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SOLUTIONS TO SOME SINGULAR NONLINEAR BOUNDARY VALUE PROBLEMS

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This article is devoted to the memory of Jerry Marsden

Abstract. We apply the so-called p-regularity theory to prove the existence of solutions to two nonlinear boundary value problems: an equation of rod bending and some nonlinear Laplace equation.

1. Introduction

The p-regularity theory is an effective apparatus to study many nonlinear mathematical, physical and numerical problems (see [3], [4]). Usually such a problem is given as a nonlinear equation

$$F(x) = 0$$

where F is a sufficiently smooth map between Banach spaces X and Y. The above equation describes a regular submanifold of X near a regular point x^*, i.e. when the operator $F'(x^*)$ is surjective.

The p-regularity theory [3]–[5], [7] deals with the irregular cases. The main idea of this construction is to replace the operator $F'(x^*)$ (which is not surjective) with another linear operator which is surjective. The latter operator, denoted by $\Psi_p(x^*, h)$, is related with the p^{th} order of the Taylor expansion of F at x^*.

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Here the vector h is taken from the tangent cone to the set $\{F(x) = 0\}$ at x^* and p is taken so large that the operator $\Psi_p(x^*, h)$ is really surjective (this is the so-called p-regularity condition). In the next section we recall the main concepts of the p-regularity theory.

In the third section we apply the p-regularity theory to the following boundary value problems:

- the equation of rod bending

$$\frac{d^2u}{dx^2} + (1+\varepsilon)(u + u^2) = 0, \quad u(0) = u(\pi) = 0; \tag{1.1}$$

- the nonlinear Laplace equation

$$\Delta u + (10+\varepsilon)\phi(u) = 0, \quad u|_{\partial\Omega} = 0 \tag{1.2}$$

  where $\Omega = [0,\pi] \times [0,\pi]$, ε is a small parameter and ϕ is some function of one variable).

These problems are related respectively with the string oscillations and the membrane oscillations (see [2]). Analogous problems were studied by M. Buchner, J. Marsden and S. Schechter [1]; they used methods of the bifurcation theory (the Lyapunov–Schmidt reduction) and results obtained are similar to ours.

2. Elements of the p-regularity theory

We begin with some notations. X and Y will denote fixed Banach spaces. If

$$B: X \times \ldots \times X = X^r \mapsto Y$$

is a symmetric r-linear continuous operator then we consider its two restrictions:

$$B \circ \Delta_r: X \mapsto Y, \qquad B \circ \Gamma_r: X \times X \mapsto Y, \tag{2.1}$$

where $\Delta_r: X \mapsto X^r$, $\Delta_r(x) = (x, \ldots, x)$, is the diagonal embedding and $\Gamma_r: X \times X \mapsto X^r$ is defined as $\Gamma_1(h,g) = g$ and $\Gamma_r(h,z) = (h, \ldots, h, g)$ for $r \geq 2$. Thus

$$B \circ \Delta_r(h) = B(h, \ldots, h), \qquad B \circ \Gamma_r(h,g) = B(h, \ldots, h, g). \tag{2.2}$$

The map $B \circ \Delta_r$ is homogeneous polynomial of degree r and the map $B \circ \Gamma_r$ is homogeneous polynomial of degree $r-1$ with respect to the first argument and is linear with respect to the second argument. Note also that $B \circ \Gamma_r$ equals the derivative of $B \circ \Delta_r$ (up to a factor).

Let $F: U \mapsto Y$ be a p times Frechet differentiable map from an open subset $U \subset X$. Let $x^* \in U$.

Definition 2.1. We say that the map F is *regular at* x^* if $\operatorname{Im} F'(x^*) = Y$; otherwise, we say that F is *degenerate at* x^* ([1]). We say that F is *completely degenerate at* x^* up to order p if $F^{(r)}(x^*) = 0$ for $r = 1, \dots, p-1$.

By the classical Lyusternik theorem the *solution set*

$$M = M(x^*) = \{F(x) = F(x^*)\} \tag{2.3}$$

is a submanifold near x^* if F is regular at x^*. Moreover, the tangent space

$$T_{x^*} M = \ker F'(x^*). \tag{2.4}$$

Since the point x^* is fixed below the derivatives $F^{(j)}(x^*)$ will be denoted simply by $F^{(j)}$.

Let us pass to the definition of p-regularity. Assume that F is degenerate at x^*. Therefore

$$Y_1 = \operatorname{cl} \operatorname{Im} F' \neq Y$$

(where cl denotes the closure and the derivative is taken at x^*).

We define two series $Z_2, Z_3, \dots$ and $Y_2, Y_3, \dots$ of subspaces of Y as follows. We put Z_2 as some closed subspace complementary to Y_1. Let $P_2 \colon Y \mapsto Z_2$ be the projection to Z_2 along Y_1. We then put

$$Y_2 = \operatorname{cl} \operatorname{span} \operatorname{Im} P_2 F^{(2)} \circ \Delta_2.$$

Next, we define Z_3 as a closed complementary to $Y_1 \oplus Y_2$ with a corresponding projection P_3 onto Z_3 and $Y_3 = \operatorname{cl} \operatorname{span} \operatorname{Im} P_3 F^{(3)} \circ \Delta_3$. Further subspaces are defined along this scheme: Z_i is complementary to $Y_1 \oplus \dots \oplus Y_{i-1}$ with corresponding projection P_i and $Y_i = \operatorname{cl} \operatorname{span} \operatorname{Im} P_i F^{(i)} \circ \Delta_i$.

Assume that this construction ends-up at some moment, thus

$$Y = Y_1 \oplus \dots \oplus Y_p \tag{2.5}$$

for some finite p. Denote also $Q_j \colon Y \mapsto Y_j$ the projections corresponding to the above decomposition. Then we have the maps

$$f_j \colon U \mapsto Y_j, \qquad f_j(x) = Q_j F(x).$$

Definition 2.2. For a fixed $h \in X$ the linear operator

$$\Psi_p(h) = \Psi_p(x^*, h) \colon X \mapsto X, \qquad \Psi_p(h) g = \sum_{j=1}^{p} f_j^{(j)} \circ \Gamma_j(h, g), \tag{2.6}$$

(see (2.1)) is called the *p-factor operator*. We say that F is *p-regular at* x^* *along vector* h if $\operatorname{Im} \Psi_p(h) = Y$

([1]) Usually, e.g. in the finite dimensional case, the notion of critical point x_* of F is used. It is such a point that $F'(x_*)$ is neither injective nor surjective; $\operatorname{rank} F'(x_*) < \min(\dim X, \dim Y)$ if $\dim X, \dim Y < \infty$. Definition 2.1 is specific for this paper and is slightly different.

Using the decomposition (2.5) the operator (2.6) can be written as follows

$$g \mapsto (Q_1 F' g, Q_2 F^{(2)} \circ \Gamma_2(h, g), \dots , Q_p F^{(p)} \circ \Gamma_p(h, g)).$$

Below the vector h is chosen from the following generalization of the kernel of $F'(x^*)$.

Definition 2.3. The *p-kernel* of Ψ_p is the set

$$H(x^*) = \{h : \Psi_p(x^*, h)h = 0\}.$$

In other words, it is the intersection

$$H(x^*) = \bigcap_{j=1}^{p} \{f_j^{(j)} \circ \Delta_j(h) = 0\}$$

of p cones corresponding to the zero loci of the homogeneous polynomials $h \mapsto f_j^{(j)} \circ \Delta_j(h)$. In the completely degenerate case we have

$$H(x^*) = \{F^{(p)} \circ \Delta_p(h) = 0\}.$$

Definition 2.4. We say that F is *p-regular at* x^* if either $H(x^*) = 0$ or F is p-regular at x^* along every $h \in H(x^*) \setminus 0$.

We can regard the p-regularity as the usual regularity of the map $h \mapsto \Psi_p(h)h$ along the punctured p-kernel; thus $\ker^p \Psi_p \setminus 0$ is a smooth (and homogeneous) submanifold of X.

The following generalization of the Lyusternik theorem holds.

Theorem 2.5 ([4], [5]). *If F is p-regular at x^* then the tangent cone $C_{x_*}M$ to the level set* (2.2) *equals $H(x^*)$. In particular, the solution set M is either reduced to $\{x^*\}$ or is higher dimensional and each component of $C_{x_*}M$ corresponds to a local branch of M of the same dimension.*

In the sequel we shall use the following standard results from analysis.

Remark 2.6. A linear bounded operator $A : X \mapsto Y$ is called *Fredholm* if its kernel $\ker A$ and cokernel $\operatorname{coKer} A = Y/\operatorname{Im} A$ have finite dimension. Recall that in such a case $\operatorname{Im} A$ is closed and equals to the annihilator of $\ker A^*$, i.e. $\operatorname{Im} A = (\ker A^*)^\top$.

In this paper we consider second order differential operators acting on functions on a domain $\Omega \subset \mathbb{R}^n$. There we have the scalar product

$$\langle u, v\rangle = \int_\Omega u(x)v(x)d^n x.$$

The operators considered are symmetric, i.e. $\langle Au, v\rangle = \langle u, Av\rangle$ for $u, v \in X$. The standard theory (see [6]) says that in this case we have the decomposition

$$Y = \operatorname{Im} A \oplus \ker A \quad \text{and} \quad \dim \ker A = \dim \operatorname{coKer} A.$$

REMARK 2.7. Let A and B be bounded operators between Banach spaces X and Y. Let $Y = Y_1 \oplus Y_2$ where $Y_1 = \operatorname{Im} A$. Let also Q_2 be the projection onto Y_2 along Y_1. Then the operator $A + Q_2 B$ is surjective if and only if the operator $Q_2 B|_{\ker A} \colon \ker A \mapsto Y_2$ is surjective.

3. Applications

3.1. The equation of rod bending. The differential equation (1.1) can be written in the form

$$F(u,\varepsilon) := \frac{d^2u}{dx^2} + (1+\varepsilon)(u+u^2) = 0,$$

where F acts between the Banach spaces

$$X = C_0^2([0,\pi]) \times \mathbb{R} \quad \text{and} \quad Y = C([0,\pi]),$$

where $C_0^r(\Omega) = C^r(\Omega) \cap \{u|_{\partial\Omega} = 0\}$. Of course, $(u,\varepsilon) = (0,0)$ is a solution to this equation. Our aim is to solve this equation for small and nonzero ε.

The first derivative of F at $(0,0)$ is

$$F' = (F'_u, F'_\varepsilon) = \left(\frac{d^2}{dx^2} + 1, 0\right) \tag{3.1}$$

and the second derivative at $(0,0)$ is

$$F'' = \begin{pmatrix} F''_{uu} & F''_{u\varepsilon} \\ F''_{\varepsilon u} & F''_{\varepsilon\varepsilon} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix},$$

thus

$$F''(h,g) = 2h_u(x)g_u(x) + h_\varepsilon g_u(x) + g_\varepsilon h_u(x) \tag{3.2}$$

for $h = (h_u(x), h_\varepsilon)$ and $g = (g_u(x), g_\varepsilon)$ from X (which consists of functions and constants).

We easily find that

$$\ker F' = \mathbb{R} \cdot \sin x \times \mathbb{R}. \tag{3.3}$$

The image of F' consists of those function $v(x)$ for which the equation

$$d^2u/dx^2 + u = v$$

admits a solution $u(x)$ with the Dirichlet boundary condition. The general solution to the latter equation (which we find using the variation of constants method) takes the form

$$u(x) = C_1 \cos x + C_2 \sin x - \cos x \int_0^x v(s) \sin s \, ds + \sin x \int_0^x v(s) \cos s \, ds$$

and the boundary condition implies $C_1 = 0$ and the following:

$$Y_1 = \operatorname{Im} F' = \left\{ v : \langle v, \sin x \rangle = \int_0^\pi v(s) \sin x \, dx = 0 \right\}. \tag{3.4}$$

Of course, the function $v(x) = \sin x$ does not belong to $\operatorname{Im} F'(0,0) = \operatorname{Im} F'$ which means that the point $(0,0) \in X$ is irregular for F. We choose

$$Z_2 = \mathbb{R} \cdot \sin x \subset Y, \tag{3.5}$$

which satisfies $Y = Y_1 \oplus Z_2$ (this agrees with Remark 2.6). The projection operator P_2 to Z_2 along Y_1 can be described as

$$P_2 v = \frac{2}{\pi} \sin x \cdot \langle v, \sin x \rangle. \tag{3.6}$$

Note that

$$F'' \circ \Delta_2(h_u, h_\varepsilon) = 2h_u^2(x) + 2h_\varepsilon h_u(x)$$

(see (3.2)), hence the subspace $Y_2 =$span $\operatorname{Im} P_2 F'' \circ \Delta_2$ equals Z_2. So we have the expansion (2.5) with $p = 2$, i.e. $Y = Y_1 \oplus Y_2$; we recall the corresponding projectors $Q_{1,2} \colon Y \mapsto Y_{1,2}$ where $Q_2 = P_2$ is defined above.

Let us pass to the description if the 2-factor operator and the examination of the 2-regularity condition. For $h = (h_u(x), h_\varepsilon)$ and $g = (g_u(x), g_\varepsilon)$ we have

$$\begin{aligned} \Psi_2(h)g &= Q_1 F' g + Q_2 F'' \circ \Gamma_2(h, g) \\ &= \{d^2 g_u(x)/dx^2 + g_u(x)\} + P_2\{2h_u(x) g_u(x) + h_\varepsilon g_u(x) + g_\varepsilon h_u(x)\}. \end{aligned}$$

The determination of the 2-kernel of Ψ_2, i.e. $\{\Psi_2(h)h = 0\}$, runs as follows:

$$\begin{aligned} h_u &= C \sin x, \\ \langle \sin x, h_u^2(x) + h_\varepsilon h_u(x) \rangle &= C^2 \int_0^\pi \sin^3 x + h_\varepsilon C \int_0^\pi \sin^2 x = 0. \end{aligned}$$

Calculation of the above integrals gives two possibilities (which correspond to two $1-$dimensional components of $\ker^2 \Psi_2$):

1. $C = 0$, i.e. $h_u(x) \equiv 0$ and h_ε arbitrary;
2. $h_\varepsilon = -8C/(3\pi)$.

Recall that the 2-regularity means that the linear operator $\Psi_2(h)$ is surjective for any $h \in \ker^2 \Psi_2$. In the both cases of the choice of h the operator $\Psi_2(h)$ has the form $A + P_2 B$, where $Y_1 = \operatorname{Im} A$ is complementary to $Y_2 = \operatorname{Im} P_2$. By Remark 2.7 it is enough to show that $\operatorname{Im}(P_2 B|_{\ker A}) = Y_2$, i.e. that the integral

$$\langle \sin x, 2h_u(x) g_u(x) + h_\varepsilon g_u(x) + g_\varepsilon h_u(x) \rangle$$

is nonzero for $g_u = \sin x$ and typical constant g_ε.

In the case 1 the problem reduces to the nonvanishing of the integral

$$\langle \sin x, g_u \rangle = \int_0^\pi \sin^2 x.$$

But this case is non-interesting, because it corresponds to the obvious 1-dimensional family of solutions to equation (1.1) of the form

$$u(x) \equiv 0, \quad \varepsilon - \text{arbitrary}.$$

In the case 2 we reduce the problem to the case

$$h_u = g_u = \sin x, \quad h_\varepsilon = -8/3\pi, \quad g_\varepsilon - \text{arbitrary}$$

and to non-vanishing of the integral

$$\left\langle \sin x, 2\sin^2 x + \left(g_\varepsilon - \frac{8}{3\pi}\right)\sin x \right\rangle. \tag{3.7}$$

Of course, for a typical constant g_ε the latter expression is nonzero.

Now Theorem 2.5 applied to the second component of the tangent cone to M implies the following.

THEOREM 3.1. *For sufficiently small* $|\varepsilon|$ *the rod bending equation* (1.1) *has a unique nonzero solution* $u(x, \varepsilon)$ *such that*

$$u(x, \varepsilon) = \frac{3\pi}{8}\,\varepsilon \sin x + o(\varepsilon).$$

3.2. The nonlinear membrane equation. Like in the rod bending case equation (1.2) takes the form:

$$F(u, \varepsilon) := \Delta u + (10 + \varepsilon)\phi(u) = 0,$$

where $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ is the Laplacian, ε is a small constant and the function ϕ satisfies the following properties:

$$\phi(0) = 0, \quad \phi'(0) = 1, \quad 10\,\phi''(0) = a \neq 0. \tag{3.8}$$

Above the nonlinear operator F acts between the Banach spaces

$$X = C_0^2(\Omega) \times \mathbb{R}, \quad \Omega = [0, \pi] \times [0, \pi] \quad \text{and} \quad Y = C(\Omega).$$

Of course, $(u, \varepsilon) = (0, 0)$ is a solution.

Moreover, we have the following 1-parameter family of solutions:

$$u(x) \equiv 0, \quad \varepsilon - \text{arbitrary}. \tag{3.9}$$

The first and the second derivatives of F at $(0, 0)$ are following:

$$F' = (F'_u, F'_\varepsilon) = (\Delta + 10, 0), \tag{3.10}$$

$$F'' = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.11}$$

We see that $\ker F'$ consists of pairs $h = (h_u(x), h_\varepsilon)$ such that $h_\varepsilon \in \mathbb{R}$ and $h_u(x)$ is an eigenfunction of the Laplacian in Ω with the Dirichlet boundary conditions with the eigenvalue $-10 = -1 - 3^2$. Therefore

$$\ker F' = \{\mathbb{R} \cdot u_1(x) + \mathbb{R} \cdot u_2(x)\} \times \mathbb{R}, \tag{3.12}$$

where

$$u_1 = \frac{2}{\pi} \sin x_1 \cdot \sin(3x_2), \quad u_2 = \frac{2}{\pi} \sin x_2 \cdot \sin(3x_1) \tag{3.13}$$

are orthonormal with respect to the scalar product

$$\langle u, v \rangle = \int_\Omega u(x)v(x)\, d^2x.$$

The operator $F'_u = \Delta + 10$ is symmetric and Fredholm. From the general theory (see Remark 2.6) it follows that $\dim \ker F'_u = \dim \ker(F'_u)^* < \infty$ and that

$$Y = \operatorname{Im} F'_u \oplus \ker F'_u = Y_1 \oplus Y_2,$$

where the latter decomposition is orthogonal with respect to the above scalar product. As usually, we denote by $Q_{1,2}$ the projectors corresponding to the above decomposition. We have

$$Q_2 v = \langle v, u_1 \rangle u_1 + \langle v, u_2 \rangle\, u_2. \tag{3.14}$$

Of course, the above means that the point $(u, \varepsilon) = (0, 0)$ is irregular for F.

Now we pass to application of the 2-regularity theory. By (3.11) we have

$$F''(h, g) = a h_u(x) g_u(x) + h_\varepsilon g_u(x) + g_\varepsilon h_u(x)$$

where $h = (h_u(x), h_\varepsilon)$ and $g = (g_u(x), g_\varepsilon)$ are vectors from $X = C_0^2(\Omega) \times \mathbb{R}$. Therefore we get

$$\Psi_2(h)g = \{\Delta + 10\} g_u + Q_2\{a h_u(x) g_u(x) + h_\varepsilon g_u(x) + g_\varepsilon h_u(x)\}, \tag{3.15}$$

hence $\ker^2 \Psi_2$ consists of

$$h = (h_u, h_\varepsilon) = (H_1 u_1(x) + H_2 u_2(x), h_\varepsilon)$$

such that

$$\langle u_j, a h_u^2(x) + 2 h_\varepsilon h_u(x) \rangle = 0, \quad j = 1, 2. \tag{3.16}$$

Let

$$\alpha = \int_\Omega u_{1,2}^3 = \frac{16}{27} \cdot \left(\frac{2}{\pi}\right)^3, \quad \beta = \int_\Omega u_1^2 u_2 = \int_\Omega u_1 u_2^2 = -\frac{48}{175} \cdot \left(\frac{2}{\pi}\right)^3 \tag{3.17}$$

(as one can calculate). Then system (3.16) takes the form:

$$a(\alpha H_1^2 + 2\beta H_1 H_2 + \beta H_2^2) + 2 h_\varepsilon H_1 = 0, \tag{3.18}$$

$$a(\beta H_1^2 + 2\beta H_1 H_2 + \alpha H_2^2) + 2 h_\varepsilon H_2 = 0. \tag{3.19}$$

Using the matrices $H = (H_1, H_2)^\top$ and

$$\mathcal{M}(H) = a\begin{pmatrix} \alpha H_1 + \beta H_2 & \beta H_1 + \beta H_2 \\ \beta H_1 + \beta H_2 & \beta H_1 + \alpha H_2 \end{pmatrix}$$

we rewrite it in the following form:

$$(\mathcal{M}(H) + 2h_\varepsilon)H = 0. \tag{3.20}$$

On the other hand, the Y_2- part (where $Y_2 \simeq \mathbb{R}^2$) of the 2-factor operator $\Psi_2(h)$ takes the form:

$$(G, g_\varepsilon) \mapsto (\mathcal{M}(H) + h_\varepsilon)G + g_\varepsilon H = [\mathcal{M}(H) + h_\varepsilon, H](G, g_\varepsilon)^\top, \tag{3.21}$$

where $g = (G_1u_1(x) + G_2u_2(x), g_\varepsilon)$ and $G = (G_1, G_2)^\top$; accordingly with Remark 2.7 g is taken from $\ker F'$.

Equation (3.20) has two types of possible solutions:

1. $H = 0$, $h_\varepsilon \neq 0$;
2. $\det(\mathcal{M}(H) + 2h_\varepsilon) = 0$, $H \in \ker(\mathcal{M}(H) + 2h_\varepsilon) \setminus 0$.

In the case 1 the operator (3.21) is obviously surjective; but this case corresponds to the tangent cone to the solution (3.9).

In the case 2 we multiply the equations (3.18)–(3.19) by H_2 and H_1, respectively. Then we take the difference which implies the following equation:

$$(H_2 - H_1)(\beta H_1^2 + (3\beta - \alpha)H_1H_2 + \beta H_2^2) = \beta(H_2 - H_1)(H_1 + \kappa H_2)(H_1 + H_2/\kappa) = 0$$

where

$$\kappa = \frac{209}{81} + \frac{16}{81}\sqrt{145} \approx 4.9588 \tag{3.22}$$

is the greater root of the equation $\beta\kappa^2 - (3\beta - \alpha)\kappa + \beta = 0$, i.e. $\kappa + 1/\kappa = 418/81$. Thus we have three possibilities:

$$\begin{aligned} H_2 &= H_1, & h_\varepsilon &= -(a/2)(\alpha + 3\beta)H_1, \\ H_2 &= -\kappa H_1, & h_\varepsilon &= -(a/2)(\alpha - \beta)(\kappa - 1)H_1, \\ H_1 &= -\kappa H_2, & h_\varepsilon &= -(a/2)(\alpha - \beta)(\kappa - 1)H_2. \end{aligned} \tag{3.23}$$

The 2-regularity condition along any of the latter solution means that the linear operators (3.21) is surjective. Of course, this is equivalent to the property that the 2×3 matrix

$$[\mathcal{M}(H) + h_\varepsilon, H] \tag{3.24}$$

has maximal rank (equal 2) when (H_1, H_2) satisfies one of the conditions (3.23). But a sufficient condition for this is that

$$\det(\mathcal{M}(H) + h_\varepsilon) \neq 0; \tag{3.25}$$

(we recall that $\det(\mathcal{M}(H) + 2h_\varepsilon) = 0$ in the case 2). Since $\det(\mathcal{M} + \lambda) = (\lambda + \operatorname{tr}\mathcal{M}/2)^2 + \det\mathcal{M} - \operatorname{tr}^2\mathcal{M}/4$ we have $\det(\mathcal{M}(H)+2h_\varepsilon) - \det(\mathcal{M}(H)+h_\varepsilon) = h_\varepsilon(3h_\varepsilon + \operatorname{tr}\mathcal{M})$. So, if (3.25) does not hold then it must be

$$h_\varepsilon = -\frac{\operatorname{tr}\mathcal{M}}{3} = -\frac{a}{3}(\alpha+\beta)(H_1+H_2).$$

It is easy that this contradicts (3.23) for nonzero H.

Like in the previous section we can conclude this section with the following

Theorem 3.2. *For sufficiently small* $|\varepsilon|$ *the membrane equation* (1.2) *has three nonzero solution* $u(x,\varepsilon)$ *such that*

$$\begin{aligned} u(x,\varepsilon) &= \frac{-2/a}{\alpha+3\beta}\varepsilon\cdot\{u_1(x)+u_2(x)\} + o(\varepsilon),\\ u(x,\varepsilon) &= \frac{-2/a}{(\alpha-\beta)(\kappa-1)}\,\varepsilon\cdot\{u_1(x)-\kappa u_2(x)\} + o(\varepsilon),\\ u(x,\varepsilon), &= \frac{-2/a}{(\alpha-\beta)(\kappa-1)}\,\varepsilon\cdot\{u_2(x)-\kappa u_1(x)\} + o(\varepsilon), \end{aligned}$$

where a, α, β, κ, $u_{1,2}(x)$ *are given in equations* (3.8), (3.16), (3.17) *and* (3.22).

3.3. Comparison with the bifurcation theory approach. The whole our paper was inspired by the paper [1]. There the authors consider the bifurcation problem for a map of the form

$$F(u,\lambda) = Lu + (\lambda-\lambda_0)u + R(u),$$

where L is an elliptic selfadjoint operator, with a domain $X \subset Y$ being a suitable Sobolev space, $R\colon X \mapsto Y$ is a smooth map with $R(0)=0, R'(0)=0$ and λ_0 is an eigenvalue of L of multiplicity $n \geq 1$. They use the Lyapunov–Schmidt procedure to arrive at a system of finite dimensional algebraic equations. In our examples we arrive at a similar system of equations, but in somewhat different way.

Our 2-regularity condition corresponds to the following regularity hypothesis in [1] (denoted by (R)):

Let $C_i^{jk} = \langle R''(0)(u_j,u_k),u_i\rangle$, *where* $\{u_j\}_{j=1,\dots,n}$ *is an orthogonal basis of* $\ker(L-\lambda_0)$. *Then for each nonzero* $(x,\lambda)\in\mathbb{R}^n\times\mathbb{R}$ *satisfying*

$$2\lambda x_i + \sum_{j,k} C_i^{jk}x_jx_k = 0, \quad i=1,\dots,n,$$

the $n\times(n+1)$ *matrix* $\Big[\sum_j C_i^{jk}x_k + \lambda\delta_i^k, x_i\Big]$ *has maximal rank.*

In the above membrane case $C_i^{jk} = a\,\langle u_ju_k,u_i\rangle = a\alpha$ or $= a\beta$, x_i correspond to H_i and λ corresponds to h_ε. Moreover, in the case of equation (1.2) the authors of [1] do not get as precise leading terms as in Theorem 3.2.

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