is an open semi-disc around $(5, -2)$, as in Figure 1.

Let $r: V \to K$ be the retraction given by

$$\begin{aligned} r(\rho\cos t, \rho\sin t) &= (2\cos t, 2\sin t) && \text{if } 0 \leq t \leq 3\pi/2, \\ r(x, y) &= (x, -2) && \text{if } 0 \leq x \leq 5 \text{ and } -3 < y < -1/2, \\ r(x, y) &= (2, 0) && \text{if } (x, y) \in V_1, \\ r(x, y) &= (5, -2) && \text{if } (x, y) \in V_2. \end{aligned}$$

Thus, r is a convenient retraction but it is not a proximative retraction.

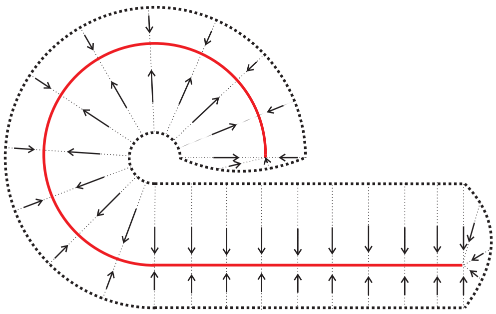

Figure 1

A clear property of the ENR's with the p.c.r. is that the cartesian product of a finite number of spaces with the p.c.r. also has the p.c.r.

A more general formulation of Theorem 2.1 is the following:

Theorem 3.7. *Let* $\mathbf{G} = \{p_1, \dots, p_n : S_1 \times \dots \times S_n \to \mathbb{R}\}$ *be a game which each* $S_i \subset \mathbb{R}^{m_i}$ *is a compact ENR with the p.c.r.,* $1 \leq i \leq n$. *Also, suppose* $p_i : S \to \mathbb{R}$ *continuous as a function of* n *variables and* $p_i(s_1, \dots, s_n)$ *continuously differentiable in a neighbourhood of* s_i *when the other variables are kept fixed,* $1 \leq i \leq n$. *If* $\chi(S_i) \neq 0$ *for* $1 \leq i \leq n$ *then the game* $\mathbf{G}$ *has at least one w.l.e.*

In order to prove Theorem 3.7, we need the following lemma.

Lemma 3.8. *Let* X *be a compact subset of* $\mathbb{R}^m$ *and let* V *be an open neighbourhood of* X *in* $\mathbb{R}^m$. *Then, given a continuous vector field* $v : X \to \mathbb{R}^m$, *there exists* $t_1 > 0$ *such that* $x + tv(x) \in V$ *for all* $x \in X$ *and all* $t \in [0, t_1]$.

Proof. Suppose $v : X \to \mathbb{R}^m$ a continuous vector field. If $v(x) = 0$ for every $x \in X$, there is nothing to prove. Suppose $v(x) \neq 0$ for some $x \in X$. Then, the real number $u = \max\limits_{x \in X}\{\|v(x)\|\}$ is a finite positive number. For every $x \in X$, there is $\varepsilon_x > 0$ such that $B(x, \varepsilon_x) \subset V$. Since X is compact, we obtain a finite open subcover $\{B(x_i, \varepsilon_{x_i}/4)\}_{i=1}^{l}$ with

$$X \subset \bigcup_{i=1}^{l} B\left(x_i, \frac{\varepsilon_{x_i}}{4}\right) \subset \bigcup_{i=1}^{l} B(x_i, \varepsilon_{x_i}) \subset V.$$

Let $\varepsilon = \min\limits_{1 \leq i \leq l}\{\varepsilon_{x_i}/4\}$ and $t_1 = \varepsilon/u$. Thus, $x + tv(x) \in V$ for all $x \in X$ and all $t \in [0, t_1]$. In fact, given $x \in X$, we have $x \in B(x_i, \varepsilon_{x_i}/4)$ for some x_i. If $v(x) = 0$ the conclusion is obvious. If $v(x) \neq 0$ then, given $t \in [0, t_1]$, we have

$$t \leq t_1 = \frac{\varepsilon}{u} \leq \frac{\varepsilon_{x_i}}{4u} \leq \frac{\varepsilon_{x_i}}{4\|v(x)\|}.$$

It follows that

$$\|x + tv(x) - x_i\| \leq \|x - x_i\| + t\|v(x)\| \leq \frac{\varepsilon_{x_i}}{4} + \frac{\varepsilon_{x_i}}{4\|v(x)\|}\|v(x)\| = \frac{\varepsilon_{x_i}}{2} < \varepsilon_{x_i}.$$

Therefore, $x + tv(x) \in B(x_i, \varepsilon_{x_i}) \subset V$. Hence, $x + tv(x) \in V$ for all $x \in X$ and all $t \in [0, t_1]$. □

Proof of Theorem 3.7. Consider the vector field $v: S \to \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n}$ defined by

$$s = (s_1, \ldots, s_n) \to (v_1(s), \ldots, v_n(s)) \in \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n},$$

where the component $v_i(s)$ in direction $\mathbb{R}^{m_i}$ is the gradient vector $\overrightarrow{\nabla}_{s_i} p_i(s)$ at the point s_i for the function $p_i(s_1, \ldots, s_n)$ considered as a function of s_i, with s_j fixed for $j \neq i$. Since $S_1, \ldots, S_n$ have the p.c.r., the cartesian product $S = S_1 \times \ldots \times S_n$ also has. Let $r : V \to S$ be a convenient retraction. By Lemma 3.8, let $t_1 > 0$ be such that $s + tv(s) \in V$ for all $s \in S$ and all $t \in [0, t_1]$. Thus, we have well defined the function $f: S \to S$ given by

$$f(s) = r(s + t_1 v(s)).$$

Note that f is homotopic to the identity map, $\mathrm{id}_S: S \to S$, via homotopy $H: X \times [0, t_1] \to X$ given by $H(x, t) = r(x + tv(x))$, for all $x \in X$ and all $t \in [0, t_1]$. Hence, $\Lambda(f) = \chi(S)$.

If we suppose $\chi(S_i) \neq 0$, $1 \leq i \leq n$, then, $\chi(S) = \chi(S_1) \ldots \chi(S_n) \neq 0$. It follows that $\Lambda(f) \neq 0$. Thus, by Lefschetz fixed point theorem, f has at least one fixed point. Let $\widetilde{s} \in S$ be such a point. We verify that $\widetilde{s}$ is a w.l.e. for $p_1, \ldots, p_n$. In fact, we have that $\widetilde{s} = f(\widetilde{s}) = r(\widetilde{s} + t_1 v(\widetilde{s}))$ and, since r is a convenient retraction, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x - r(\widetilde{s} + t_1 v(\widetilde{s}))\| = \|x - \widetilde{s}\| < \delta$$

implies that

$$\langle \widetilde{s} + t_1 v(\widetilde{s}) - r(\widetilde{s} + t_1 v(\widetilde{s})), x - r(\widetilde{s} + t_1 v(\widetilde{s}))\rangle = t_1 \langle v(\widetilde{s}), x - \widetilde{s}\rangle \leq \frac{t_1 \varepsilon}{2} \|x - \widetilde{s}\|.$$

Moreover, from definition of $v(\widetilde{s})$, we can assume that if $\|\widetilde{s} - s\| < \delta$ then

$$p_i(\widetilde{s}_1, \ldots, \widetilde{s}_{i-1}, s_i, \widetilde{s}_{i+1}, \ldots, \widetilde{s}_n) \leq p_i(\widetilde{s}) + \langle v_i(\widetilde{s}), s_i - \widetilde{s}_i\rangle + \frac{\varepsilon}{2} \|s_i - \widetilde{s}_i\|,$$

$1 \leq i \leq n$. It follows that, if $s \in S$ and $\|s - \widetilde{s}\| < \delta$ then

$$p_i(\widetilde{s}_1, \ldots, \widetilde{s}_{i-1}, s_i, \widetilde{s}_{i+1}, \ldots, \widetilde{s}_n) \leq p_i(\widetilde{s}) + \varepsilon \|s_i - \widetilde{s}_i\|,$$

$1 \leq i \leq n$. Hence, $\widetilde{s}$ is a w.l.e. for $p_1, \ldots, p_n$. □

Corollary 3.9. *Let $\mathbf{G} = \{p_1, \ldots, p_n: S_1 \times \ldots \times S_n \to \mathbb{R}\}$ be a game with $S_i \subset \mathbb{R}^{m_i}$ being a compact proximative neighbourhood retract, $1 \leq i \leq n$. Also, suppose $p_i: S \to \mathbb{R}$ continuous as a function of n variables and $p_i(s_1, \ldots, s_n)$*

continuously differentiable in s_i when the other variables are kept fixed, $1\leq i\leq n$. If $\chi(S_i)\neq 0$ for $1\leq i\leq n$ then the game has at least one w.l.e.

Proof. From Proposition 3.4, every compact proximative neighbourhood retract has the p.c.r. Thus, it is just to apply Theorem 3.7. □

Corollary 3.10. *Let $\mathbf{G}=\{p_1,\dots,p_n\colon S_1\times\dots\times S_n\to\mathbb{R}\}$ be a game with $S_i\subset\mathbb{R}^{m_i}$ being a compact differentiable C^2-manifold, with or without boundary, $1\leq i\leq n$. Also, suppose $p_i\colon S\to\mathbb{R}$ continuous as a function of n variables and $p_i(s_1,\dots,s_n)$ continuously differentiable in s_i when the other variables are kept fixed, $1\leq i\leq n$. If $\chi(S_i)\neq 0$ for $1\leq i\leq n$ then the game has at least one w.l.e.*

Proof. From Proposition 3.3, every manifold of class C^2 has the p.c.r. Thus, it is just to apply Theorem 3.7. Another argument is that every compact C^2-manifold, with or without boundary, is a proximative neighbourhood retract (see [4, p. 21]). Thus, it is just to apply Corollary 3.9. □

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