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INFINITE MANY POSITIVE SOLUTIONS FOR NONLINEAR FIRST-ORDER BVPS WITH INTEGRAL BOUNDARY CONDITIONS ON TIME SCALES

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Abstract. In this paper, we investigate the existence of infinite many positive solutions for the nonlinear first-order BVP with integral boundary conditions

$$\begin{cases} x^{\Delta}(t) + p(t)x^{\sigma}(t) = f(t, x^{\sigma}(t)), & t \in (0, T)_{\mathbb{T}}, \\ x(0) - \beta x^{\sigma}(T) = \alpha \displaystyle\int_0^{\sigma(T)} x^{\sigma}(s)\Delta g(s), \end{cases}$$

where $x^{\sigma} = x \circ \sigma$, $f\colon [0,T]_{\mathbb{T}} \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, p is regressive and rd-continuous, $\alpha, \beta \geq 0$, $g\colon [0,T]_{\mathbb{T}} \to \mathbb{R}$ is a nondecreasing function. By using the fixed-point index theory and a new fixed point theorem in a cone, we provide sufficient conditions for the existence of infinite many positive solutions to the above boundary value problem on time scale $\mathbb{T}$.

1. Introduction

In recent years, boundary value problems with integral boundary conditions constitute a very interesting and important class of problems and have attracted the attention of Khan [7], Gallardo [4], Karakostas and Tsamatos [6], Lomtatidze

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and Malaguti [16] and the references therein. Also, more and more attentions have paid on discussing the solutions for boundary value problems on time scales and have obtained many good results of the existence of solutions for two-, three- and multi-point boundary value problems. To mention a few, we refer the reader to some recent contributions [1], [3], [5], [8], [9], [12]–[15], [17], [21]–[27] and references therein. For example:

In [22], [3], the authors studied the following first-order BVP on time scales:

$$\begin{cases} x^{\Delta}(t) = f(x^{\sigma}(t)), & t \in [0,T]_{\mathbb{T}}, \\ x(0) = \beta x^{\sigma}(T), \end{cases} \tag{1.1}$$

where $x^{\sigma} = x \circ \sigma$. By using the twin-fixed-point theorem due to Avery and Henderson, Sun [22] investigated the existence of at least two positive solutions for BVP (1.1) when $0 < \beta < 1$. Cabada [3] developed the method of lower and upper solutions coupled with the monotone iterative techniques to obtain the existence of extremal solutions for BVP (1.1) when $\beta = 1$.

Sun and Li [23] studied the following BVP on time scales:

$$\begin{cases} x^{\Delta}(t) + p(t)x^{\sigma}(t) = g(t, x^{\sigma}(t)), & t \in [0,T]_{\mathbb{T}}, \\ x(0) = x^{\sigma}(T). \end{cases} \tag{1.2}$$

By applying novel inequalities and the Schaefer fixed point theorem, the existence of at least one solution for BVP (1.2) is obtained.

In [24], Ge and Tian studied the following first-order three-point BVP on time scales:

$$\begin{cases} x^{\Delta}(t) + p(t)x^{\sigma}(t) = f(t, x^{\sigma}(t)), & t \in [0,T]_{\mathbb{T}}, \\ x(0) - ax(\xi) = \beta x^{\sigma}(T), \end{cases} \tag{1.3}$$

where $\alpha, \beta \geq 0$ with $\alpha/e_p(\xi,0) + \beta/e_p(\sigma(T),0) < 1$. By using several fixed point theorems, the existence of at least one positive solution and multiple positive solutions for BVP (1.3) are obtained.

In [1], Anderson interested in the following first-order $(n+2)$-point BVP on time scales:

$$\begin{cases} y^{\Delta}(t) + p(t)y^{\sigma}(t) = \lambda f(t, y^{\sigma}(t)), & t \in (0,T)_{\mathbb{T}}, \\ y(0) = y^{\sigma}(T) + \sum\limits_{i=1}^{n} \gamma_i y(t_i). \end{cases} \tag{1.4}$$

In the study, conditions for the existence of at least one positive solution for BVP (1.4) is discussed by using the Guo–Krasnosel'skiĭ fixed point theorem.

To the best of our knowledge, up to the present, few papers have been published on the existence of solutions of BVPs with integral boundary conditions on time scales (see [11]).

In this paper, we are concerned with the following nonlinear first-order BVP with integral boundary conditions on time scale $\mathbb{T}$:

$$\tag{1.5} \begin{cases} x^{\Delta}(t) + p(t)x^{\sigma}(t) = f(t, x^{\sigma}(t)), & t \in (0,T)_{\mathbb{T}}, \\ x(0) - \beta x^{\sigma}(T) = \alpha \displaystyle\int_0^{\sigma(T)} x^{\sigma}(s)\Delta g(s), \end{cases}$$

where $f\colon [0,T]_{\mathbb{T}} \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, $p\colon [0,T]_{\mathbb{T}} \times \mathbb{R}^+ \to \mathbb{R}^+$ is regressive and rd-continuous, $\alpha, \beta \geq 0$, $g\colon [0,T]_{\mathbb{T}} \to \mathbb{R}$ is an increasing function. The integral in (1.5) is a Riemann–Stieltjes one on time scales, which is introduced in [18].

For convenience, we introduce the following notation:

$$\begin{aligned} \Gamma &= \left[1 - \alpha \int_0^{\sigma(T)} e_p(0,\sigma(s))\Delta g(s) - \beta e_p(0,\sigma(T))\right]^{-1}, \quad \gamma = \frac{m}{M}, \\ m &= \Gamma \beta e_p^2(0,\sigma(T)), \\ M &= \Gamma e_p(\sigma(T),0)\left(1 + \beta e_p(0,\sigma(T)) + \alpha \int_0^{\sigma(T)} g^{\Delta}(s) e_p(0,\sigma(s))\Delta s\right). \end{aligned}$$

Throughout this paper, we always assume that $\Gamma > 0$.

The main purpose of this paper is to establish some criteria for the existence of infinite many solutions for BVP (1.5) by using the fixed-point index theory and a new fixed-point theorem in a cone.

Remark 1.1. Let $\{t_i\}_{i=1}^n (n \in \mathbb{N})$ be a finite sequence of distinct points in $(0,T)_{\mathbb{T}}$ satisfying $t_1 < \ldots < t_n$, $t_{n+1} = T$. In system (1.5), we set

$$g(s) = \sum_{i=1}^n a_i \chi(s - t_i),$$

where $a_1, \ldots, a_n$ are nonnegative constants, and $\chi(s)$ is a characteristic function, that is,

$$\chi(s) = \begin{cases} 1 & \text{for } s \geq 0, \\ 0 & \text{for } s < 0, \end{cases}$$

which implies that $g(0) = 0$, $g(T) = \sum\limits_{i=1}^n a_i$ and

$$g(s) = \sum_{i=1}^n a_i \chi(s - t_i) = \begin{cases} 0 & \text{for } s \in [0, t_1)_{\mathbb{T}}, \\ \sum\limits_{i=1}^l a_i & \text{for } s \in [t_l, t_{l+1})_{\mathbb{T}},\ l = 1, \ldots, n. \end{cases}$$

By some basic concepts and time-scale calculus formulae in [2], one can easily obtain that

$$\int_0^T x^{\sigma}(s)\Delta g(s) = \sum_{i=1}^n a_i x(t_i).$$

Then system (1.5) reduces to the following $(n+2)$-point BVP

$$\begin{cases} x^{\Delta}(t) + p(t)x^{\sigma}(t) = f(t, x^{\sigma}(t)), & t \in (0,T)_{\mathbb{T}}, \\ x(0) - \beta x^{\sigma}(T) = \alpha \sum\limits_{i=1}^{n} a_i x(t_i). \end{cases}$$

Obviously, BVP (1.1)–(1.4) are special cases of BVP (1.5). Therefore, the BVPs with integral boundary conditions on time scales include two-, three-, multi-point and nonlocal boundary value problems as special cases.

The paper is organized as follows. In Section 2, some basic definitions and lemmas on time scales are introduced. In Section 3, some useful lemmas and theorems are established. In Section 4, by using the fixed point index theory and a new fixed-point theorem in cones, some sufficient conditions for the existence of infinite many positive solutions for BVP (1.5) are obtained.

2. Preliminaries

In this section, we shall first recall some basic definitions and lemmas which are used in what follows.

Definition 2.1 ([2]). A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real set $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho\colon \mathbb{T} \to \mathbb{T}$ and the graininess $\mu\colon \mathbb{T} \to \mathbb{R}^{+}$ are defined, respectively, by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) := \sigma(t) - t.$$

In this definition we put $\inf\emptyset = \sup\mathbb{T}$ (i.e. $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup\emptyset = \inf\mathbb{T}$ (i.e. $\rho(t) = t$ if $\mathbb{T}$ has a minimum t). The point $t \in \mathbb{T}$ is called *left-dense*, *left-scattered*, *right-dense* or *right-scattered* if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$ or $\sigma(t) > t$, respectively. Points that are right-dense and left-dense at the same time are called *dense*. If $\mathbb{T}$ has a *left-scattered maximum* m_1, defined $\mathbb{T}^k = \mathbb{T} - \{m_1\}$; otherwise, set $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a *right-scattered minimum* m_2, defined $\mathbb{T}_k = \mathbb{T} - \{m_2\}$; otherwise, set $\mathbb{T}_k = \mathbb{T}$.

Definition 2.2 ([2]). A function f is rd-*continuous* provided it is continuous at each right-dense point in $\mathbb{T}$ and has a left-sided limit at each left-dense point in $\mathbb{T}$. The set of rd-continuous functions f will be denoted by $C_{\mathrm{rd}}(\mathbb{T})$. A function g is *left-dense continuous* (i.e. ld-*continuous*), if g is continuous at each left-dense point in $\mathbb{T}$ and its right-sided limit exists (finite) at each right-dense point in $\mathbb{T}$. The set of left-dense continuous functions g will be denoted by $C_{\mathrm{ld}}(\mathbb{T})$.

Definition 2.3 ([2]). A function $p\colon \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$, where $\mu(t) = \sigma(t) - t$ is the graininess function. The set of all regressive rd-continuous functions $f\colon \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^+$ is given by $\{f \in \mathcal{R}: 1 + \mu(t)f(t) > 0\}$ for all $t \in \mathbb{T}$. Let $p \in \mathcal{R}$. The exponential function is defined by

$$e_p(t,s) = \exp\bigg(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\bigg),$$

where $\xi_{h(z)}$ is the so-called *cylinder transformation*.

Definition 2.4 ([2]). If $F^\Delta(t) = f(t)$, then we define the delta integral by

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

Lemma 2.5 ([2]). *Let p, $q \in \mathcal{R}$. Then*

(a) $e_p(t,s) = 1/e_p(s,t)$;
(b) $e_p(t,s)e_p(s,r) = e_p(t,r)$;
(c) $e_p^\Delta(\,\cdot\,,s) = pe_p(\,\cdot\,,s)$.

Lemma 2.6 ([2]). *Let $a \in \mathbb{T}^k$, $b \in \mathbb{T}$ and assume that $f: \mathbb{T} \times \mathbb{T}^k \to \mathbb{R}$ is continuous at (t,t), where $t \in \mathbb{T}^k$ with $t > a$. Also assume that $f^\Delta(t,\,\cdot\,)$ is* rd*-continuous on $[a,\sigma(t)]$. Suppose that for each $\varepsilon > 0$ there exists a neighbourhood U of t, independent of $\tau \in [a,\sigma(t)]$, such that*

$$|f(\sigma(t),\tau) - f(s,\tau) - f^\Delta(t,\tau)(\sigma(t) - s)| \le \varepsilon|\sigma(t) - s|, \quad \textit{for all } s \in U,$$

where f^Δ denotes the derivative of f with respect to the first variable. Then:

(a) $g(t) := \displaystyle\int_a^t f(t,\tau)\Delta\tau$ *implies* $g^\Delta(t) = \displaystyle\int_a^t f^\Delta(t,\tau)\Delta\tau + f(\sigma(t),t)$;

(b) $h(t) := \displaystyle\int_t^b f(t,\tau)\Delta\tau$ *implies* $h^\Delta(t) = \displaystyle\int_t^b f^\Delta(t,\tau)\Delta\tau - f(\sigma(t),t)$.

Lemma 2.7 ([2]). *If $a, b \in \mathbb{T}$, $f, g \in C_{\mathrm{rd}}$, then*

$$\int_a^b f(\sigma(t))g^\Delta(t)\Delta t = [fg]_a^b - \int_a^b f^\Delta(t)g(t)\Delta t.$$

Lemma 2.8 ([2]). *If $f, g: \mathbb{T} \to \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}^k$, then*

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t).$$

Lemma 2.9 ([18]). *Let $a, b, c \in \mathbb{T}$ with $a < b < c$. If f is bounded on $[a,c]_{\mathbb{T}}$ and g is monotonically increasing on $[a,c]_{\mathbb{T}}$, then*

$$\int_a^c f\Delta g = \int_a^b f\Delta g + \int_b^c f\Delta g.$$

The set of all functions that are Δ-integrable with respect to g in the Riemann–Stietjes sense will be denoted by $\mathcal{R}_\Delta(g,I)$.

Lemma 2.10 ([18]). *Let $I = [a,b]_{\mathbb{T}}$, $a,b \in \mathbb{T}$. Then, the condition $f \in \mathcal{R}_{\Delta}(g,I)$ is equivalent to each one of the following items:*

(a) *f is a monotonic function on I;*
(b) *f is a continuous function on I;*
(c) *f is regulated on I;*
(d) *f is a bounded and has a finite number of discontinuity points on I.*

Lemma 2.11 ([18]). *Let $I = [a,b]_{\mathbb{T}}$, $a,b \in \mathbb{T}$. Suppose that g is an increasing function such that g^{Δ} is continuous on $(a,b)_{\mathbb{T}}$ and f^{σ} is a real bounded function on I. Then $f^{\sigma} \in \mathcal{R}_{\Delta}(g,I)$ if and only if $f^{\sigma}g^{\Delta} \in \mathcal{R}_{\Delta}(g,I)$. Moreover,*

$$\int_a^b f^{\sigma}(t)\Delta g(t) = \int_a^b f^{\sigma}(t)g^{\Delta}(t)\Delta t.$$

Lemma 2.12 ([18]). *Let $I = [a,b]_{\mathbb{T}}$, $a,b \in \mathbb{T}$, suppose that g is an increasing function such that g^{Δ} is continuous on $(a,b)_{\mathbb{T}}$ and f^{σ} is a real bounded function on I. Then*

$$\int_a^b f^{\sigma}\Delta g = [fg]_a^b - \int_a^b g\Delta f.$$

3. Foundational lemmas

In this section, we first introduce some background definitions, the fixed-point index theorem and a new fixed point theorem in a cone. Then present basic lemmas that are very crucial in the proof of the main results.

Definition 3.1. Let $(E, \|\cdot\|)$ be a real Banach space. A nonempty, closed, convex set $P \subset E$ is said to be a *cone* provided the following are satisfied:

(a) if $y \in P$ and $\lambda \geq 0$, then $\lambda y \in P$;
(b) if $y \in P$ and $-y \in P$, then $y = 0$.

If $P \subset E$ is a cone, we denote the order induced by P on E by $\leq$, that is, $x \leq y$ if and only if $y - x \in P$.

Theorem 3.2 ([10]). *Let E be a Banach space and $P \subset E$ be a cone in E. Let $r > 0$ and define $\Omega_r = \{x \in P : \|x\| < r\}$. Assume that $T\colon P\bigcap\overline{\Omega}_r \to P$ is completely continuous operator such that $Tx \neq x$ for $x \in \partial\Omega$.*

(a) *If $\|Tx\| \leq \|x\|$ for $x \in \partial\Omega_r$, then $i(T,\Omega_r,P) = 1$.*
(b) *If $\|Tx\| \geq \|x\|$ for $x \in \partial\Omega_r$, then $i(T,\Omega_r,P) = 0$.*

Here $i(T,\Omega_r,P)$ is the index of operator T with respect to Ω_r in P, which can be found in [10].

THEOREM 3.3 ([19]). *Let P be a cone in a Banach space E. Let α, β and γ be three increasing, nonnegative and continuous functionals on P satisfying, for some c and $M > 0$,*

$$\gamma(x) \le \beta(x) \le \alpha(x), \quad \|x\| \le M\gamma(x) \quad \textit{for all } x \in P(\gamma, c),$$

where $P(\gamma, c) = \{x \in P : \gamma(x) < c\}$ ($P(\beta, b)$ and $P(\alpha, a)$ are similarly defined). Suppose that there exists a completely continuous operator $T\colon \overline{P(\gamma, c)} \to P$ and $0 < a < b < c$ such that:

(a) *$\gamma(Tx) < c$, for all $x \in \partial P(\gamma, c)$;*
(b) *$\beta(Tx) > b$, for all $x \in \partial P(\beta, b)$;*
(c) *$P(\alpha, a) \neq \emptyset$, and $\alpha(Tx) < a$, for all $x \in P(\alpha, a)$.*

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ such that

$$0 \le \alpha(x_1) < a < \alpha(x_2), \quad \beta(x_2) < b < \beta(x_3), \quad \gamma(x_3) < c.$$

Let $E = C([0, \sigma(T)]_{\mathbb{T}}, \mathbb{R})$ with the norm $\|x\| = \sup_{t \in [0, \sigma(T)]_{\mathbb{T}}} |x(t)|$. Then it is a Banach space.

For $h \in E$, we consider the following linear BVP:

$$\begin{cases} x^{\Delta}(t) + p(t)x^{\sigma}(t) = h(t), & t \in (0, T)_{\mathbb{T}}, \\ x(0) - \beta x^{\sigma}(T) = \alpha \displaystyle\int_0^{\sigma(T)} x^{\sigma}(s)\Delta g(s). \end{cases}$$

LEMMA 3.4. *Suppose $h \in E$, then x is a solution of*

$$x(t) = \int_0^{\sigma(T)} G(t, s)h(s)\Delta s, \quad t \in [0, \sigma(T)]_{\mathbb{T}},$$

where

$$G(t, s) = \begin{cases} \Gamma e_p(s, t)\Big[1 - \alpha \displaystyle\int_0^{\sigma(s)} g^{\Delta}(r)e_p(0, \sigma(r))\Delta r\Big], & 0 \le s \le t \le \sigma(T), \\ \Gamma e_p(s, t)\Big[\beta e_p(0, \sigma(T)) + \alpha \displaystyle\int_{\sigma(s)}^{\sigma(T)} g^{\Delta}(r)e_p(0, \sigma(r))\Delta r\Big], & \\ & 0 \le t \le s \le \sigma(T), \end{cases}$$

if and only if x is a solution of BVP (3.1).

PROOF. Assume that $x(t)$ is a solution of (3.1). By the first equation in (3.1), we have

$$[x(t)e_p(t, 0)]^{\Delta} = e_p(t, 0)h(t).$$

Integrating the above equation from 0 to t leads to

$$x(t)e_p(t, 0) = x(0) + \int_0^t e_p(s, 0)h(s)\Delta s,$$

and so

$$x(t)=e_p(0,t)\bigg[x(0)+\int_0^t e_p(s,0)h(s)\Delta s\bigg].$$

By the boundary condition in BVP (3.1), one has

$$x(0)=\Gamma\bigg\{\alpha\int_0^{\sigma(T)}\Delta g(s)\int_0^{\sigma(s)}e_p(r,\sigma(s))h(r)\Delta r+\beta\int_0^{\sigma(T)}e_p(s,\sigma(T))h(s)\Delta s\bigg\}.$$

Then

$$\begin{aligned}
x(t) &= e_p(0,t)\Gamma\bigg\{\beta\int_0^{\sigma(T)}e_p(s,\sigma(T))h(s)\Delta s+\alpha\int_0^{\sigma(T)}\int_0^{\sigma(T)}g^{\Delta}(r)e_p(s,\sigma(r))\\
&\qquad -\alpha\int_0^{\sigma(T)}\int_0^{\sigma(s)}g^{\Delta}(r)e_p(s,\sigma(r))\Delta r h(s)\Delta s\bigg\}+\int_0^t e_p(s,t)h(s)\Delta s\\
&=\int_0^{\sigma(T)}G(t,s)h(s)\Delta s.
\end{aligned}$$

This means that if x is a solution of (3.1) then x satisfies (3.2).

On the other hand, if x satisfies (3.2), we have

$$x(t)=\int_0^{\sigma(T)}G(t,s)h(s)\Delta s,\quad t\in[0,\sigma(T)]_{\mathbb{T}}.$$

Then

$$x(t)e_p(t,0)=\int_0^{\sigma(T)}H(t,s)h(s)\Delta s,\quad t\in[0,\sigma(T)]_{\mathbb{T}},$$

where

$$H(t,s)=\begin{cases}\Gamma e_p(s,0)\bigg[1-\alpha\displaystyle\int_0^{\sigma(s)}g^{\Delta}(r)e_p(0,\sigma(r))\Delta r\bigg], & 0\le s\le t\le\sigma(T),\\ \Gamma e_p(s,0)\bigg[\beta e_p(0,\sigma(T))+\alpha\displaystyle\int_{\sigma(s)}^{\sigma(T)}g^{\Delta}(r)e_p(0,\sigma(r))\Delta r\bigg], & \\ & 0\le t\le s\le\sigma(T).\end{cases}$$

Notice that

$$\begin{aligned}
&\bigg[\int_0^{\sigma(T)}H(t,s)h(s)\Delta s\bigg]^{\Delta}\\
&=\Gamma\bigg[\int_0^t e_p(s,0)\bigg(1-\alpha\int_0^{\sigma(s)}g^{\Delta}(r)e_p(0,\sigma(r))\Delta r\bigg)h(s)\Delta s\bigg]^{\Delta}\\
&\quad+\Gamma\bigg[\int_t^{\sigma(T)}e_p(s,0)\bigg(\beta e_p(0,\sigma(T))+\alpha\int_{\sigma(s)}^{\sigma(T)}g^{\Delta}(r)e_p(0,\sigma(r))\Delta r\bigg)h(s)\Delta s\bigg]^{\Delta}\\
&=\Gamma\bigg[e_p(t,0)\bigg(1-\alpha\int_0^{\sigma(t)}g^{\Delta}(r)e_p(0,\sigma(r))\Delta r\bigg)h(t)\bigg]\\
&\quad-\Gamma\bigg[e_p(t,0)\bigg(\beta e_p(0,\sigma(T))+\alpha\int_{\sigma(t)}^{\sigma(T)}g^{\Delta}(r)e_p(0,\sigma(r))\Delta r\bigg)h(t)\bigg]\\
&=e_p(t,0)h(t).
\end{aligned}$$

Hence, we get from (3.3) that

$$(x(t)e_p(t,0))^{\Delta} = h(t)e_p(t,0),$$

that is

$$x^{\Delta}(t) + p(t)x^{\sigma}(t) = h(t), \quad t \in (0,T)_{\mathbb{T}}.$$

Finally, we can obtain from (3.2) that

$$\begin{aligned}
&x(0) - \beta x^{\sigma}(T)\\
&= \int_0^{\sigma(T)} G(0,s)h(s)\Delta s - \beta \int_0^{\sigma(T)} G(\sigma(T),s)h(s)\Delta s\\
&= \int_0^{t} \Gamma e_p(s,0)\bigg[1-\alpha\int_0^{\sigma(s)} g^{\Delta}(r)e_p(0,\sigma(r))\Delta r\bigg]h(s)\Delta s\\
&\quad + \int_t^{\sigma(T)} \Gamma e_p(s,0)\bigg[\beta e_p(0,\sigma(T)) + \alpha\int_{\sigma(s)}^{\sigma(T)} g^{\Delta}(r)e_p(0,\sigma(r))\Delta r\bigg]h(s)\Delta s\\
&\quad - \beta\int_0^{t} \Gamma e_p(s,\sigma(T))\bigg[1-\alpha\int_0^{\sigma(s)} g^{\Delta}(r)e_p(0,\sigma(r))\Delta r\bigg]h(s)\Delta s\\
&\quad - \beta\bigg\{\int_t^{\sigma(T)} \Gamma e_p(s,\sigma(T))\bigg[\beta e_p(0,\sigma(T)) + \alpha\int_{\sigma(s)}^{\sigma(T)} g^{\Delta}(r)e_p(0,\sigma(r))\Delta r\bigg]h(s)\Delta s\bigg\}\\
&= \alpha\int_0^{\sigma(T)} g^{\Delta}(s)\bigg[\int_0^{\sigma(T)} G(x^{\sigma}(s),r)h(r)\Delta r\bigg]\Delta s\\
&= \alpha\int_0^{\sigma(T)} g^{\Delta}(s)x^{\sigma}(s)\Delta s = \alpha\int_0^{\sigma(T)} x^{\sigma}(s)\Delta g(s).
\end{aligned}$$

So the proof of this lemma is complete. □

Lemma 3.5. *Let $G(t,s)$ be defined in Lemma 3.1, then*:

(a) $G(t,s) \geq 0$ *for all* $t,s \in [0,\sigma(T)]_{\mathbb{T}}$;
(b) $m \leq G(t,s) \leq M$ *for all* $t,s \in [0,\sigma(T)]_{\mathbb{T}}$;
(c) $G(t,s) \geq \gamma \sup_{(t,s)\in[0,\sigma(T)]_{\mathbb{T}}\times[0,\sigma(T)]_{\mathbb{T}}} G(t,s)$ *for all* $t,s \in [0,\sigma(T)]_{\mathbb{T}}$.

Proof. Since

$$\bigg[1-\alpha\int_0^{\sigma(T)} e_p(0,\sigma(s))\Delta g(s) - \beta e_p(0,\sigma(T))\bigg]^{-1} > 0,$$

then it is clear that (1) holds. Now we will show that (b) holds.

$$G(t,s) = \begin{cases} \Gamma e_p(s,t)\bigg[1-\alpha\displaystyle\int_0^{\sigma(s)} g^{\Delta}(r)e_p(0,\sigma(r))\Delta r\bigg], & 0 \leq s \leq t \leq \sigma(T),\\ \Gamma e_p(s,t)\bigg[\beta e_p(0,\sigma(T)) + \alpha\displaystyle\int_{\sigma(s)}^{\sigma(T)} g^{\Delta}(r)e_p(0,\sigma(r))\Delta r\bigg], & \\ & 0 \leq t \leq s \leq \sigma(T),\end{cases}$$

$$\geq \begin{cases} \Gamma e_p(s,0)e_p(0,t)\Big[1-\alpha\displaystyle\int_0^{\sigma(s)} g^{\Delta}(r)e_p(0,\sigma(r))\Delta r\Big], \\ \hfill 0 \leq s \leq t \leq \sigma(T), \\ \Gamma e_p(s,0)e_p(0,t)\beta e_p(0,\sigma(T)), \hfill 0 \leq t \leq s \leq \sigma(T), \end{cases}$$

$$\geq \begin{cases} \Gamma e_p(0,\sigma(T))\Big[1-\alpha\displaystyle\int_0^{\sigma(s)} g^{\Delta}(r)e_p(0,\sigma(r))\Delta r], & 0 \leq s \leq t \leq \sigma(T), \\ \Gamma\beta e_p^2(0,\sigma(T)), & 0 \leq t \leq s \leq \sigma(T), \end{cases}$$

$$\geq \Gamma\beta e_p^2(0,\sigma(T)) = m.$$

Hence, the left-hand side of (b) holds. And it is easy to show that the right-hand side of (b) also holds.

Now we will show that (c) holds. For all $t, s \in [0,\sigma(T)]_{\mathbb{T}}$, it follows that

$$G(t,s) \geq m = \frac{m}{M} \times M \geq \frac{m}{M} \sup_{(t,s)\in[0,\sigma(T)]_{\mathbb{T}}\times[0,\sigma(T)]_{\mathbb{T}}} G(t,s),$$

which implies that (c) holds. This completes the proof of Lemma 3.2. □

Define a cone $P \subset E$ by

$$P = \{x \in E : x(t) \geq 0,\ x(t) \geq \gamma\|x\|,\ t \in [0,\sigma(T)]_{\mathbb{T}}\}$$

and an operator $A\colon P \to P$ by

$$(Ax)(t) = \int_0^{\sigma(T)} G(t,s)f(s,x^{\sigma}(s))\Delta s,$$

where G is defined the same as that in Lemma 3.1.

LEMMA 3.6. *If $x \in P$, then $Ax \in P$.*

PROOF. Clearly, $(Ax)(t) \geq 0$, for all $t \in [0,\sigma(T)]_{\mathbb{T}}$. On the other hand, we have

$$\begin{aligned} (Ax)(t) &= \int_0^{\sigma(T)} G(t,s)f(s,x^{\sigma}(s))\Delta s \\ &\geq m\int_0^{\sigma(T)} f(s,x^{\sigma}(s))\Delta s = \gamma M\int_0^{\sigma(T)} f(s,x^{\sigma}(s))\Delta s \\ &\geq \gamma \sup_{(t,s)\in[0,\sigma(T)]_{\mathbb{T}}\times[0,\sigma(T)]_{\mathbb{T}}} G(t,s)\int_0^{\sigma(T)} f(s,x^{\sigma}(s))\Delta s \\ &\geq \gamma \sup_{t\in[0,\sigma(T)]_{\mathbb{T}}}\int_0^{\sigma(T)} G(t,s)f(s,x^{\sigma}(s))\Delta s = \gamma\|Ax\|. \end{aligned}$$

So it is easy to see $Ax \in P$. The proof is complete. □

Clearly, the fixed points of the operator A are the solutions of BVP (1.5).

Lemma 3.7. *$A\colon P \to P$ is completely continuous.*

Proof. Firstly, we will show that A is continuous. Let $x_n,\ x \in A$ and $\lim\limits_{n\to\infty} x_n = x$. For $t \in [0,\sigma(T)]_{\mathbb{T}}$, we have

$$\begin{aligned}|(Ax_n)(t)-(Ax)(t)| &\le \int_0^{\sigma(T)} G(t,s)|f(s,x_n^\sigma(s))-f(s,x^\sigma(s))|\Delta s\\ &\le M\int_0^{\sigma(T)} |f(s,x_n^\sigma(s))-f(s,x^\sigma(s))|\Delta s.\end{aligned}$$

Since $f\colon [0,T]_{\mathbb{T}} \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, we have $\|Ax_n - Ax\| \to 0$ as $n \to \infty$. That is, A is continuous.

Secondly, we will show that A is compact. Let $\Omega \subset P$ be a bounded set, that is, there exists an $L > 0$ such that for any $x \in \Omega$, $\|x(t)\| \le L$, for all $t \in [0,\sigma(T)]_{\mathbb{T}}$. Then, for $x \in \Omega$, we have

$$|(Ax)(t)| \le \left|\int_0^{\sigma(T)} G(t,s)f(s,x^\sigma(s))\Delta s\right| \le M\sigma(T) \sup_{s\in[0,\sigma(T)]_{\mathbb{T}},\|x\|<L} f(s,x^\sigma(s)),$$

which shows that $A(\Omega)$ is bounded. Finally, we will show that A is equicontinuous. Let $x \in \Omega$, $t_1,t_2 \in [0,\sigma(T)]_{\mathbb{T}}$. It follows that

$$\begin{aligned}
&|(Ax)(t_2)\ -(Ax)(t_1)|\\
&\quad= \frac{1}{e_p(t_1,0)}\bigg\{\Gamma\bigg[\alpha\int_0^{\sigma(T)}\int_0^{\sigma(s)} e_p(s,\sigma(r))f(s,x^\sigma(s))\Delta s\Delta g(s)\\
&\qquad+\beta\int_0^{\sigma(T)} e_p(s,\sigma(T))f(s,x^\sigma(s))\Delta s\bigg] + \int_0^{t_2} e_p(s,0)f(s,x^\sigma(s))\Delta s\bigg\}\\
&\qquad-\bigg|\frac{1}{e_p(t_1,0)}\bigg\{\Gamma\bigg[\alpha\int_0^{\sigma(T)}\int_0^{\sigma(s)} e_p(s,\sigma(r))f(s,x^\sigma(s))\Delta s\Delta g(s)\\
&\qquad+\beta\int_0^{\sigma(T)} e_p(s,\sigma(T))f(s,x^\sigma(s))\Delta s\bigg] + \int_0^{t_1} e_p(s,0)f(s,x^\sigma(s))\Delta s\bigg\}\bigg|\\
&\quad\le \left|\frac{1}{e_p(t_2,0)}-\frac{1}{e_p(t_1,0)}\right|\Gamma\bigg[\alpha\int_0^{\sigma(T)}\int_0^{\sigma(s)} e_p(s,\sigma(r))f(s,x^\sigma(s))\Delta s\Delta g(s)\\
&\qquad+\beta\int_0^{\sigma(T)} e_p(s,\sigma(T))f(s,x^\sigma(s))\Delta s\bigg]\\
&\qquad+\left|\frac{1}{e_p(t_2,0)}-\frac{1}{e_p(t_1,0)}\right|\int_0^{t_2} e_p(s,0)f(s,x^\sigma(s))\Delta s\\
&\qquad+\frac{1}{e_p(t_1,0)}\left|\int_{t_1}^{t_2} e_p(s,0)f(s,x^\sigma(s))\Delta s\right|\\
&\quad\le \left|\frac{1}{e_p(t_2,0)}-\frac{1}{e_p(t_1,0)}\right|\Gamma\bigg[\alpha\int_0^{\sigma(T)}\int_0^{\sigma(s)} e_p(s,\sigma(r))f(s,x^\sigma(s))\Delta s\Delta g(s)
\end{aligned}$$

$$
\begin{aligned}
&+\beta\int_0^{\sigma(T)} e_p(s,\sigma(T))f(s,x^\sigma(s))\Delta s\bigg]\\
&+\bigg|\frac{1}{e_p(t_2,0)}-\frac{1}{e_p(t_1,0)}\bigg|\Gamma\int_0^{t_2} e_p(s,0)f(s,x^\sigma(s))\Delta s\\
&+\frac{|t_2-t_1|}{e_p(t_1,0)}e_p(t_2,0)\sup_{s\in[0,\sigma(T)]_{\mathbb{T}},\|x\|\le L} f(s,x)\\
=&\bigg|\frac{1}{e_p(t_2,0)}-\frac{1}{e_p(t_1,0)}\bigg|M_1+\bigg|\frac{1}{e_p(t_2,0)}-\frac{1}{e_p(t_1,0)}\bigg|M_2\\
&+|t_2-t_1|e_p(t_2,t_1)M_3\\
=&\bigg|\frac{1}{e_p(t_2,0)}-\frac{1}{e_p(t_1,0)}\bigg|(M_1+M_2)+|t_2-t_1|e_p(t_2,t_1)M_3\\
\le&\frac{1}{e_p(t_1,0)e_p(t_2,0)}\sup_{t\in[0,\sigma(T)]_{\mathbb{T}}} pe_p(t,0)|t_2-t_1|(M_1+M_2)\\
&+|t_2-t_1|e_p(t_2,t_1)M_3,
\end{aligned}
$$

which shows that $|(Ax)(t_2)-(Ax)(t_1)|$ tends uniformly to 0 as $|t_2-t_1|\to 0$. Using the Mean Value Theorem [2] and Arzela–Ascoli Theorem [20], so the operator $A\colon P\to P$ is completely continuous. This completes the proof. □

4. Main results

The first main result of this paper is the following Theorem 4.1, which provides sufficient conditions for BVP (1.5) to have infinite many positive solutions.

The following conditions will be used in later statements.

- (H_1) $f\in C([0,T]_{\mathbb{T}}\times\mathbb{R}^+,\mathbb{R}^+)$ and $f(t,x)$ is bounded on $[0,\sigma(T)]_{\mathbb{T}}$ when x is bounded.
- (H_2) There exists a sequence $\{t_i\}_{i=1}^\infty$ such that $1<t_i<t_{i+1}<T/2$, $\lim_{i\to\infty} t_i = t_0<T/2$ and $t_0\in[0,\sigma(T)]_{\mathbb{T}}$.

THEOREM 4.1. *Assume that* (H_1)–(H_2) *hold. Let* $\{\theta_k\}_{k=1}^\infty$ *be such that* $\theta_k\in(t_k,t_{k+1})$, $k=1,2,\ldots$ *Let* $\{r_k\}_{k=1}^\infty$ *and* $\{R_k\}_{k=1}^\infty$ *be positive numbers such that*

$$
R_{k-1}<\gamma r_k<r_k<\frac{r_k}{m\sigma(T)}<R_k,\quad k=2,3,\ldots
$$

Furthermore, for each k, we assume that f satisfies:

- (H_3) *$f(t,x)\ge r_k/(m\sigma(T))$ for all $1/\theta_k\le t\le\theta_k$, $\gamma r_k\le x\le r_k$;*
- (H_4) *$f(t,x)\le R_k/M\sigma(T)$ for all $0\le t\le T$, $0\le x\le R_k$.*

Then BVP (1.5) *has infinite many solutions* $\{x^{[k]}\}_{k=1}^\infty$ *such that*

$$
r_k\le\|x^{[k]}\|\le R_k,\quad k=1,2,\ldots
$$

Proof. Since $1 < t_k < \theta_k < t_{k+1} \leq t_0 < T$, $k = 1, 2, \ldots$, then for for all $k \in N^+$ and $x \in P$, by the definition of cone we have

$$x(t) \geq \gamma \|x\|, \quad t \in \left[\frac{1}{\theta_k}, \theta_k\right].$$

Consider the sequence $\{\Omega_{1,k}\}_{k=1}^{\infty}$ and $\{\Omega_{2,k}\}_{k=1}^{\infty}$ of open subsets of E defined by

$$\begin{aligned}\Omega_{1,k} &= \{x \in P : \|x\| < r_k\}, \quad k = 1, 2, \ldots, \\ \Omega_{2,k} &= \{x \in P : \|x\| < R_k\}, \quad k = 1, 2, \ldots\end{aligned}$$

For a fixed k and $x \in \partial\Omega_{1,k}$, from (4.1) we have

$$r_k = \|x\| = \sup_{t \in [0, \sigma(T)]} |x(t)| \geq \sup_{t \in [1/\theta_k, \theta_k]} x(t) \geq \gamma \|x\| = \gamma r_k.$$

By (H_3), we have

$$f(t, x) \geq \frac{r_k}{m\sigma(T)} \quad \text{for all } t \in \left[\frac{1}{\theta_k}, \theta_k\right]$$

and

$$\begin{aligned}\|A(x)(t)\| &= \left| \int_0^{\sigma(T)} G(t, s) f(s, x^\sigma(s)) \Delta s \right| \\ &\geq m \int_0^{\sigma(T)} f(s, x^\sigma(s)) \Delta s \geq m\sigma(T) \cdot \frac{r_k}{m\sigma(T)} = r_k = \|x\|.\end{aligned}$$

Thus, an application of Theorem 3.1 implies that $i(A, \Omega_{1,k}, P) = 0$.

On the other hand, let $x \in \partial\Omega_{2,k}$, we have

$$x(t) \leq \sup_{t \in [0, \sigma(T)]} |x(t)| = \|x\| = R_k.$$

By (H_4), we have

$$f(t, x) \leq \frac{R_k}{M\sigma(T)} \quad \text{for all } t \in [0, T].$$

So

$$\begin{aligned}\|A(x)(t)\| &= \left| \int_0^{\sigma(T)} G(t, s) f(s, x^\sigma(s)) \Delta s \right| \\ &\leq M \int_0^{\sigma(T)} f(s, x^\sigma(s)) \Delta s \leq M\sigma(T) \cdot \frac{R_k}{M\sigma(T)} = R_k = \|x\|.\end{aligned}$$

Thus Theorem 3.1 implies that $i(A, \Omega_{1,k}, P) = 1$. Hence, since $r_k < R_k$ for $k \in N^+$, it follows from additivity of the fixed-point index that

$$i(A, \Omega_{2,k} \setminus \overline{\Omega}_{1,k}, P) = 1, \quad k \in N^+.$$

Thus A has a fixed point $x^{[k]}$ in $\Omega_{2,k} \setminus \overline{\Omega}_{1,k}$ such that $r_k \leq \|x^k\| \leq R_k$ with $Ax^{[k]} = x^k$, i.e.

$$x^{[k]}(t) = \int_0^{\sigma(T)} G(t,s)f(s,x^\sigma(s))\Delta s, \quad t \in [0,\sigma(T)]_{\mathbb{T}}.$$

Since $k \in N$ is arbitrary, $x^{[k]}$ are positive solutions of BVP (1.5). The proof is complete. □

Our next result uses Theorem 3.2. Let $r_k \in (1/\theta_k, \theta_k)$ where $\theta_k \in (t_k, t_{k+1})$, $k = 1, 2 \ldots$

We define the following nonnegative increasing continuous functions α_k, β_k and μ_k by

$$\begin{aligned} \alpha_k(x) &= \max_{t\in[1/\theta_k,\theta_k]_{\mathbb{T}}} x(t) = x(\theta_k), \\ \beta_k(x) &= \min_{t\in[r_k,\theta_k]_{\mathbb{T}}} x(t) = x(r_k), \\ \mu_k(x) &= \max_{t\in[1/\theta_k,r_k]_{\mathbb{T}}} x(t) = x(r_k). \end{aligned}$$

It is obvious that, for each $x \in P$, $\mu_k(x) \leq \beta_k(x) \leq \alpha_k(x)$. In addition, for each $x \in P$, $\mu_k(x) = x(r_k) \geq \gamma\|x\|$. Thus

$$\|x\| \leq \frac{\mu_k(x)}{\gamma}, \quad \text{for all } x \in P.$$

In the next result, we let $\rho_k = M\theta_k$, $\eta_k = mr_k$.

THEOREM 4.2. *Suppose that conditions* (H$_1$) *and* (H$_2$) *hold. Let* $\{\theta_k\}_{k=1}^\infty$ *be such that* $\theta_k \in (t_k, t_{k+1})$, $k = 1, 2 \ldots$ *Let* $\{a_k\}_{k=1}^\infty$, $\{b_k\}_{k=1}^\infty$ *and* $\{c_k\}_{k=1}^\infty$ *be positive numbers such that*

$$c_{k-1} < a_k < \gamma b_k < b_k < \frac{c_k}{\gamma}, \quad \rho_k b_k < \eta_k c_k, \quad k = 2, 3 \ldots$$

Furthermore for each natural number k we assume that f satisfies:

- (H$_5$) $f(t,x) < c_k/\rho_k$ *for all* $1/\theta_k \leq t \leq \theta_k$, $0 \leq x \leq c_k/\gamma$;
- (H$_6$) $f(t,x) \geq b_k/\eta_k$ *for all* $0 \leq t < T$, $b_k \leq x \leq b_k/\gamma$;
- (H$_7$) $f(t,x) < a_k/\rho_k$ *for all* $0 \leq t < T$, $0 \leq x \leq a_k/\gamma$.

Then BVP (1.5) *has three infinite families of solutions* $\{x^{[1k]}\}_{k=1}^\infty$, $\{x^{[2k]}\}_{k=1}^\infty$, $\{x^{[3k]}\}_{k=1}^\infty$, *for* $k = 1, 2 \ldots$ *satisfying:*

$$0 \leq \alpha(x^{[1k]}) \leq a_k \leq \alpha(x^{[2k]}), \quad \beta(x^{[2k]}) \leq b_k \leq \beta(x^{[3k]}), \quad \mu(x^{[3k]}) < c_k.$$

PROOF. We define the completely continuous operator A by (3.4). It is easy to check that $A: \overline{P(\mu_k, c_k)} \to P$ for $k \in N^+$.

We shall show that all the conditions of Theorem 3.2 are satisfied. To make use of condition (a) of Theorem 3.2, we choose $x \in \partial P(\mu_k, c_k)$. Then $\mu_k(x) =$

$\max_{1/\theta_k \le t \le r_k} x(t) = x(r_k) = c_k$, this implies that $0 \le x(t) \le c_k$, for $t \in [0, r_k]$. Recall that $\|x\| \le \mu_k(x)/\gamma = c_k/\gamma$. So we have $0 \le x(t) \le c_k/\gamma$, $t \in [0, \sigma(T)]_{\mathbb{T}}$. Then assumption (H$_5$) implies $f(t,x) < c_k/\rho_k$, $t \in [1/\theta_k, \theta_k]$. Therefore

$$\begin{aligned}\mu_k(Ax) &= \max_{1/\theta_k \le t \le r_k} (Ax)(t) = (Ax)(r_k) = \int_0^{r_k} G(t,s) f(s, x^\sigma(s)) \Delta s \\ &\le \int_0^{\theta_k} G(t,s) f(s, x^\sigma(s)) \Delta s \le \theta_k \cdot M \cdot \frac{c_k}{\rho_k} = c_k.\end{aligned}$$

Hence condition (a) is satisfied.

Secondly, we shall show that condition (b) of Theorem 3.2 is fulfilled. For this end, we select $x \in \partial P(\beta_k, b_k)$, then $\beta_k(x) = \min_{r_k \le t \le \theta_k} x(t) = x(r_k) = b_k$, this means $x(t) \ge b_k$, for $t \in [r_k, \theta_k]_{\mathbb{T}}$. So we have $\|x\| \ge b_k$, $t \in [r_k, \theta_k]_{\mathbb{T}}$. Noting that $\|x\| \le \mu_k/\gamma \le \beta_k/\gamma = b_k/\gamma$, we have $b_k \le x(t) \le b_k/\gamma$, $t \in [r_k, \theta_k]_{\mathbb{T}}$.

By (H$_6$), we have $f(t,x) > b_k/\eta_k$, $t \in [r_k, \theta_k]_{\mathbb{T}}$. Therefore

$$\begin{aligned}\beta_k(Ax) &= \min_{r_k \le t \le \theta_k} (Ax)(t) = (Ax)(r_k) \\ &= \int_0^{r_k} G(t,s) f(s, x^\sigma(s)) \Delta s \ge \frac{b_k}{\eta_k} \cdot r_k \cdot m = b_k,\end{aligned}$$

and so the condition (b) is satisfied.

Finally, we verify that condition (c) of Theorem 3.2 is also met. We note that $x_0(t) \equiv a_k/2$ is an element of $P(\alpha_k, a_k)$ and $\alpha_k(x_0) = a_k/2 < a_k$. So $P(\alpha_k, a_k) \ne \emptyset$. Now let $x \in \partial P(\alpha_k, a_k)$, then $\alpha_k(x) = \max_{t \in [1/\theta_k, \theta_k]} x(t) = a_k$. This implies that $0 \le x(t) \le a_k$, for $t \in [1/\theta_k, \theta_k]$. Together with $\|x\| \le \mu_k(x)/\gamma \le \alpha_k(x)/\gamma = a_k/\gamma$. Then we get $0 \le x(t) \le a_k/\gamma$, $t \in [0, T]$.

By (H_7), we have $f(t,x) < a_k/\rho_k$, $t \in [0,T]$. As before, we get

$$\begin{aligned}\alpha_k(Ax) &= \max_{1/\theta_k \le t \le \theta_k} (Ax)(t) = (Ax)(\theta_k) \\ &= \int_0^{\theta_k} G(t,s) f(s, x^\sigma(s)) \Delta s \le \theta_k \cdot \frac{a_k}{\rho_k} \cdot M = a_k.\end{aligned}$$

Thus condition (c) of Theorem 3.2 is satisfied. Since all the hypotheses of Theorem 3.2 are satisfied, A has three families of solutions $\{x^{[1k]}\}_{k=1}^\infty$, $\{x^{[2k]}\}_{k=1}^\infty$, $\{x^{[3k]}\}_{k=1}^\infty$ satisfying

$$0 \le \alpha(x^{[1k]}) \le a_k \le \alpha(x^{[2k]}), \quad \beta(x^{[2k]}) \le b_k \le \beta(x^{[3k]}), \quad \mu(x^{[3k]}) < c_k,$$

for $k = 1, 2 \dots$ The proof is complete. □

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