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NONTRIVIAL SOLUTIONS OF FOURTH-ORDER SINGULAR BOUNDARY VALUE PROBLEMS WITH SIGN-CHANGING NONLINEAR TERMS

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Abstract. In this paper, the fourth-order singular boundary value problem (BVP)

$$u^{(4)}(t) = h(t)f(u(t)), \quad t \in (0,1),$$
$$u(0) = u(1) = u'(0) = u'(1) = 0$$

is considered under some conditions concerning the first characteristic value corresponding to the relevant linear operator, where h is allowed to be singular at both $t = 0$ and $t = 1$. In particular, $f\colon (-\infty,\infty) \to (-\infty,\infty)$ may be a sign-changing and unbounded function from below, and it is not also necessary to exist a control function for f from below. The existence results of nontrivial solutions and positive-negative solutions are given by the topological degree theory and the fixed point index theory, respectively.

1. Introduction

It is well known that the bending of elastic beam can be described with some fourth-order boundary value problems. Much attention has been given to fourth-order boundary value problems with diverse boundary conditions via many methods (see [1]–[5], [10], [12]–[16], [19], [21], [22]). In order to use the fixed point index theory on a cone, the condition $f(u) \geq 0$ is often imposed on

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the nonlinearity f, i.e. f is nonnegative (see [1], [4], [5], [13]–[16]). However, many nonlinear problems which arise in applications do not have nonnegative nonlinearity. When the nonlinearity is allowed to take on both positive and negative values, such problems arise naturally in chemical reactor theory, design of suspension bridges, combustion and management of natural resources (see [2], [3], [10], [12], [19], [21], [22]). [2], [19] discussed semipositone higher-order differential equations and singular superlinear Sturm–Liouville problems, respectively. In order to get the existence of nontrivial solutions, they demand that f is bounded from below, i.e. $f(t,u) \geq -M$ for $(t,u) \in [0,1] \times [0,\infty)$. [10], [12] studied the nontrivial solutions for a kind of four-order and two-order nonlinear boundary value problem with the following more general conditions: f is an odd continuous function, $f(u,v) \geq bu + cv - C$ for $t \in [0,1]$, $u \geq 0$, $v \leq 0$, and $f(u) \geq -C|u|^\alpha - b$ for $u \in (-\infty,\infty)$, respectively, where b, c, C are positive constants and $\alpha \in (0,1)$, i.e. f is bounded from below or has a control function from below.

Many papers (see, for instance, [1], [15], [21] and references therein) discussed the existence of positive solutions of the following

$$\begin{cases} u^{(4)}(t) = f(t,u(t)), & t \in (0,1), \\ u(0) = u(1) = u'(0) = u'(1) = 0. \end{cases} \tag{1.1}$$

They all studied the case that f is a nonnegative continuous function, and the result obtained is the existence of positive solutions using fixed point theorems on a cone. As far as we know, there are few papers to study the existence of nontrivial solutions to BVP (1.1) under the condition that $f(u)$ may be a sign-changing function. Motivated by [3], [9], [10], [12], [19], [21], [22], the aim of this paper is to study the following singular boundary value problem of fourth-order equation:

$$\begin{cases} u^{(4)}(t) = h(t)f(u(t)), & t \in (0,1), \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases} \tag{1.2}$$

where $h \in C((0,1),[0,\infty))$ and may be singular at both $t = 0$ and $t = 1$, $f\colon (-\infty,\infty) \to (-\infty,\infty)$ may be a sign-changing and unbounded function from below, and it is not also necessary to exist a control function for f from below. We obtain the existence results of nontrivial solutions by means of the topological degree theory under some conditions on $f(u)$ concerning the first characteristic value corresponding to the relevant linear operator. In the meantime, existence of positive and negative solutions is also discussed in this paper.

This paper is organized as follows. Section 2 gives preliminaries and some lemmas. Section 3 is devoted to our main results. Section 4 discusses the existence of positive and negative solutions of BVP (1.1). Section 5 gives some examples to indicate the applications of our main results.

2. Preliminaries and some lemmas

Definition 2.1 ([11]). Let E be a real Banach space and $A: E \to E$ be a nonlinear operator. A nonzero solution to the equation $x = \lambda Ax$ is called an *eigenvector of the nonlinear operator* A; the corresponding number λ is called a *characteristic value* (λ^{-1} is called *eigenvalue*).

Let E be a real Banach space, E^* the dual space of E, P a total cone in E (i.e. P is a positive cone and $\overline{P-P} = E$), and P^* the dual cone of P, namely $P^* = \{f \in E^* : f(x) \geq 0, \text{ for all } x \in P\}$.

Theorem 2.2 (Krein–Rutman, [6], [11]). *Let K be a linear compact positive operator with $r(K) > 0$, where $r(K)$ denotes the spectral radius of K. Then $r(K)$ is an eigenvalue of K with a positive eigenvector. Meanwhile, $r(K)$ is an eigenvalue of K^*, the dual operator of K, with positive eigenvector in P^*.*

Let $B: P \to P$ be a completely continuous linear positive operator, then B can be uniquely extended as a completely continuous linear positive operator which maps E into E. On account of Krein–Rutman's theorem, if $r(B) > 0$, then there exist $\varphi \in P \setminus \{\theta\}$ and $g^* \in P^* \setminus \{\theta\}$, such that

$$B\varphi = r(B)\varphi, \qquad B^* g^* = r(B) g^*. \tag{2.1}$$

Choosing such an element $g^* \in P^* \setminus \{\theta\}$ such that the latter equality in (2.1) holds, for a given number $\delta > 0$, set

$$P(g^*, \delta) = \{u \in P : g^*(u) \geq \delta \|u\|\}. \tag{2.2}$$

Definition 2.3 ([18]). If there exists $g^* \in P^* \setminus \{\theta\}$ such that $B^* g^* = r(B) g^*$ holds, and there exists $\delta > 0$ such that $B(P) \subset P(g^*, \delta)$, then we call that the completely continuous linear positive operator $B: P \to P$ satisfies (H)-condition.

Lemma 2.4 ([11]). *Suppose that E is a Banach space, $A_n: E \to E$ (for $n = 1, 2, \ldots$) are completely continuous operators, $A: E \to E$, and*

$$\lim_{n\to\infty} \max_{\|u\|<r} \|A_n u - Au\| = 0, \quad \textit{for all } r > 0,$$

then A is a completely continuous operator.

LEMMA 2.5 ([9], [11]). *Suppose that $T\colon C[0,1] \to C[0,1]$ is a completely continuous linear operator. If there exist $v \in C[0,1] \setminus (-P)$ and a constant $c > 0$ such that $cTv \geq v$, then the spectral radius $r(T) \neq 0$ and T has a positive eigenvector corresponding to its first characteristic value $\lambda_1 = (r(T))^{-1}$.*

Let $G(t,s)$ be the Green's function for $u^{(4)}(t) = 0$ for all $t \in [0,1]$ subject to $u(0) = u(1) = u'(0) = u'(1) = 0$, i.e.

$$G(t,s) = \frac{1}{6}\begin{cases} s^2(1-t)^2[(t-s) + 2(1-s)t], & 0 \leq s \leq t \leq 1, \\ t^2(1-s)^2[(s-t) + 2(1-t)s], & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.3}$$

It is easy to verify that, for all $t, s \in [0,1]$,

$$\frac{1}{3}t^2s^2(1-t)^2(1-s)^2 \leq G(t,s) \leq \frac{2}{3}s^2(1-s)^2 \leq s(1-s). \tag{2.4}$$

In order to state our main theorems in this paper, we make the following assumptions:

(H_1) $h\colon (0,1) \to [0,\infty)$ is continuous, $h(t) \not\equiv 0$ for $t \in (0,1)$ and

$$\int_0^1 t(1-t)h(t)\,dt < \infty; \tag{2.5}$$

(H_2) $f\colon (-\infty,\infty) \to (-\infty,\infty)$ is continuous.

In the following, we introduce some notations. Let $E = C[0,1]$ and define the norm $\|u\| = \max_{0\leq t\leq 1} |u(t)|$, $u \in E$. It is easy to see that $(E, \|\cdot\|)$ is a Banach space. Define

$$P = \{u \in E : u(t) \geq 0,\ t \in [0,1]\},$$

then P is a positive cone of E. Denote by $\Omega_r = \{u \in E : \|u\| < r\}$ $(r > 0)$ the open ball of radius r and by θ the zero function in E.

Let

$$(Bu)(t) = \int_0^1 G(t,s)h(s)u(s)\,ds, \quad t \in [0,1], \text{ for all } u \in E, \tag{2.6}$$

$$(Fu)(t) = f(u(t)), \quad t \in [0,1], \text{ for all } u \in E,$$

$$(Au)(t) = (BFu)(t) = \int_0^1 G(t,s)h(s)f(u(s))\,ds, \tag{2.7}$$

where $G(t,s)$ is given by (2.3). It follows from (2.4) and (2.5) that for $u \in E$, $t \in [0,1]$,we have

$$\begin{aligned} |(Au)(t)| &\leq \int_0^1 G(t,s)h(s)|f(u(s))|\,ds \\ &\leq \Big\{\max_{-\|u\|\leq x\leq\|u\|} |f(x)|\Big\} \int_0^1 s(1-s)h(s)\,ds < \infty. \end{aligned}$$

So the operator A is well defined, and it is well known that the solution of singular BVP (1.2) is equivalent to the fixed point of A in $C[0,1]$. By (H_2), $F\colon E \to E$ is a nonlinear bounded continuous operator.

As the linear operator B, we have the following properties.

Lemma 2.6. *Suppose that* (H_1) *is satisfied, then for the operator* B *defined by* (2.6), *one has*

(a) $B\colon E \to E$ *is a completely continuous positive linear operator.*
(b) $B\colon E \to E$ *satisfies* (H)*-condition.*

Proof. (a) It follows from (2.4) and (2.5) that for $u \in E$, $t \in [0,1]$, one can get

$$|(Bu)(t)| \leq \int_0^1 G(t,s)h(s)|u(s)|\,ds \leq \|u\| \int_0^1 s(1-s)h(s)\,ds < \infty, \quad t \in [0,1].$$

Hence $B\colon E \to E$ is well defined. It follows from $G(t,s) \geq 0$ and $h(t) \geq 0$ that $B(P) \subset P$. Obviously, B is a linear operator, namely B is a positive linear operator. In the following, we shall prove that B is a completely continuous operator.

For any natural number n ($n \geq 2$), set

$$h_n(t) = \begin{cases} \inf_{t<s\leq 1/n} h(s) & \text{for } 0 \leq t \leq 1/n, \\ h(t) & \text{for } 1/n \leq t \leq (n-1)/n, \\ \inf_{(n-1)/n \leq s < t} h(s) & \text{for } (n-1)/n \leq t \leq 1. \end{cases} \tag{2.8}$$

Then $h_n\colon [0,1] \to [0,\infty)$ is continuous and $h_n(t) \leq h(t)$, for all $t \in (0,1)$. Let

$$(B_n u)(t) = \int_0^1 G(t,s)h_n(s)u(s)\,ds. \tag{2.9}$$

It is obvious that $B_n\colon E \to E$ is completely continuous. For any $r > 0$ and $u \in \Omega_r$, by (2.4)–(2.6), (2.8) and (2.9), we have

$$\begin{aligned} \max_{\|u\|<r} \|B_n u - Bu\| &= \max_{t\in[0,1]} \left| \int_0^1 G(t,s)(h_n(s) - h(s))u(s)\,ds \right| \\ &\leq \|u\| \max_{t\in[0,1]} \int_0^1 G(t,s)(h(s) - h_n(s))\,ds \\ &= \|u\| \max_{t\in[0,1]} \int_{e(n)} G(t,s)(h(s) - h_n(s))\,ds \\ &\leq \|u\| \int_{e(n)} s(1-s)(h(s) - h_n(s))\,ds \\ &\leq \|u\| \int_{e(n)} s(1-s)h(s)\,ds, \end{aligned}$$

where $e(n) = [0, 1/n] \cup [(n-1)/n, 1]$. From (2.5) and the absolute continuity of integral, one can get

$$\lim_{n\to\infty} \int_{e(n)} s(1-s)h(s)\,ds = 0,$$

which, together with above inequality, implies that

$$\lim_{n\to\infty} \max_{\|u\|<r} \|B_n u - Bu\| = 0.$$

Therefore, it follows from Lemma 2.4 that $B\colon E \to E$ is a completely continuous positive linear operator.

(b) By (H_1), there exists $[\alpha, \beta] \subset (0, 1)$ such that $h(t) > 0$, for all $t \in [\alpha, \beta]$. Let $\omega(t) = t^2(1-t)^2$, then $\omega \in P$. Because $G(t, s) \geq \frac{1}{3} t^2 s^2 (1-t)^2 (1-s)^2$, so

$$(B\omega)(t) = \int_0^1 G(t, s)h(s)s^2(1-s)^2\,ds \geq \frac{1}{3} \int_\alpha^\beta s^4(1-s)^4 h(s)\,ds \cdot \omega(t),$$

for all $t \in [0, 1]$. Let $c = \left(\frac{1}{3} \int_\alpha^\beta s^4(1-s)^4 h(s)\,ds\right)^{-1}$, then $c(B\omega)(t) \geq \omega(t)$ for all $t \in [0, 1]$. From Lemma 2.5, we know that the spectral radius $r(B) \neq 0$. Thus, corresponding to $\lambda_1 = (r(B))^{-1}$, the first characteristic value of B, B has a positive eigenvector φ, i.e.

$$B\varphi = r(B)\varphi. \tag{2.10}$$

Since φ is a positive eigenvector of B, we know from maximus principle that $\varphi(t) > 0$, for all $t \in (0, 1)$ (see [17]). It follows from $G(0, s) = G(1, s) = 0$, $s \in [0, 1]$ that $\varphi(0) = \varphi(1) = 0$, which implies that $\varphi'(0) > 0$ and $\varphi'(1) < 0$. Define a function Φ on $[0, 1]$ by

$$\Phi(s) = \begin{cases} \varphi'(0) & \text{for } s = 0, \\ \dfrac{\varphi(s)}{s(1-s)} & \text{for } s \in (0, 1), \\ -\varphi'(1) & \text{for } s = 1. \end{cases}$$

Then it is easy to see that $\Phi(s)$ is continuous on $[0, 1]$ and $\Phi(s) > 0$ for all $s \in [0, 1]$. So there exist $\delta_1, \delta_2 > 0$ such that $\delta_1 \leq \Phi(s) \leq \delta_2$ for all $s \in [0, 1]$. Thus

$$\delta_1 G(t, s) \leq \delta_1 s(1-s) \leq \varphi(s) \leq \delta_2 s(1-s), \quad \text{for all } t, s \in [0, 1]. \tag{2.11}$$

Define g^* by

$$g^*(u) = \int_0^1 h(t)\varphi(t)u(t)\,dt, \quad u \in E. \tag{2.12}$$

For any $u \in E$, by (2.5) and (2.11), we have

$$\int_0^1 h(t)u(t)\varphi(t)\,dt \leq \delta_2 \|u\| \int_0^1 t(1-t)h(t)\,dt < \infty$$

and then g^* is well defined. We now prove that

$$B^* g^* = r(B) g^*. \tag{2.13}$$

In fact, For any $u \in E$, noticing that $G(t,s) = G(s,t)$ for all $t, s \in [0,1]$, one gets

$$\begin{aligned}
r(B)g^*(u) &= \int_0^1 h(t)(r(B)\varphi(t))u(t)\,dt \\
&= \lim_{n\to\infty} \int_{1/n}^{(n-1)/n} h(t)(B\varphi)(t)u(t)\,dt \\
&= \lim_{n\to\infty} \int_{1/n}^{(n-1)/n} h(t)u(t) \int_0^1 G(t,s)h(s)\varphi(s)\,ds\,dt \\
&= \lim_{n\to\infty} \iint_{[1/n,(n-1)/n]\times[0,1]} G(t,s)h(t)u(t)h(s)\varphi(s)\,dt\,ds \\
&= \lim_{n\to\infty} \int_0^1 h(s)\varphi(s)\,ds \int_{1/n}^{(n-1)/n} G(s,t)h(t)u(t)\,dt \\
&= \int_0^1 h(s)\varphi(s)\,ds \int_0^1 G(s,t)h(t)u(t)\,dt \\
&= \int_0^1 h(s)\varphi(s)(Bu)(s)\,ds = g^*(Bu) = (B^*g^*)(u).
\end{aligned}$$

So (2.13) holds.

Take $\delta = r(B)\delta_1$ in (2.2). For any $u \in P$, by (2.10)–(2.13), we have

$$\begin{aligned}
g^*(Bu) = (B^*g^*)(u) = r(B)g^*(u) &= r(B)\int_0^1 h(s)\varphi(s)u(s)\,ds \\
&\geq r(B)\delta_1 \int_0^1 G(t,s)h(s)u(s)\,ds = r(B)\delta_1(Bu)(t) = \delta(Bu)(t),
\end{aligned}$$

for all $t \in [0,1]$. Hence $g^*(Bu) \geq \delta\|Bu\|$, i.e. $B(P) \subset P(g^*, \delta)$. In conclusion, the linear operator $B\colon E \to E$ satisfies (H)-condition. □

Similar to the proof of part (a) in Lemma 2.6, we also can obtain the following lemma.

Lemma 2.7. *Suppose that* (H_1) *and* (H_2) *are satisfied, Then* $A\colon E \to E$ *is a completely continuous operator.*

Lemma 2.8 ([20]). *Let E be a Banach space and P be a total cone in E. Suppose $T\colon P \to P$ is a bounded linear operator (therefore, T can be uniquely extended to a bounded linear operator on $\overline{P-P} = E$, and the extended operator is denoted by T again) with the spectral radius $r(T) < 1$. If $w, w_0 \in E$ such that $w \leq Tw + w_0$, then $w \leq (I-T)^{-1}w_0$, where $(I-T)^{-1}$ is the inverse operator of the operator $I - T$.*

LEMMA 2.9 ([7]). *Let E be a Banach space and Ω be a bounded open set in E. Suppose that $A\colon \overline{\Omega} \to E$ is a completely continuous operator. If there exists $u_0 \neq \theta$ such that*

$$u - Au \neq \mu u_0, \quad \text{for all } u \in \partial\Omega,\ \mu \geq 0,$$

then the topological degree $\deg(I - A, \Omega, \theta) = 0$.

LEMMA 2.10 ([7]). *Let E be a Banach space and Ω be a bounded open set in E with $\theta \in \Omega$. Suppose that $A\colon \overline{\Omega} \to E$ is a completely continuous operator. If*

$$Au \neq \mu u, \quad \text{for all } u \in \partial\Omega,\ \mu \geq 1,$$

then the topological degree $\deg(I - A, \Omega, \theta) = 1$.

LEMMA 2.11 ([8]). *Let E be a Banach space, and P be a cone in E, and Ω be a bounded open set in E. Suppose that $A\colon P \bigcap \overline{\Omega} \to P$ is a completely continuous operator. If there exists $u_0 \in P \setminus \{\theta\}$ such that*

$$u - Au \neq \mu u_0, \quad u \in P \bigcap \partial\Omega,\ \mu \geq 0,$$

then the fixed point index $i(A, P \cap \Omega, P) = 0$.

LEMMA 2.12 ([8]). *Let E be a Banach space, and P be a cone in E, and Ω be a bounded open set in E. Suppose that $A\colon P \cap \overline{\Omega} \to P$ is a completely continuous operator. If*

$$Au \neq \mu u, \quad u \in P \cap \partial\Omega,\ \mu \geq 1,$$

then the fixed point index $i(A, P \cap \Omega, P) = 1$.

REMARK 2.13. If P is replaced by $-P$ in Lemmas 2.11 and 2.12, the conclusions are still true.

3. Main results

THEOREM 3.1. *Suppose that the conditions* (H_1) *and* (H_2) *are satisfied. If*

$$\liminf_{u \to 0} \frac{f(u)}{|u|} > \lambda_1, \tag{3.1}$$

$$\limsup_{u \to \infty} \left| \frac{f(u)}{u} \right| < \lambda_1, \tag{3.2}$$

where λ_1 is the first characteristic value of B defined by (2.6)*, then the singular* BVP (1.2) *has at least one nontrivial solution.*

PROOF. It follows from (3.1) that there exist $\varepsilon_0 > 0$ and $r_1 > 0$ such that

$$f(u) \geq (\lambda_1 + \varepsilon_0)|u|, \quad |u| \leq r_1. \tag{3.3}$$

Now we show that

$$u - Au \neq \mu\varphi, \quad \text{for all } u \in \partial\Omega_{r_1},\ \mu \geq 0, \tag{3.4}$$

where φ is given by (2.10).

If otherwise, there exist $u_1 \in \partial\Omega_{r_1}$ and $\mu_1 \geq 0$ such that

$$u_1 - Au_1 = \mu_1 \varphi, \tag{3.5}$$

then by the definition of A, B, F and (3.3), (3.5), we have

$$u_1 = B\bigg(Fu_1 + \frac{\mu_1}{r(B)}\,\varphi\bigg) \in B(P) \subset P(g^*, \delta).$$

Therefore, it follows from (2.2) and (2.12) that

$$\|u_1\| \leq \frac{1}{\delta}\, g^*(u_1) = \frac{1}{\delta} \int_0^1 h(t)\varphi(t)u_1(t)\,dt. \tag{3.6}$$

Combining (3.3) and (3.5) leads to

$$\begin{aligned} u_1(t) = (Au_1)(t) + \mu_1\varphi(t) &\geq \int_0^1 G(t,s)h(s)f(u_1(s))\,ds \\ &\geq (\lambda_1 + \varepsilon_0) \int_0^1 G(t,s)h(s)u_1(s)\,ds. \end{aligned}$$

Multiplying the both sides of above inequality by $h(t)\varphi(t)$ and integrating on $[0,1]$, one gets

$$\begin{aligned} \int_0^1 h(t)\varphi(t)u_1(t)\,dt &\geq (\lambda_1 + \varepsilon_0) \int_0^1 h(t)\varphi(t) \int_0^1 G(t,s)h(s)u_1(s)\,ds\,dt \\ &= (\lambda_1 + \varepsilon_0) \lim_{n\to\infty} \int_{1/n}^{(n-1)/n} h(t)\varphi(t) \int_0^1 G(t,s)h(s)u_1(s)\,ds\,dt \\ &= (\lambda_1 + \varepsilon_0) \lim_{n\to\infty} \iint_{[1/n,(n-1)/n]\times[0,1]} G(t,s)h(t)\varphi(t)h(s)u_1(s)\,dt\,ds \\ &= (\lambda_1 + \varepsilon_0) \lim_{n\to\infty} \int_0^1 h(s)u_1(s)\,ds \int_{1/n}^{(n-1)/n} G(s,t)h(t)\varphi(t)\,dt \\ &= (\lambda_1 + \varepsilon_0) \int_0^1 h(s)u_1(s)\,ds \int_0^1 G(s,t)h(t)\varphi(t)\,dt \\ &= (\lambda_1 + \varepsilon_0) \int_0^1 h(s)u_1(s)(B\varphi)(s)\,ds \\ &= (\lambda_1 + \varepsilon_0)r(B) \int_0^1 h(s)\varphi(s)u_1(s)\,ds. \end{aligned} \tag{3.7}$$

From (3.6), we know that $\int_0^1 h(s)\varphi(s)u_1(s)\,ds > 0$, which together with (3.7) imply that $(\lambda_1 + \varepsilon_0)r(B) \leq 1$, which is a contradiction. Hence (3.4) holds and by Lemma 2.9 we have

$$\deg(I - A, \Omega_{r_1}, \theta) = 0. \tag{3.8}$$

On the other hand, it follows from (3.2) that there exist $\varepsilon_1 \in (0, \lambda_1)$ and $R_0 > 0$ such that

$$|f(u)| \leq (\lambda_1 - \varepsilon_1)|u|, \quad |u| \geq R_0.$$

Setting $C_0 = \max_{|u| \leq R_0} ||f(u)| - (\lambda_1 - \varepsilon_1)|u|| + 1$, it is clear that

$$|f(u)| \leq (\lambda_1 - \varepsilon_1)|u| + C_0, \quad \text{for all } u \in (-\infty, \infty). \tag{3.9}$$

Let $D = \{u \in E : \mu u = Au, \ \mu \geq 1\}$. In the following, we shall prove that D is bounded. In fact, for $u \in D$, then there exists $\mu \geq 1$ such that

$$\mu u(t) = Au(t) = \int_0^1 G(t,s)h(s)f(u(s))\,ds. \tag{3.10}$$

It follows from (3.9), (3.10) and the the additivity of Leray–Schauder degree that

$$\begin{aligned} |u(t)| \leq \mu |u(t)| &\leq \int_0^1 G(t,s)h(s)|f(u(s))|\,ds \\ &\leq (\lambda_1 - \varepsilon_1)\int_0^1 G(t,s)h(s)|u(s)|\,ds + C_0 \int_0^1 G(t,s)h(s)\,ds \\ &= (\lambda_1 - \varepsilon_1)(B|u|)(t) + \psi(t), \end{aligned}$$

where $\psi(t) = C_0 \int_0^1 G(t,s)h(s)\,ds$, then $\psi \in P$.

Since $r((\lambda_1 - \varepsilon_1)B) = (\lambda_1 - \varepsilon_1)r(B) < 1$, therefore from Lemma 2.8, we have $|u(t)| \leq (I - (\lambda_1 - \varepsilon_1)B)^{-1}\psi(t)$. So D is bounded and there exists sufficiently large number $R_1 > r_1$ such that

$$\mu u \neq Au, \quad \text{for all } u \in \partial\Omega_{R_1}, \ \mu \geq 1.$$

We get from Lemma 2.10 that

$$\deg(I - A, \Omega_{R_1}, \theta) = 1. \tag{3.11}$$

From (3.8) and (3.11), it follows that

$$\deg(I - A, \Omega_{R_1} \setminus \overline{\Omega}_{r_1}, \theta) = \deg(I - A, \Omega_{R_1}, \theta) - \deg(I - A, \Omega_{r_1}, \theta) = 1.$$

Therefore, A has at least one fixed point on $\Omega_{R_1} \setminus \overline{\Omega}_{r_1}$, i.e. the singular BVP (1.2) has at least one nontrivial solution. □

THEOREM 3.2. *Suppose that the conditions* (H$_1$) *and* (H$_2$) *are satisfied. If*

$$\liminf_{u \to +\infty} \frac{f(u)}{u} > \lambda_1, \qquad \limsup_{u \to -\infty} \frac{f(u)}{u} < \lambda_1, \tag{3.12}$$

$$\limsup_{u \to 0} \left|\frac{f(u)}{u}\right| < \lambda_1, \tag{3.13}$$

where λ_1 *is the first characteristic value of* B *defined by* (2.6)*, then the singular* BVP (1.2) *has at least one nontrivial solution.*

Proof. It follows from (3.12) that there exist $\varepsilon \in (0, \lambda_1)$, $M_1 > 0$ and $M_2 > 0$ such that

$$f(u) \geq (\lambda_1 + \varepsilon)u, \quad u > M_1, \qquad f(u) \geq (\lambda_1 - \varepsilon)u, \quad u < -M_2.$$

Setting $C_1 = \max_{0 \leq u \leq M_1} |f(u) - (\lambda_1 + \varepsilon)u| + 1$, and $C_2 = \max_{-M_2 \leq u \leq 0} |f(u) - (\lambda_1 - \varepsilon)u| + 1$, it is clear that

$$f(u) \geq (\lambda_1 + \varepsilon)u - C_1, \quad u \geq 0, \qquad f(u) \geq (\lambda_1 - \varepsilon)u - C_2, \quad u \leq 0.$$

Let $C = \max\{C_1, C_2\}$, then

$$f(u) \geq (\lambda_1 + \varepsilon)u - C, \quad u \geq 0, \qquad f(u) \geq (\lambda_1 - \varepsilon)u - C, \quad u \leq 0.$$

It is easy to see that

$$f(u) \geq (\lambda_1 + \varepsilon)u - C, \quad \text{for all } u \in (-\infty, \infty), \tag{3.14}$$

$$f(u) \geq (\lambda_1 - \varepsilon)u - C, \quad \text{for all } u \in (-\infty, \infty). \tag{3.15}$$

Let $D_1 = \{u \in E : u = Au + \mu\varphi, \ \mu \geq 0\}$, where φ is given by (2.10). In the following, we prove that D_1 is bounded. In fact, if $u \in D_1$, then there exists $\mu \geq 0$ such that

$$u = Au + \mu\varphi. \tag{3.16}$$

Let $u_0 \equiv 1$ and from (3.14), (3.16), one has

$$u = BFu + \mu\varphi \geq (\lambda_1 + \varepsilon)Bu - C(Bu_0).$$

Because g^* is a positive linear function and $g^*(Bu) = (B^*g^*)u = r(B)g^*(u)$, we have

$$g^*(u) \geq (\lambda_1 + \varepsilon)r(B)g^*(u) - Cr(B)g^*(u_0).$$

Thus

$$g^*(u) \leq \frac{Cr(B)g^*(u_0)}{(\lambda_1 + \varepsilon)r(B) - 1}, \quad \text{for all } u \in D_1. \tag{3.17}$$

On the other hand, we get from (3.15) that

$$Fu - (\lambda_1 - \varepsilon)u + C \in P, \quad \text{for all } u \in E.$$

Which together with (3.16) imply that

$$u - (\lambda_1 - \varepsilon)Bu + C(Bu_0) = B\left(Fu - (\lambda_1 - \varepsilon)u + Cu_0 + \frac{\mu}{r(B)}\varphi\right) \in B(P) \subset P(g^*, \delta).$$

Thus, from the definition of $P(g^*, \delta)$, one can get

$$\begin{aligned} \|u - (\lambda_1 - \varepsilon)Bu + C(Bu_0)\| &\le \frac{1}{\delta} g^*(u - (\lambda_1 - \varepsilon)Bu + C(Bu_0)) \\ &= \frac{1}{\delta}[g^*(u) - (\lambda_1 - \varepsilon)r(B)g^*(u) + Cr(B)g^*(u_0)] \\ &= \frac{1 - (\lambda_1 - \varepsilon)r(B)}{\delta} g^*(u) + \frac{Cr(B)}{\delta} g^*(u_0). \end{aligned} \tag{3.18}$$

By (3.17) and (3.18), there exists $\rho_1 > 0$ such that

$$(I - (\lambda_1 - \varepsilon)B)D_1 \subset \Omega_{\rho_1}.$$

By $r((\lambda_1 - \varepsilon)B) = (\lambda_1 - \varepsilon)r(B) < 1$, we have that the inverse operator $(I - (\lambda_1 - \varepsilon)B)^{-1}$ exists and is bounded. Therefore, there exists $\rho_2 > 0$ such that

$$D_1 \subset (I - (\lambda_1 - \varepsilon)B)^{-1}\Omega_{\rho_1} \subset \Omega_{\rho_2},$$

which means that D_1 is bounded and then there exists sufficiently large number R_2 such that

$$u \ne Au + \mu\varphi, \quad \text{for all } u \in \partial\Omega_{R_2},\ \mu \ge 0.$$

From Lemma 2.9, we have

$$\deg(I - A, \Omega_{R_2}, \theta) = 0. \tag{3.19}$$

It follows from (3.13) that there exist $\varepsilon_2 \in (0, \lambda_1)$ and $0 < r_2 < R_2$ such that

$$|f(u)| \le (\lambda_1 - \varepsilon_2)|u|, \quad |u| \le r_2. \tag{3.20}$$

In the following, we shall prove that

$$\mu u \ne Au, \quad \text{for all } u \in \partial\Omega_{r_2},\ \mu \ge 1. \tag{3.21}$$

If otherwise, there exist $u_2 \in \partial\Omega_{r_2}$ and $\mu_2 \ge 1$ such that $\mu_2 u_2 = Au_2$. Then, by (3.20), we have

$$\begin{aligned} |u_2(t)| &\le |\mu_2 u_2(t)| = |Au_2(t)| \\ &\le (\lambda_1 - \varepsilon_2) \int_0^1 G(t,s)h(s)|u_2(s)|\,ds = (\lambda_1 - \varepsilon_2)(B|u_2|)(t), \end{aligned}$$

thus $|u_2| \le (\lambda_1 - \varepsilon_2)(B|u_2|)$. It follows from $r((\lambda_1 - \varepsilon_2)B) = (\lambda_1 - \varepsilon_2)r(B) < 1$ that we have $|u_2(t)| \le 0$ by Lemma 2.8. So $u_2(t) \equiv 0$, for all $t \in [0,1]$, which contradicts $u_2 \in \partial\Omega_{r_2}$. Hence by Lemma 2.10 we have

$$\deg(I - A, \Omega_{r_2}, \theta) = 1. \tag{3.22}$$

From (3.19), (3.22) and and the the additivity of Leray-Schauder degree, we have

$$\deg(I - A, \Omega_{R_2} \setminus \overline{\Omega}_{r_2}, \theta) = \deg(I - A, \Omega_{R_2}, \theta) - \deg(I - A, \Omega_{r_2}, \theta) = -1.$$

Therefore, A has at least one fixed point on $\Omega_{R_2} \setminus \overline{\Omega}_{r_2}$, which means that the singular BVP (1.2) has at least one nontrivial solution. □

By the process similar to the proof of Theorem 3.2, we also can obtain the following corollary.

Theorem 3.3. *Suppose that the conditions* (H_1), (H_2) *and* (3.12) *are satisfied. If*

$$\lim_{u\to 0} \frac{f(u)}{u} = \lambda, \tag{3.23}$$

where $\lambda \neq \lambda_n$, $\{\lambda_n : n = 1, 2, \dots\}$ *is the characteristic value set of* B *defined by* (2.6)*, then the singular* BVP (1.2) *has at least one nontrivial solution.*

Proof. Similar to the proof of Theorem 3.2, from (3.12) we know that there exists sufficiently large number R_3 such that

$$\deg(I - A, \Omega_{R_3}, \theta) = 0. \tag{3.24}$$

It follows from (3.23) that

$$(A'_\theta u)(t) = \lambda \int_0^1 G(t,s)h(s)u(s)ds = \lambda(Bu)(t).$$

Obviously, 1 is not the characteristic value of A'_θ. According to Theorem 2.6 in [7], there exists $0 < r_3 < R_3$ such that

$$\deg(I - A, \Omega_{r_3}, \theta) = \deg(I - A'_\theta, \Omega_{r_3}, \theta) = \pm 1. \tag{3.25}$$

By (3.24) and (3.25),

$$\deg(I - A, \Omega_{R_3} \setminus \overline{\Omega}_{r_3}, \theta) = \deg(I - A, \Omega_{R_3}, \theta) - \deg(I - A, \Omega_{r_3}, \theta) = \mp 1.$$

As a result, A has at least one fixed point on $\Omega_{R_3} \setminus \overline{\Omega}_{r_3}$, which implies that the singular BVP (1.2) has at least one nontrivial solution. □

Remark 3.4. In Theorems 3.1–3.3, $f\colon (-\infty,\infty) \to (-\infty,\infty)$ may be a sign-changing and unbounded function from below, and it is not also necessary to exist a control function for f from below.

4. Existence of positive and negative solutions

Theorem 4.1. *Suppose that the conditions* (H_1) *and* (H_2) *are satisfied.*

(a) *If* $uf(u) \geq 0$ *for* $u \leq 0$ *and the following conditions*

$$\liminf_{u\to 0^-} \frac{f(u)}{u} > \lambda_1, \tag{4.1}$$

$$\limsup_{u\to -\infty} \frac{f(u)}{u} < \lambda_1 \tag{4.2}$$

hold, then the singular BVP (1.2) *has at least one negative solution.*

(b) *If* $uf(u) \geq 0$ *for* $u \geq 0$ *and the following conditions*

$$\liminf_{u \to 0^+} \frac{f(u)}{u} > \lambda_1, \tag{4.3}$$

$$\limsup_{u \to +\infty} \frac{f(u)}{u} < \lambda_1 \tag{4.4}$$

hold, then the singular BVP (1.2) *has at least one positive solution, where* λ_1 *is the first characteristic value of* B *defined by* (2.6).

Proof. We only prove the case (a), the proof for case (b) is similar, so we omit it. Because $uf(u) \geq 0$ for $u \leq 0$, so we have $A\colon -P \to -P$. It follows from (4.1) that there exist $\varepsilon_3 > 0$ and $r_4 > 0$ such that

$$f(u) \leq (\lambda_1 + \varepsilon_3)u, \quad -r_4 \leq u \leq 0. \tag{4.5}$$

Now we show that

$$u - Au \neq \mu(-\varphi), \quad \text{for all } u \in (-P) \cap \partial\Omega_{r_4}, \quad \mu \geq 0, \tag{4.6}$$

where φ is given by (2.10).

If otherwise, there exist $u_3 \in (-P) \cap \partial\Omega_{r_4}$ and $\mu_3 \geq 0$ such that

$$u_3 - Au_3 = -\mu_3\varphi, \tag{4.7}$$

then by the definition of A, B, F and (4.5), we have

$$-u_3 = B\bigg(-Fu_3 + \frac{\mu_3}{r(B)}\varphi\bigg) \in B(P) \subset P(g^*, \delta).$$

Therefore, it follows from (2.2) and (2.12) that

$$\|u_3\| \leq \frac{1}{\delta}\, g^*(-u_3) = \frac{1}{\delta} \int_0^1 h(t)\varphi(t)(-u_3(t))\, dt. \tag{4.8}$$

Combining (4.5) and (4.7) leads to

$$\begin{aligned} -u_3(t) = &-(Au_3)(t) + \mu_3\varphi(t) \geq \int_0^1 G(t,s)h(s)(-f(u_3(s)))\, ds \\ &\geq (\lambda_1 + \varepsilon_3) \int_0^1 G(t,s)h(s)(-u_3(s))\, ds. \end{aligned}$$

Multiplying the both sides of above inequality by $h(t)\varphi(t)$ and integrating on $[0,1]$, similar to the proof of (3.7), one gets

$$\int_0^1 h(t)\varphi(t)(-u_3(t))\, dt \geq (\lambda_1 + \varepsilon_3)r(B) \int_0^1 h(s)\varphi(s)(-u_3(s))\, ds, \tag{4.9}$$

which is a contradiction because $(\lambda_1 + \varepsilon_3)r(B) > 1$. Hence (4.6) holds and by Remark 2.13 we have

$$i(A, (-P) \cap \Omega_{r_4}, -P) = 0. \tag{4.10}$$

On the other hand, it follows from (4.2) that there exist $\varepsilon_4 \in (0, \lambda_1)$ and $R_4 > 0$ such that

$$f(u) \geq (\lambda_1 - \varepsilon_4)u, \quad u < -R_4.$$

Setting $\widetilde{C} = \min_{-R_4 \leq u \leq 0}\{f(u) - (\lambda_1 - \varepsilon_4)u\}$, and $C_3 = min\{\widetilde{C}, 0\}$. It is clear that

$$f(u) \geq (\lambda_1 - \varepsilon_4)u + C_3, \quad \text{for all } u \in (-\infty, 0]. \tag{4.11}$$

Let $D = \{u \in -P : \mu u = Au,\ \mu \geq 1\}$. In the following, we shall prove that D is bounded. In fact, if $u \in D$, then there exists $\mu \geq 1$ such that

$$\mu u(t) = Au(t) = \int_0^1 G(t,s)h(s)f(u(s))\,ds. \tag{4.12}$$

It follows from (3.9) and (4.11) that

$$\begin{aligned} 0 \geq u(t) \geq \mu u(t) = Au(t) &= \int_0^1 G(t,s)h(s)f(u(s))\,ds \\ &\geq (\lambda_1 - \varepsilon_4)\int_0^1 G(t,s)h(s)u(s)\,ds + C_3\int_0^1 G(t,s)h(s)\,ds \\ &= (\lambda_1 - \varepsilon_4)(Bu)(t) + \rho(t), \end{aligned}$$

where $\rho(t) = C_3\int_0^1 G(t,s)h(s)\,ds$, then $\rho \in -P$, i.e.

$$0 \leq -u(t) \leq (\lambda_1 - \varepsilon_4)(B(-u))(t) - \rho(t).$$

Since $r((\lambda_1 - \varepsilon_4)B) = (\lambda_1 - \varepsilon_4)r(B) < 1$, therefore from Lemma 2.8 we have $0 \leq -u(t) \leq (I - (\lambda_1 - \varepsilon_4)B)^{-1}(-\rho(t))$. So D is bounded and there exists sufficiently large number $R_4 > r_4$ such that

$$\mu u \neq Au, \quad \text{for all } u \in -P \cap \partial\Omega_{R_4},\ \mu \geq 1.$$

We get from Remark 2.13 that

$$i(A, (-P) \cap \Omega_{R_4}, -P) = 1. \tag{4.13}$$

From (4.10) and (4.13), it follows that

$$\begin{aligned} &i(A, (-P) \cap (\Omega_{R_4} \setminus \overline{\Omega}_{r_4}), -P) \\ &\qquad = i(A, (-P) \cap \Omega_{R_4}, -P) - i(A, (-P) \cap \Omega_{r_4}, -P) = 1. \end{aligned} \tag{4.14}$$

Therefore, A has at least one fixed point on $(-P) \cap (\Omega_{R_4} \setminus \overline{\Omega}_{r_4})$, i.e. the singular BVP (1.2) has at least one negative solution.

If (4.3), (4.4) in Theorem 4.1 hold, similar to the proof of (4.14), one can get that there exist $R_5 > r_5 > 0$ such that

$$i(A, P \cap (\Omega_{R_5} \setminus \overline{\Omega}_{r_5}), P) = 1,$$

i.e. A has at least one fixed point on $P \cap (\Omega_{R_5} \setminus \overline{\Omega}_{r_5})$, that is, the singular BVP (1.2) has at least one positive solution. □

Similar to the proof of Theorem 4.1, we have the following result.

Theorem 4.2. *Suppose that the conditions* (H_1) *and* (H_2) *are satisfied. If* $uf(u) \geq 0$ *for* $u \leq 0$ *and*

$$\limsup_{u\to 0^-} \frac{f(u)}{u} < \lambda_1, \qquad \liminf_{u\to -\infty} \frac{f(u)}{u} > \lambda_1,$$

then the singular BVP (1.2) *has at least one negative solution. If* $uf(u) \geq 0$ *for* $u \geq 0$ *and*

$$\limsup_{u\to 0^+} \frac{f(u)}{u} < \lambda_1, \qquad \liminf_{u\to +\infty} \frac{f(u)}{u} > \lambda_1,$$

then the singular BVP (1.2) *has at least one positive solution, where* λ_1 *is the first characteristic value of* B *defined by* (2.6).

5. Examples

Example 5.1. Let $h(t) = t^{p-1}(1-t)^{q-1}$, where $p, q \in (0,1)$. It is clear that $h(t)$ is singular at both $t = 0$ and $t = 1$, and satisfies (H_1).

Let $f(u) = (1-u^2)/(1+u^2)$. It is easy to see that $f(u)$ is sign-changing for $u \geq 0$. In addition,

$$\lim_{u\to 0} \frac{f(u)}{|u|} = +\infty \quad \text{and} \quad \lim_{|u|\to\infty} \left|\frac{f(u)}{u}\right| = \frac{|1-u^2|}{(1+u^2)|u|} = 0 < \lambda_1.$$

Thus by Theorem 3.1, we can obtain the existence of a nontrivial solution of (1.2).

Example 5.2. Let $h(t)$ be the same as in Example 5.1 and let

$$f(u) = \begin{cases} 1 + \displaystyle\sum_{i=1}^{n} (-1)^i a_i u^i - |u|^{1/2}, & u \in (-\infty, -1], \\ \displaystyle\sum_{i=1}^{n} a_i u^i, & u \in [-1, \infty), \end{cases}$$

where $a_i > 0$ ($i = 1, \dots, n$), $b_1 \in (-\infty, \infty)$, $b_j > 0$ ($j = 2, \dots, n$) and $b_1 < \lambda_1$. Obviously, $f(u)$ is sign-changing for $u \geq 0$. Moreover,

$$\lim_{u\to +\infty} \frac{f(u)}{u} = +\infty, \quad \lim_{u\to -\infty} \frac{f(u)}{u} = -\infty \quad \text{and} \quad \lim_{u\to 0} \frac{f(u)}{u} = b_1.$$

Immediately we can apply Theorem 3.2 to obtain the existence of a nontrivial solution of (1.2).

EXAMPLE 5.3. Let $h(t)$ be the same as in Example 5.1 and let $f(u) = u^2 - u$. It is easy to see that $f(u)$ is sign-changing for $u \geq 0$. In addition,

$$\lim_{u\to+\infty} \frac{f(u)}{u} = +\infty, \quad \lim_{u\to-\infty} \frac{f(u)}{u} = -\infty, \quad \lim_{u\to 0} \frac{f(u)}{u} = -1 \neq \lambda_k$$

$(k = 1, 2, \dots)$, where λ_k is an characteristic value of B. Thus by Theorem 3.3, we can obtain the existence of a nontrivial solution of (1.2).

EXAMPLE 5.4. Let $h(t)$ be the same as in Example 5.1 and let $f(u) = \sqrt[3]{u}$. Then $uf(u) \geq 0$ for $u \in (-\infty, \infty)$. In addition,

$$\lim_{u\to 0} \frac{f(u)}{u} = \lim_{u\to 0} \frac{\sqrt[3]{u}}{u} = \infty, \qquad \lim_{u\to\infty} \frac{f(u)}{u} = \lim_{u\to\infty} \frac{\sqrt[3]{u}}{u} = 0,$$

thus, all conditions of Theorem 4.1 are satisfied, so by Theorem 4.1, we know that the equation (1.2) has at least one negative solution and one positive solution.

EXAMPLE 5.5. Let $h(t)$ be the same as in Example 5.1 and let $f(u) = u^2 \arctan u$. Then $uf(u) \geq 0$ for $u \in (-\infty, \infty)$. In addition,

$$\lim_{u\to 0} \frac{f(u)}{u} = \lim_{u\to 0} u \arctan u = 0, \qquad \lim_{u\to\infty} \frac{f(u)}{u} = \lim_{u\to\infty} u \arctan u = +\infty.$$

Then by Theorem 4.2, we know that the equation (1.2) has at least two nontrivial solutions: one negative and one positive.

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