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THE CONLEY INDEX OVER A PHASE SPACE FOR FLOWS

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Abstract. We construct the Conley index over a phase space for flows. Our definition is an alternative for the Conley index over a base defined in [5]. We also compare it to other Conley-type indices and prove its continuation property.

1. Introduction

The theory of dynamical systems provides us with several indices, such as the fixed point index or the Morse index, which allow to examine some properties of a given system. Recently, one of the most exploited ones has been the Conley index, which was defined in [2] for flows and, for example, in [3] or [11] for semidynamical systems generated by a map.

The Conley index has been used to prove the existence and behavior of an isolated invariant set in its so-called isolating neighbourhood. Due to the local character of the index a lot of information about an isolated invariant set is lost, in paticular, its position in a phase space.

The idea how to overcome this difficulty was presented in [5] where the Conley index over a base was defined for flows (see also Chapter 4). A natural problem appeared how to extend this definition for the discrete case. The solution was given in [7] and [8] where the Conley index over a phase space for discrete semidynamical systems was defined. Its construction was based on a relation of

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the so-called M-equivalence. If one assumes that a map generating the system is homotopic with identity, as it is in the case of a time-one map induced by a flow, we may introduce a new relation of a fiberwise moving homotopy (see Chapter 2). This allows to define an alternative index for flows, which also distinguishes two isolated invariant sets differently situated in a phase space and is strongly related to its discrete version.

Theorem 4.16 which shows the relation between the indices may be essential for the computer-assisted proofs concerning the isolated invariant sets for flows. In order to use a computer for precise calculations one has to discretize the problem.

2. Spaces over a base

2.1. Category of spaces over a base. For a given topological space X we define the category of spaces over a base X, which will be denoted by $\mathcal{SB}(X)$.

DEFINITION 2.1.

$$\begin{aligned}\mathrm{Ob}(\mathcal{SB}(X)) =\{&(U,r,s) : U \text{ is a topological space},\\ &r\colon U\to X,\ s\colon X\to U \text{ continuous, such that } r\circ s=\mathrm{id}_X\},\end{aligned}$$

$$\begin{aligned}\mathrm{Mor}_{\mathcal{SB}(X)}((U,r,s),(U',r',s')) = \{&(F,f) : F\colon U\to U',\ f\colon X\to X\\ &\text{continuous, such that } F\circ s=s'\circ f \text{ and } r'\circ F=f\circ r\}\end{aligned}$$

Identity:

$$\mathrm{id}_{\mathcal{SB}(X)}((U,r,s),(U,r,s)) = (\mathrm{id}\,U,\mathrm{id}_X).$$

Composition of two morphisms:

$$\begin{aligned}(F,f)&\in \mathrm{Mor}_{\mathcal{SB}(X)}((U,r,s),(U',r',s')),\\ (G,g)&\in \mathrm{Mor}_{\mathcal{SB}(X)}((U',r',s'),(U'',r'',s''))\end{aligned}$$

is defined by $(G,g)\circ(F,f)=(G\circ F, g\circ f)$.

REMARK 2.2. We may naturally identify r with a projection ($r\circ s\circ r=r$) and s with an inclusion ($s(X)\subseteq U$). Their presence in the definition of objects may seem to be redundant, but it is convenient in the later notation.

EXAMPLE 2.3. Given a pair $P=(P_1,P_2)$ of compact subsets of a metric space (X,d) satisfying $P_2\subseteq P_1$ we define $U(P)$ as the adjunction $P_1\cup_{\mathrm{id}|_{P_2}} X$, i.e.

$$U(P) := X\times 0\cup P_1\times 1/\sim,$$

where $\sim$ denotes the minimal equivalence relation such that $(x,0)\sim(x,1)$ for each $x\in P_2$. Let $[x,q]_P$ denote the equivalence class of (x,q) in $U(P)$.

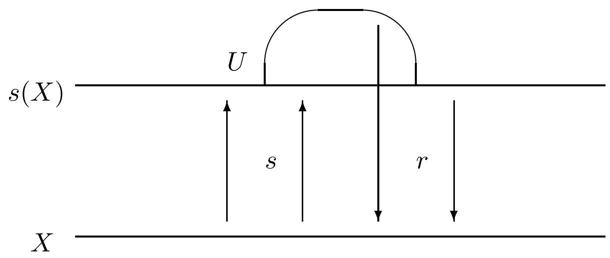

Figure 1. The space over a base

Now we have a natural inclusion $s_P\colon X \ni x \mapsto [x,0]_P \in U(P)$ and a projection $r_P\colon U(P) \ni [x,q]_P \mapsto x \in X$.

$U(P)$ is a metrizable space. In particular, if $P_2 \neq \emptyset$, then $U(P)$ is a metric space with a metric $d_P\colon U(P) \times U(P) \to [0,+\infty)$ given by a formula:

$$d_P([x_1,q_1]_P,[x_2,q_2]_P) = \begin{cases} d(x_1,x_2) & \text{for } q_1 = q_2, \\ \inf_{y\in P_2}\{d(x_1,y)+d(y,x_2)\} & \text{for } q_1 \neq q_2. \end{cases}$$

Obviously, maps r_P, s_P are continuous and $r_P \circ s_P = \mathrm{id}_X$, so $(U(P), r_P, s_P) \in \mathrm{Ob}(\mathcal{SB}(X))$.

Remark 2.4. $\mathcal{SB}$ is a well-defined category.

Note that the second element of a morphism in the category of spaces over a base is always determined by the first one:

Remark 2.5. $(F,f) \in Mor_{\mathcal{SB}(X)}((U,r,s),(U',r',s')) \Rightarrow f = r' \circ F \circ s$.

Therefore the second element in this morphism seems to be redundant. However, it is convenient to use it in order to make the notation clearer.

Denote a compact interval $[0,1]$ by $\mathbb{I}$.

For two morphisms in $\mathcal{SB}(X)$ we define the relation $\simeq_*$ of homotopy:

Definition 2.6. $(F,f),(F',f') \in \mathrm{Mor}_{\mathcal{SB}(X)}((U,r,s),(U',r',s'))$.

$$\begin{aligned}(F,f) \simeq_* (F',f') \Leftrightarrow\ & \exists\, H\colon U \times \mathbb{I} \to U',\ h\colon X \times \mathbb{I} \to X \text{ continuous:} \\ & H \circ (s \times \mathrm{id}_{\mathbb{I}}) = s' \circ h,\ r' \circ H = h \circ (r \times \mathrm{id}_{\mathbb{I}}), \\ & H(\,\cdot\,,0) = F,\ H(\,\cdot\,,1) = F',\ h(\,\cdot\,,0) = f,\ h(\,\cdot\,,1) = f'.\end{aligned}$$

A pair (H,h) will be called a homotopy joining (F,f) with (F',f').

Uwaga 2.7. $\simeq_*$ is an equivalence relation.

Remark 2.8. If (H,h) is a homotopy joining (F,f) with (F',f') – two morphisms from $\mathrm{Mor}_{\mathcal{SB}(X)}((U,r,s),(U',r',s'))$, then

$$(H(\,\cdot\,,t),h(\,\cdot\,,t)) \in \mathrm{Mor}_{\mathcal{SB}(X)}((U,r,s),(U',r',s')), \qquad \text{for every } t \in \mathbb{I}.$$

2.2. Homotopy types over a base. We recall the notion of the fiberwise deforming homotopy type defined in [5], which was a key definition for the construction of the Conley index over a base for flows.

Definition 2.9. Two objects (U, r, s) i (U', r', s') in the category $\mathcal{SB}(X)$ are said to have the same fiberwise deforming homotopy type over X, if there exist continuous maps $\Phi: U \to U'$ and $\Psi: U' \to U$ satisfying

$$\Phi \circ s = s', \qquad \Psi \circ s' = s, \tag{2.1}$$

$$r' \circ \Phi \simeq r \operatorname{rel} s(X), \qquad r \circ \Psi \simeq r' \operatorname{rel} s'(X), \tag{2.2}$$

$$\Psi \circ \Phi \simeq \mathrm{id}_U \operatorname{rel} s(X), \qquad \Phi \circ \Psi \simeq \mathrm{id}_{U'} \operatorname{rel} s'(X). \tag{2.3}$$

Now we define a different homotopy type for objects in the same category:

Definition 2.10. Two objects (U, r, s) and (U', r', s') in the category $\mathcal{SB}(X)$ are said to have the same fiberwise moving homotopy type over X, if there exist $(\Phi, \varphi) \in \mathrm{Mor}_{\mathcal{SB}(X)}((U, r, s), (U', r', s'))$ and $(\Psi, \psi) \in \mathrm{Mor}_{\mathcal{SB}(X)}((U', r', s'), (U, r, s))$ satisfying:

$$\varphi \simeq \mathrm{id}_X \simeq \psi, \tag{2.4}$$

$$(\Psi, \psi) \circ (\Phi, \varphi) \simeq_* (\mathrm{id}_U, \mathrm{id}_X), \qquad (\Phi, \varphi) \circ (\Psi, \psi) \simeq_* (\mathrm{id}_{U'}, \mathrm{id}_X). \tag{2.5}$$

The fiberwise moving homotopy type over X of (U, r, s) will be denoted by $[U, r, s]_X$.

Remark 2.11. Having the same fiberwise moving homotopy type over X is the equivalence relation.

The following examples illustrate the lack of relation between the fiberwise deforming homotopy type and the fiberwise moving homotopy type. The spaces $U(P)$ and $U(Q)$ are constructed according to the Example 2.3.

Example 2.12. Let $X = \mathbb{R}$, $P_1 = \{0\} \cup \{1/2^n : n \in \mathbb{N}\}$, $P_2 = \{0\}$, $Q_1 = \{1\} \cup \{1 + 1/2^n : n \in \mathbb{N}\}$, $Q_2 = \{1\}$.

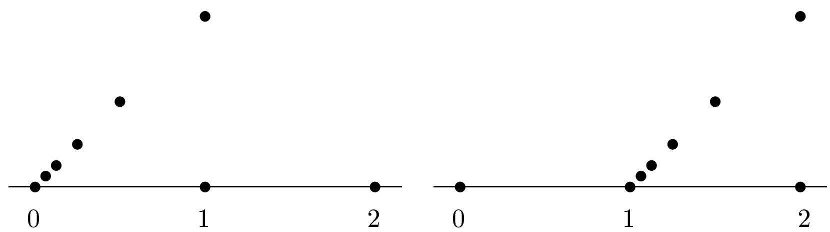

Figure 2. $U(P)$ and $U(Q)$ have the same fiberwise moving homotopy types and different fiberwise deforming homotopy types

One can easily notice that fiberwise moving homotopy types over X for $\mathcal{U} := (U(P), r_P, s_P)$ and $\mathcal{U}' := (U(Q), r_Q, s_Q)$ are equal. The maps $\Phi, \Psi, \varphi, \psi$ from Definition 2.10 are given as follows:

$$\Phi\colon U(P) \ni [u,q]_P \mapsto [u+1,q]_Q \in U(Q), \qquad \varphi\colon X \ni u \mapsto u+1 \in X,$$
$$\Psi\colon U(Q) \ni [u,q]_Q \mapsto [u-1,q]_P \in U(P), \qquad \psi\colon X \ni u \mapsto u-1 \in X.$$

On the other hand, $\mathcal{U}$ and $\mathcal{U}'$ have different fiberwise deforming homotopy types over X. Otherwise, there would exist maps $\Phi\colon U(P) \to U(Q)$ and $\Psi\colon U(Q) \to U(P)$ from Definition 2.9. Then $\Phi([0,1]_P) = \Phi([0,0]_P) = [0,0]_Q$ and there would exist a neighbourhood V of the point $[0,1]_P$ in $U(P)$ such that $\Phi(V) \subseteq s_Q(X)$, so the image of "almost all" points from $U(P) \setminus s_P(X)$ by the map $\Psi \circ \Phi$ would be contained in $s_P(X)$. That would contradict $\Psi \circ \Phi \simeq \mathrm{id}_{U(P)}$.

EXAMPLE 2.13. Consider $A := \{1/2^n : n \in \mathbb{N}\}$. Now take $X = [-1,0] \cup A$, $P_1 = \{0\}$, $Q_1 = \{-1\}$, $P_2 = Q_2 = \emptyset$.

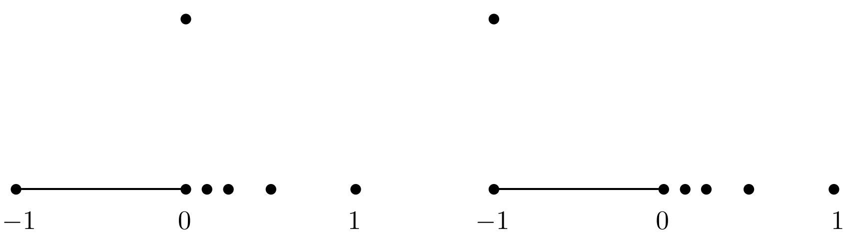

FIGURE 3. $U(P)$ and $U(Q)$ have the same fiberwise deforming homotopy types and different fiberwise moving homotopy types

This time $\mathcal{U} := (U(P), r_P, s_P)$ and $\mathcal{U}' := (U(Q), r_Q, s_Q)$ have the same fiberwise deforming homotopy types over X. Indeed, the maps $\Phi\colon U(P) \to U(Q)$ and $\Psi\colon U(Q) \to U(P)$ from Definition 2.9 are given by formulas:

$$\Phi\colon U(P) \ni [u,q]_P \mapsto \begin{cases} [-1,1]_Q & \text{if } [u,q]_P = [0,1]_P, \\ [u,0]_Q & \text{otherwise,} \end{cases} \in U(Q),$$
$$\Psi\colon U(Q) \ni [u,q]_Q \mapsto \begin{cases} [0,1]_P & \text{if } [u,q]_Q = [-1,1]_Q, \\ [u,0]_P & \text{otherwise,} \end{cases} \in U(P).$$

On the other hand, $\mathcal{U}$ and $\mathcal{U}'$ have different fiberwise moving homotopy types over X. Assume the opposite. Then there exist the maps $\Phi\colon U(P) \to U(Q)$, $\Psi\colon U(Q) \to U(P)$ and $\varphi, \psi\colon X \to X$ from Definition 2.9. We have $\Phi([0,1]_P) = [-1,1]_Q$. Otherwise, $(\Psi \circ \Phi)([0,1]_P) \in s_P(X)$ and $\Psi \circ \Phi \not\simeq \mathrm{id}_{U(P)}$. This implies

$$\varphi(0) = (\varphi \circ r_P)([0,1]_P) = (r_Q \circ \Phi)([0,1]_P) = -1.$$

We have $\varphi \simeq \mathrm{id}_X$, so $\varphi|_A = \mathrm{id}_A$ and from the continuity of φ we get a contradiction $\varphi(0) = 0$.

3. Isolated invariant sets

We start with a simple definition of a section, which will be necessary to formulate and prove the property of continuation for the index.

Definition 3.1. Let $\Lambda \subseteq \mathbb{R}$ be a compact interval. For $K \subseteq X \times \Lambda$ i $\lambda \in \Lambda$ we define its section $K_\lambda := \{x \in X : (x, \lambda) \in K\}$.

3.1. The continuous case. Definitions and theorems recalled in this subsection come, for example, from [2], [1], [6] and [5].

Let X be a locally compact metric space with a flow $\rho: X \times \mathbb{R} \to X$. To simplify notation, for $x \in X$ and $a, b, t \in \mathbb{R}$ we will write $x \cdot t$ instead of $\rho(x, t)$ and $x \cdot [a, b]$ instead of $\rho(x, [a, b])$.

Definition 3.2. For a given subset $N \subseteq X$ sets

$$\begin{aligned} \operatorname{Inv} N &:= \{x \in N : x \cdot \mathbb{R} \subseteq N\}; \\ \operatorname{Inv}^+ N &:= \{x \in N : x \cdot \mathbb{R}^+ \subseteq N\}; \\ \operatorname{Inv}^- N &:= \{x \in N : x \cdot \mathbb{R}^- \subseteq N\}; \end{aligned}$$

are called respectively an invariant, positively invariant, and negatively invariant part of N.

Definition 3.3. A given set $N \subseteq X$ is called invariant when $N = \operatorname{Inv} N$.

Remark 3.4. For every $N \subseteq X$ the set $\operatorname{Inv} N$ is the biggest invariant subset of N.

Definition 3.5. A compact set $N \subseteq X$ is called an isolating neighbourhood for S if $S = \operatorname{Inv} N$ and $S \subseteq \operatorname{int}(N)$. Then we call the set S an isolated invariant set.

Fix S – an isolated invariant set for S and N – its isolating neighbourhood.

Definition 3.6. A pair $P = (P_1, P_2)$ of compact subsets of N is called an index pair for S in N, if the following conditions are satisfied:

(a) $S = \operatorname{Inv} N \subseteq \operatorname{int}(P_1 \setminus P_2)$,
(b) $x \in P_i$, $t > 0$, $x \cdot [0, t] \subseteq N \Rightarrow x \cdot [0, t] \subseteq P_i$ $(i = 1, 2)$,
(c) $x \in P_1$, $t > 0$, $x \cdot t \notin N \Rightarrow \exists\, t' \in [0, t] : x \cdot t' \in P_2$, $x \cdot [0, t'] \subseteq N$.

The set of all index pairs for S in N will be denoted by $\mathrm{IP}(S, N)$.

For an index pair $P \in \mathrm{IP}(S,N)$ we may define maps $\sigma_P, \tau_P : P_1 \to [0,\infty]$

$$\sigma_P(x) = \begin{cases} \sup\{t \geq 0 : x \cdot [0,t] \subseteq \operatorname{cl} P_1 \setminus P_2\} & \text{if } x \in \operatorname{cl} P_1 \setminus P_2, \\ 0 & \text{otherwise,} \end{cases}$$

$$\tau_P(x) = \begin{cases} \sup\{t \geq 0 : x \cdot [0,t] \subseteq P_1 \setminus P_2\} & \text{if } x \in P_1 \setminus P_2, \\ 0 & \text{otherwise.} \end{cases}$$

DEFINITION 3.7. A pair $P \in \mathrm{IP}(S,N)$ is called regular, if $\sigma_P = \tau_P$.

The set of all regular index pairs for S in N will be denoted by $\mathrm{RIP}(S,N)$.
The following theorem and remark come from [5]:

THEOREM 3.8. $\mathrm{RIP}(S,N) \neq \emptyset$.

REMARK 3.9. If ρ is a flow and $P \in \mathrm{RIP}(S,N)$ then τ_P is continuous.

Now recall the notion of an isolating block defined, for example, in [6].

DEFINITION 3.10. Let $\Sigma \subset X$. If for some $\delta > 0$ the map

$$\rho_\delta : \Sigma \times (-\delta,\delta) \ni (x,t) \mapsto x \cdot t \in X$$

is a homeomorphism onto image, then Σ is called a local δ-section.

DEFINITION 3.11. Let B be a closure of an open set in X, Σ^+ and Σ^- be two disjoint local δ-sections satisfying:

(a) $[(\operatorname{cl} \Sigma^{\pm}) \setminus \Sigma^{\pm}] \cap B = \emptyset$.
(b) $\Sigma^+ \cdot (-\delta,\delta) \cap B = (\Sigma^+ \cap B) \cdot [0,\delta)$.
(c) $\Sigma^- \cdot (-\delta,\delta) \cap B = (\Sigma^- \cap B) \cdot (-\delta,0]$.
(d) If $x \in (\operatorname{bd} B) \setminus (\Sigma^- \cup \Sigma^+)$, then there exist $\varepsilon_1, \varepsilon_2 > 0$ such that $x \cdot [-\varepsilon_1, \varepsilon_2] \subset \operatorname{bd} B$ and $x \cdot -\varepsilon_1 \in \Sigma^+$, $x \cdot \varepsilon_2 \in \Sigma^-$.

A set B satisfying the above conditions is called an isolating block for a flow ρ. Any number δ satysfying the above conditions is called a collar size of block B.

REMARK 3.12. An isolating block B is an isolating neighbourhood for $S := \operatorname{Inv} B$.

The set of all isolating blocks for S will be denoted by $\mathrm{IB}(S)$.
The following theorem is proved in [6]:

THEOREM 3.13. *For every isolated invariant set S and its neighbourhood V there exists an isolating block $B \in \mathrm{IB}(S)$ contained in V.*

Definition 3.14. For an isolating block B and $y \in Y \subseteq B$ we define the sets:

$$\begin{aligned}
B^{\pm} &:= B \cap \Sigma^{\pm} \quad \text{(the sets of entrance and exit points)},\\
A(B) &:= \operatorname{Inv}^{+} B \cup \operatorname{Inv}^{-} B,\\
a^{\pm}(B) &:= \operatorname{bd} B \cap \operatorname{Inv}^{\pm} B,\\
O(y,B) &:= \text{the component of } (y \cdot \mathbb{R}) \cap B \text{ containing } y,\\
O(Y,B) &:= \bigcup_{y \in Y} O(y,B).
\end{aligned}$$

From [1] and Remark 3.9 we get

Remark 3.15. If $B \in \mathrm{IB}(S)$, then $(B, B^{-}) \in \mathrm{RIP}(S,B)$. In particular, functions:

$$\begin{aligned}
\sigma_B^{-}&: B \ni y \mapsto \sup\{t \geq 0 : y \cdot [0,t] \subset B\} \in [0,+\infty],\\
\sigma_B^{+}&: B \ni y \mapsto \sup\{t \geq 0 : y \cdot [-t,0] \subset B\} \in [0,+\infty]
\end{aligned}$$

are continuous.

As a consequence of the definition of an isolating block we have:

Remark 3.16. Let $U^{\pm}$ be open neighbourhoods of $a^{\pm}(B)$ contained in $B^{\pm}$. Then there exist open neighbourhoods $V^{\pm}$ of sets $B^{\pm} \setminus U^{\pm}$ in $\Sigma^{\pm}$ and continuous maps $\varepsilon^{\mp}: \operatorname{cl}_{\Sigma^{\pm}} V^{\pm} \to \mathbb{R}_{+}$, equal to $\sigma_B^{\pm}$ on the intersection of their domains such that

$$\begin{aligned}
x \in \operatorname{cl}_{\Sigma^{+}} V^{+} &\Rightarrow x \cdot \varepsilon^{-}(x) \in \Sigma^{-},\\
x \in \operatorname{cl}_{\Sigma^{-}} V^{-} &\Rightarrow x \cdot -\varepsilon^{+}(x) \in \Sigma^{+},\\
x \in B^{+} \cap \operatorname{cl}_{\Sigma^{+}} V^{+} &\Rightarrow x \cdot [0, \varepsilon^{-}(x)] \subset B,\\
x \in B^{-} \cap \operatorname{cl}_{\Sigma^{-}} V^{-} &\Rightarrow x \cdot [-\varepsilon^{+}(x), 0] \subset B.
\end{aligned}$$

Finally, recall the definition of continuation of isolated invariant sets.

Definition 3.17. Let $\rho_1, \rho_2: X \times \mathbb{R} \to X$ be two flows S_1 and S_2 – isolated invariant sets for ρ_1 and ρ_2, respectively. We say that there is a continuation between (ρ_1, S_1) and (ρ_2, S_2), if there exists $\Lambda \subseteq \mathbb{R}$ – a compact interval, $\rho: X \times \Lambda \times \mathbb{R} \to X \times \Lambda$ – a flow such that $\rho(x, \lambda, t) = (\rho_\lambda(x,t), \lambda) \in X \times \Lambda$, for each $x \in X$, $t \in \mathbb{R}$ and $\lambda \in \Lambda$, S – an isolated invariant set for ρ and $a, b \in \Lambda$ such that $\rho_1 = \rho_a$, $\rho_2 = \rho_b$, $S_1 = S_a$ and $S_2 = S_b$.

3.2. The discrete case. In this subsection we recall the basic concepts from the theory of isolated invariant sets for discrete semidynamical systems; cf. [3], [4] and [11].

Let X be a locally compact metric space, $f: X \to X$ – a continuous map.

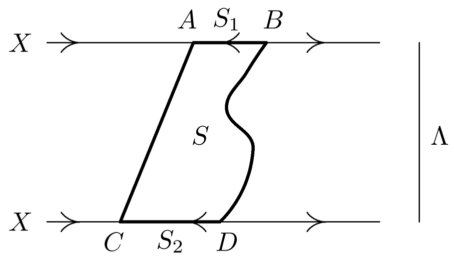

Figure 4. Two isolated invariant sets S_1 and S_2 related by continuation. A, B denote fixed points for a flow ρ_1 (top line), C, D denote fixed points for a flow ρ_2 (bottom line), $S_1(S_2)$ is an interval between A and B (C and D), S is an isolated invariant set for an extended flow on $X \times \Lambda$.

Definition 3.18. For a set $N \subseteq X$ we define an invariant part of N:

$$\operatorname{Inv} N = \{x \in N : \exists\colon \{x_k\}_{k\in\mathbb{Z}} \subseteq N\ x_0 = x \text{ and } f(x_k) = x_{k+1} \text{ for } k \in \mathbb{Z}\}.$$

Definition 3.19. A set $N \subseteq X$ is called an invariant set when $N = \operatorname{Inv} N$.

Definition 3.20. A compact set $N \subseteq X$ is called an isolating neighborhood for $S := \operatorname{Inv} N$ if $S \subseteq \operatorname{int}(N)$. The set S is called an isolated invariant set.

Fix S – an isolated invariant set.

Definition 3.21. A pair $P = (P_1, P_2)$ of compact subsets of X, is called an index pair for S if and only if

(a) $S = \operatorname{Inv} \operatorname{cl}(P_1 \setminus P_2) \subseteq \operatorname{int}(P_1 \setminus P_2)$,
(b) $f(P_2) \cap P_1 \subseteq P_2$,
(c) $f(P_1 \setminus P_2) \subseteq P_1$.

The set of all index pairs for S will be denoted by $\mathrm{IP}(S)$.

Assume $\Lambda \subseteq \mathbb{R}$ is a compact interval and $f\colon X \times \Lambda \to X \times \Lambda$ is a continuous map such that $f(X \times \{\lambda\}) \subseteq X \times \{\lambda\}$, for all $\lambda \in \Lambda$. For $\lambda \in \Lambda$ we define a map $f_\lambda\colon X \to X$ satisfying $f(x, \lambda) = f_\lambda(x)$, for all $x \in X$ and $\lambda \in \Lambda$.

The following remark is a part of Proposition 5.2 from [11].

Remark 3.22. If $\lambda \in \Lambda$, S is an isolated invariant set for f and P is an index pair for S, then S_λ is an isolated invariant set for f_λ and $P_\lambda = (P_{1\lambda}, P_{2\lambda})$ is an index pair for S_λ.

The remark is directly related to the definition of continuation of isolated invariant sets.

Definition 3.23. Assume $f_1, f_2\colon X \to X$ are continuous maps, S_1 and S_2 are isolated invariant set for f_1 and f_2, respectively. We say that there is a continuation between (f_1, S_1) and (f_2, S_2), if there exists $\Lambda \subseteq \mathbb{R}$ – a compact interval, $f: X \times \Lambda \to X \times \Lambda$ – a discrete semidynamical system such that $f(x, \lambda) = (f_\lambda(x), \lambda) \in X \times \Lambda$, for all $x \in X$ and $\lambda \in \Lambda$, S – an isolated invariant set for f and $a, b \in \Lambda$ such that $f_1 = f_a$, $f_2 = f_b$, $S_1 = S_a$ and $S_2 = S_b$.

Recall Theorem 1 from [4]:

Theorem 3.24. *If $\rho: X \times \mathbb{R} \to X$ is a flow, ρ_T is a time-T-map ($\rho_T = \rho(\,\cdot\,, T)$) where $T \in \mathbb{R}$, $S \subseteq X$ is a compact set then the following conditions are equivalent:*

(a) *S is an isolated invariant set for ρ.*
(b) *S is an isolated invariant set for ρ_T.*
(c) *S is an isolated invariant set for ρ_t for all $t > 0$.*

From Theorem 3.24 and definitions of continuation of isolated invariant sets in a continuous and a discrete case one may easily prove the following lemma:

Lemma 3.25. *If $\rho, \rho': X \times \mathbb{R} \to X$ are flows, $T \in \mathbb{R}$, $\rho_T = \rho(\,\cdot\,, T)$, $\rho'_T = \rho'(\,\cdot\,, T')$, $S, S' \subseteq X$ are isolated invariant sets for ρ and ρ', then continuation between (ρ, S) and (ρ', S') imples continuation between (ρ_T, S) and (ρ'_T, S').*

As we have the same isolated invariant set for a flow and its discretization, we may ask if there exists a common index pair and how it relates to isolating blocks. The answer is given by the following theorem:

Theorem 3.26. *If $\rho: X \times \mathbb{R} \to X$ is a flow, S – an isolated invariant set for ρ and B – an isolating block for S, then there exists $T_0 > 0$ such that for all $T \in (0, T_0]$ a pair $P = (P_1, P_2) := (B \cup B^- \cdot [0, T_0], B^- \cdot [0, T_0])$ is an index pair for S (in P_1) both with respect to ρ and to $\rho_T = \rho(\,\cdot\,, T)$. Moreover, the pair is regular with respect to ρ.*

Proof. If δ is a collar size of block B, then we may choose T_0 to be an arbitrary number from the interval $(0, \delta)$. Now take $T \in (0, T_0]$.

First, notice that

$$\operatorname{Inv} P_1 = \operatorname{Inv} B = S \subseteq \operatorname{int} \operatorname{cl}(B \setminus B^-) = \operatorname{int} \operatorname{cl}(P_1 \setminus P_2) \subseteq \operatorname{int} P_1.$$

Thus, P_1 is an isolating neighbourhood for S and condition (a) from Definitions 3.6 and 3.21 is satisfied.

In order to get condition (b) of both Definitions take $x \in P_2$. There exists $y \in B^-$ and $t_0 \in [0, T_0]$ such that $x = y \cdot t_0$. Now if $x \cdot [0, t] = y \cdot [t_0, t + t_0] \subseteq P_1$, then $[t_0, t + t_0] \subseteq [0, T_0]$ and $x \cdot [0, t] \subseteq P_2$. Similarly, if $\rho_T(x) = x \cdot T = y \cdot (T + t_0) \in P_1$, then $T + t_0 \in [0, T_0]$ and $\rho_T(x) \in P_2$.

Finally, take $x \in P_1 \setminus P_2 = B \setminus B^-$. If for some $t > 0$, $x \cdot t \notin P_1$, then from the definition of an isolating block there exists $t' \in [0, t]$ such that $x' := x \cdot t' \in$

$B^- \subseteq P_2$ and $x \cdot [0, t'] \subseteq B \subseteq P_1$. We have $x' \cdot [0, T_0] \subseteq P_1$, so $t > t' + T_0 > T$. Thus, $\rho_T(x) \notin P_1$ would mean a contradiction $T > T$. Therefore, condition (c) from Definition 3.6 and 3.21 is also satisfied and $P \in \mathrm{IP}(S, P_1) \cap \mathrm{IP}(S)$.

Regularity of P follows from regularity of B (see Remark 3.15). $\square$

4. The Conley index over a phase space and its properties

4.1. The Conley index – a continuous case. Consider a flow $\rho: X \times \mathbb{R} \to X$, an isolated invariant set S, an isolating neighbourhood N for S and an index pair $P \in IP(S, N)$ for S in N.

The classical Conley index $h(S, \rho)$ for S was defined in [2] as the homotopy class of a space $P_1/P_2 = (P_1 \setminus P_2) \cup [P_2]$. This homotopy invariant is quite simple, however, it does not detect some important features of S such as its position in a phase space.

This is why a new homotopy invariant, namely the Conley index over a base, was defined in [5]. Let us recall that definition in the most general case of a base equal to a phase space X and a gluing map id_X.

Define a space $U(P)$ and maps $s_P: X \to U(P)$ and $r_P: U(P) \to X$ as in Example 2.3. The triple $(U(P), r_P, s_P)$ will be called an index space over X. The fiberwise deforming homotopy class of $(U(P), r_P, s_P)$ is independent on a choice of an isolating neighbourhood N and of a regular index pair $P \in \mathrm{RIP}(S, N)$ and is called the Conley index of S over a base X. It will be denoted by $h_{\mathrm{id}_X}(S, \rho)$.

It appears that using the notion of the fiberwise moving homotopy type one may define an alternative Conley index over base X.

To that end, take an isolating block B with its exit set B^- and using the construction from the Example 2.3 define a block space $(U(B), r_B, s_B)$.

In order to prove the correctness of the definition of the Conley index over a base the authors of [5] use Proposition 4.3 several times. The following remark is its equivalent:

Remark 4.1. If $P \in \mathrm{IP}(S, P_1)$, $Q \in \mathrm{IP}(S, Q_1)$ and $P_1 \setminus P_2 = Q_1 \setminus Q_2$, then $(U(P), r_P, s_P)$ and $(U(Q), r_Q, s_Q)$ have the same fiberwise moving homotopy class over X.

Proof. The desired maps $\Phi: U(P) \to U(Q)$, $\Psi: U(Q) \to U(P)$, $\varphi, \psi: X \to X$ from Definition 2.10 are given by trivial formulas:

$$\Phi([u, q]_P) = [u, q]_Q, \qquad \Psi([u, q]_Q) = [u, q]_P, \qquad \varphi(u) = \psi(u) = u. \qquad \square$$

Now recall two operations defined in [6] which are performed on isolating blocks: "shaving" and "squeezing".

Fix an isolating block B and $S = \mathrm{Inv}\, B$.

Let U be an open set in Σ^+ such that $a^+(B) \subset U \subset B^+$. Take $Y := B^+ \setminus \operatorname{cl} U$. The set

$$B_1 := B \setminus O(Y, B)$$

created by "shaving" B is also an isolating block for S.

LEMMA 4.2. *$(U(B_1), r_{B_1}, s_{B_1})$ and $(U(B), r_B, s_B)$ have the same fiberwise moving homotopy types over X.*

PROOF. Notice that $P := (B_1 \cup B^-, B^-)$ is an index pair for S. We also have

$$P_1 \setminus P_2 = (B_1 \cup B^-) \setminus B^- = B_1 \setminus B_1^-$$

and by the Remark 4.1 it follows that $(U(P), r_P, s_P)$ and $(U(B_1), r_{B_1}, s_{B_1})$ have the same fiberwise moving homotopy types.

We have $a^+(B_1) \subset \operatorname{int}_{\Sigma^+} B_1^+ \subset B_1^+ \subset B^+$ hence, by the Remark 3.16 there exists an open neighbourhood V of $B^+ \setminus \operatorname{int}_{\Sigma^+} B_1^+$ in Σ^+ and a continuous function $\varepsilon^-\colon \operatorname{cl}_{\Sigma^+} V \to \mathbb{R}_+$ such that if $x \in \operatorname{cl}_{\Sigma^+} V$ then $x \cdot \varepsilon^-(x) \in \Sigma^-$ and if $x \in B^+ \cap \operatorname{cl}_{\Sigma^+} V$ then $x \cdot [0, \varepsilon^-(x)] \subset B$.

The Urysohn lemma implies that there exists a continuous function $v\colon \Sigma^+ \to [0,1]$ such that $v|_{\Sigma^+ \setminus V} \equiv 0$ and $v|_{B^+ \setminus \operatorname{int}_{\Sigma^+} B_1^+} \equiv 1$.

Take $\delta > 0$ – the collar size of block B and a number $\delta' \in (0, \delta)$. Define two sets

$$\begin{aligned} W &:= \{y \cdot \tau : y \in \operatorname{cl}_{\Sigma^+} V,\ \tau \in [0, v(y)\varepsilon^-(y)]\}, \\ Z &:= \{y \cdot -\tau : y \in \operatorname{cl}_{\Sigma^+} V,\ \tau \in [0, v(y)\delta']\}. \end{aligned}$$

Now we define maps

$$\Phi\colon U(B) \to U(P), \qquad \Psi\colon U(P) \to U(B), \qquad \varphi, \psi\colon X \to X$$

by formulas:

$$\Phi([x,q]_B) = \begin{cases} [y \cdot (v(y)\varepsilon^-(y)), q]_P & \text{if } x = y \cdot \tau \in W, \\ \left[y \cdot (v(y)\varepsilon^-(y) - \dfrac{\varepsilon^-(y) + \delta'}{\delta'}\tau), 0\right]_P & \text{if } x = y \cdot -\tau \in Z, \\ & q = 0, \\ [x,q]_P & \text{if } x \notin W \cup Z, \end{cases}$$

$$\Psi([x,q]_P) = [x,q]_B,$$

$$\varphi(x) = \begin{cases} y \cdot (v(y)\varepsilon^-(y)) & \text{if } x = y \cdot \tau \in W, \\ y \cdot \left(v(y)\varepsilon^-(y) - \dfrac{\varepsilon^-(y) + \delta'}{\delta'}\tau\right) & \text{if } x = y \cdot -\tau \in Z, \\ x & \text{if } x \notin W \cup Z. \end{cases}$$

$$\psi(x) = x.$$

One can easily check that the above maps are well-defined and continuous. Moreover, they satisfy the following:

$$(\Phi,\varphi)\in\mathrm{Mor}_{\mathcal{SB}(X)}((U(B),r_B,s_B),(U(P),r_P,s_P)),$$
$$(\Psi,\psi)\in\mathrm{Mor}_{\mathcal{SB}(X)}((U(P),r_P,s_P),(U(B),r_B,s_B)).$$

In order to check the condition (2.5) define auxiliary maps:

$$\begin{aligned} k\colon W\times\mathbb{I}\ni(y\cdot\tau,t)&\mapsto(1-t)v(y)\varepsilon^-(y)+t\tau\in\mathbb{R},\\ l\colon Z\times\mathbb{I}\ni(y\cdot-\tau,t)&\mapsto(1-t)\biggl(v(y)\varepsilon^-(y)-\frac{\varepsilon^-(y)+\delta'}{\delta'}\tau\biggr)-t\tau\in\mathbb{R}.\end{aligned}$$

Now define

$$H\colon U(B)\times\mathbb{I}\to U(B),\qquad H'\colon U(P)\times\mathbb{I}\to U(P),\qquad h\colon X\times\mathbb{I}\to X,$$

by formulas:

$$H([x,q]_B,t)=\begin{cases}[y\cdot k(y\cdot\tau,t),q]_B & \text{if } x=y\cdot\tau\in W,\\ [y\cdot l(y\cdot-\tau,t),0]_B & \text{if } x=y\cdot-\tau\in Z,\ q=0,\\ [x,q]_B & \text{if } x\notin W\cup Z,\end{cases}$$

$$H'([x,q]_P,t)=\begin{cases}[y\cdot k(y\cdot\tau,t),q]_P & \text{if } x=y\cdot\tau\in W,\\ [y\cdot l(y\cdot-\tau,t),0]_P & \text{if } x=y\cdot-\tau\in Z,\ q=0,\\ [x,q]_P & \text{if } x\notin W\cup Z,\end{cases}$$

$$h(x,t)=\begin{cases}y\cdot k(y\cdot\tau,t) & \text{if } x=y\cdot\tau\in W,\\ y\cdot l(y\cdot-\tau,t) & \text{if } x=y\cdot-\tau\in Z,\\ x & \text{if } x\notin W.\end{cases}$$

Notice that for all $t\in\mathbb{I}$ we have: if $x=y\cdot\tau\in W\cap B$ then and if $x=y\cdot\tau\in W\cap B_1$ then $k(x,t)\in B_1$, so the above maps are well-defined and continuous. Homotopy h joins φ with id_X, so $\varphi\simeq\mathrm{id}_X\simeq\psi$. Finally, (H,h) and (H',h) are homotopies joining $(\Psi\circ\Phi,\psi\circ\varphi)$ with $(\mathrm{id}_{U(B)},\mathrm{id}_X)$ and $(\Phi\circ\Psi,\varphi\circ\psi)$ with $(\mathrm{id}_{U(P)},\mathrm{id}_X)$. As a consequence, $(U(B),r_B,s_B)$ has the same fiberwise moving homotopy type as $(U(P),r_P,s_P)$ and $(U(B_1),r_{B_1},s_{B_1})$. □

If $x\in B\setminus A(B)$, then $x\cdot-\sigma_B^+(x)\in\Sigma^+$, $x\cdot\sigma_B^-(x)\in\Sigma^-$. Put

$$\sigma_B(x):=\begin{cases}\sigma_B^-(x)+\sigma_B^+(x) & \text{if } x\in B\setminus A(B),\\ +\infty & \text{if } x\in A(B).\end{cases}$$

and

$$T:=\inf\{\sigma_B(x):\ x\in B\}.$$

Take $T'\in(0,T)$. Now the set $B_2:=B\setminus(B^-\cdot(-T',0])$ created by "squeezing" B is also an isolating block for S.

Lemma 4.3. *$(U(B_2), r_{B_2}, s_{B_2})$ and $(U(B), r_B, s_B)$ have the same fiberwise moving homotopy types over X.*

Proof. Take $T'' \in (T', T)$. We have $a^-(B) \subset \mathrm{int}_{\Sigma^-} : B^-$ so, by the Remark 3.16, there exists an open neighbourhood V of $\mathrm{bd}_{\Sigma^-} B^-$ in Σ^- and a continuous function $\varepsilon^+ : \mathrm{cl}_{\Sigma^-} V \to \mathbb{R}_+$ such that if $x \in \mathrm{cl}_{\Sigma^-} V$ then $x \cdot -\varepsilon^+(x) \in \Sigma^+$ and if $x \in B^- \cap \mathrm{cl}_{\Sigma^-} V$ then $x \cdot [-\varepsilon^+(x), 0] \subset B$. Moreover, making V smaller, if necessary, we may assume that the values of ε^+ on $\mathrm{cl}_{\Sigma^-} V$ are greater than T''.

By the Urysohn lemma there exists a continuous function $v : \Sigma^- \to [0,1]$ such that $v|_{\Sigma^- \setminus (V \cup B^-)} \equiv 0$ and $v|_{B^-} \equiv 1$. Define the set

$$W := \{y \cdot -\tau : y \in \mathrm{cl}_{\Sigma^-} V \setminus \mathrm{int}_{\Sigma^-} B^-,\ \tau \in [0, v(y)\varepsilon^+(y)]\}.$$

Notice that $B \setminus \mathrm{Inv}^+ B = \{y \cdot -\tau : y \in B^-,\ \tau \in [0, \varepsilon^+(y)]\}$.

Let $\beta : W \cup (B \setminus \mathrm{Inv}^+ B) \to \mathbb{R}$ be a continuous function given by formula:

$$\beta(y \cdot -\tau) := \begin{cases} 0 & \text{if } 0 \le \tau \le v(y)T', \\ \dfrac{T''}{T'' - T'}(\tau - v(y)T') & \text{if } v(y)T' \le \tau \le v(y)T'', \\ \tau & \text{if } \tau \ge v(y)T''. \end{cases}$$

Now we define maps

$$\Phi : U(B_2) \to U(B), \qquad \Psi : U(B) \to U(B_2), \qquad \varphi, \psi : X \to X$$

by formulas:

$$\Phi([x,q]_{B_2}) = \begin{cases} [y \cdot (-\beta(y \cdot -\tau)), q]_B & \text{if } x = y \cdot -\tau \in W \cup (B \setminus \mathrm{Inv}^+ B), \\ [x,q]_B & \text{if } x \notin W \cup B \text{ or } x \in \mathrm{Inv}^+ B, \end{cases}$$

$$\Psi([x,q]_B) = [x,q]_{B_2},$$

$$\varphi(x) = \begin{cases} y \cdot (-\beta(y \cdot -\tau)) & \text{if } x = y \cdot -\tau \in W \cup (B \setminus \mathrm{Inv}^+ B), \\ x & \text{if } x \notin W \cup B \text{ or } x \in \mathrm{Inv}^+ B, \end{cases}$$

$$\psi(x) = x.$$

One can easily check that the above maps are well-defined and continuous. Moreover, they satisfy the following:

$$(\Phi, \varphi) \in \mathrm{Mor}_{\mathcal{SB}(X)}((U(B_2), r_{B_2}, s_{B_2}), (U(B), r_B, s_B))$$
$$(\Psi, \psi) \in \mathrm{Mor}_{\mathcal{SB}(X)}((U(B), r_B, s_B), (U(B_2), r_{B_2}, s_{B_2})).$$

In order to check condition (2.5) define an auxiliary map

$$k : W \cup (B \setminus \mathrm{Inv}^+ B) \times \mathbb{I} \to \mathbb{R}$$

by formula:

$$k(y \cdot -\tau, t) := \begin{cases} t\tau & \text{if } 0 \le \tau \le v(y)T', \\ \dfrac{T'' - tT'}{T'' - T'}(\tau - v(y)T') + tv(y)T' & \text{if } v(y)T' \le \tau \le v(y)T'', \\ \tau & \text{if } \tau \ge v(y)T''. \end{cases}$$

Now define

$$H: U(B_2) \times \mathbb{I} \to U(B_2), \quad H': U(B) \times \mathbb{I} \to U(B), \quad h: X \times \mathbb{I} \to X$$

by formulas:

$$H([x,q]_{B_2}, t) = \begin{cases} [y \cdot (-k(y \cdot -\tau, t)), q]_{B_2} & \text{if } x = y \cdot -\tau \in W \cup (B \setminus \operatorname{Inv}^+ B), \\ [x,q]_{B_2} & \text{if } x \notin W \cup B \text{ lub } x \in \operatorname{Inv}^+ B, \end{cases}$$

$$H'([x,q]_B, t) = \begin{cases} [y \cdot (-k(y \cdot -\tau, t)), q]_B & \text{if } x = y \cdot -\tau \in W \cup (B \setminus \operatorname{Inv}^+ B), \\ [x,q]_B & \text{if } x \notin W \cup B \text{ lub } x \in \operatorname{Inv}^+ B, \end{cases}$$

$$h(x,t) = \begin{cases} y \cdot (-k(y \cdot -\tau, t)) & \text{if } x = y \cdot -\tau \in W \cup (B \setminus \operatorname{Inv}^+ B), \\ x & \text{if } x \notin W \cup B \text{ or } x \in \operatorname{Inv}^+ B. \end{cases}$$

The above maps are well-defined and continuous. The homotopy h joins φ with id_X, so $\varphi \simeq \mathrm{id}_X \simeq \psi$. Finally, (H, h) and (H', h) are homotopies joining $(\Psi \circ \Phi, \psi \circ \varphi)$ with $(\mathrm{id}_{U(B_2)}, \mathrm{id}_X)$ and $(\Phi \circ \Psi, \varphi \circ \psi)$ with $(\mathrm{id}_{U(B)}, \mathrm{id}_X)$. Hence, $(U(B), r_B, s_B)$ and $(U(B_2), r_{B_2}, s_{B_2})$ have the same fiberwise moving homotopy types. □

The following theorem enables us to define a new index:

THEOREM 4.4. *For any two isolating blocks B, C of S the fiberwise moving homotopy types of $(U(B), r_B, s_B)$ and $(U(C), r_C, s_C)$ over X are equal.*

PROOF. The proof is an exact repetition of the proof of Theorem 22.29 from [6], where the operations of "shaving" and "squeezing" are used. The only difference is that we use Lemmas 4.2 and (4.3) instead of the fact that the quotient spaces B_1/B_1^- and B_2/B_2^- have the same homotopy type as B/B^-. Therefore, we omit the proof. □

Now we may define an alternative Conley index of S over a phase space X:

DEFINITION 4.5. The Conley mh-index of an isolated invariant set S over a phase space X is the fiberwise moving homotopy type of a block space $(U(B), r_B, s_B)$ over X, for any isolating block $B \in \mathrm{IB}(S)$.

The index will be denoted by $\widehat{h}(S, \rho)$.

4.2. The Conley index – a discrete case. Let X be a fixed locally compact metric space, $f: X \to X$ – a continuous map, S – an isolated invariant set for f, $P = (P_1, P_2) \in \mathrm{IP}(S)$ – an index pair for S.

Let us briefly recall the definition and the continuation property of the Conley index over a phase space for discrete semidynamical systems from [7] and [8].

The definition is based on the notions of M-equivalence, of the index space and of the index map.

Consider (U, r, s) and (U', r', s') two objects in $\mathcal{SB}(X)$ and two morphisms $(F, f) \in \mathrm{Mor}_{\mathcal{SB}(X)}((U, r, s), (U, r, s))$, $(F', f') \in \mathrm{Mor}_{\mathcal{SB}(X)}((U', r', s'), (U', r', s'))$.

DEFINITION 4.6. Two pairs $((U, r, s), (F, f))$ and $((U', r', s'), (F', f'))$ are M-equivalent over a base X, if $f \simeq f'$ and there exist $m, n \in \mathbb{N}$, $(\Phi, \varphi) \in \mathrm{Mor}_{\mathcal{SB}(X)}((U, r, s), (U', r', s'))$ and $(\Psi, \psi) \in \mathrm{Mor}_{\mathcal{SB}(X)}((U', r', s'), (U, r, s))$ such that $\varphi \simeq f^m$, $\psi \simeq f'^n$ and there exists $k \in \mathbb{N}$ such that

$$\begin{aligned}
(\Phi, \varphi) \circ (F, f) &\simeq_* (F', f') \circ (\Phi, \varphi), \\
(\Psi, \psi) \circ (F', f') &\simeq_* (F, f) \circ (\Psi, \psi), \\
(\Psi, \psi) \circ (\Phi, \varphi) \circ (F, f)^k &\simeq_* (F, f)^{m+n+k}, \\
(\Phi, \varphi) \circ (\Psi, \psi) \circ (F', f')^k &\simeq_* (F', f')^{m+n+k}.
\end{aligned}$$

The class of M-equivalence $((U, r, s), (F, f))$ over X will be denoted by $[((U, r, s), (F, f))]_X$.

REMARK 4.7. M-equivalence over a given base is an equivalence relation.

A close relation between M-equivalence over a base and a fiberwise moving homotopy type over a base is illustrated by the following lemma:

LEMMA 4.8. *If $(U, r, s), (U', r', s') \in \mathrm{Ob}(\mathcal{SB}(X))$, then*

$$\begin{gathered}
[(U, r, s), (\mathrm{id}_U, \mathrm{id}_X)]_X = [(U', r', s'), (\mathrm{id}_{U'}, \mathrm{id}_X)]_X, \\
\Updownarrow \\
[U, r, s]_X = [U', r', s']_X.
\end{gathered}$$

PROOF. The lemma follows easily from definitions of both homotopy types. $\square$

The index space over X is a triple $(U(P), r_P, s_P)$ constructed according to the Example 2.3. The index map $f_P: U(P) \to U(P)$ is defined by formula:

$$f_P([x, q]_P) := \begin{cases} [f(x), 1]_P & \text{if } q = 1,\ x, f(x) \in P_1 \setminus P_2, \\ [f(x), 0]_P & \text{otherwise.} \end{cases}$$

Recall the main theorem of [8]:

THEOREM 4.9. *For any $P, P' \in \mathrm{IP}(S)$ pairs $((U(P), r_P, s_P), (f_P, f))$ and $((U(P'), r_{P'}, s_{P'}), (f_{P'}, f))$ are M-equivalent over a phase space X.*

DEFINITION 4.10. The Conley M-index over a phase space $\widehat{h}_d(S,f)$ of an isolated invariant set S is an M-equivalence class of $((U(P),r_P,s_P),(f_P,f))$ over X, for any index pair $P\in \mathrm{IP}(S)$:

$$\widehat{h}_d(S,f)=[((U(P),r_P,s_P),(f_P,f))]_X.$$

Recall the Theorem 5.1 from [10]:

THEOREM 4.11 (The continuation property of the discrete index). *If $\Lambda\subseteq\mathbb{R}$ is a compact interval and S is an isolated invariant set for a continuous map $f\colon X\times\Lambda\to X\times\Lambda$ satisfying $f(x,\lambda)=(f_\lambda(x),\lambda)\subseteq X\times\Lambda$, for each $x\in X$ and $\lambda\in\Lambda$, then for any $\lambda,\nu\in\Lambda$,*

$$\widehat{h}_d(S_\lambda,f_\lambda)=\widehat{h}_d(S_\nu,f_\nu).$$

Thus, indices $\widehat{h}_d(S_\lambda,f_\lambda)$ do not depend on the choice of λ.

4.3. The comparison of indices. At first, we will show that the Conley mh-index over a phase space is more general than the classical Conley index defined in [2].

THEOREM 4.12. *Assume $\rho,\rho'\colon X\times\mathbb{R}\to X$ are flows S, S' are isolated invariant sets, respectively for ρ and ρ'. Then*

$$\widehat{h}(S,\rho)=\widehat{h}(S',\rho')\Rightarrow h(S,\rho)=h(S',\rho').$$

PROOF. Let B and B' be isolating blocks for S and S', respectively. There exist $(\Phi,\varphi)\in\mathrm{Mor}_{\mathcal{SB}(X)}((U(B),r_B,s_B),(U(B'),r_{B'},s_{B'}))$, $(\Psi,\psi)\in\mathrm{Mor}_{\mathcal{SB}(X)}((U(B'),r_{B'},s_{B'}),(U(B),r_B,s_B))$ satisfying (2.5).

Define maps $\widehat{\varphi}\colon B/B^-\to B'/B'^-$ and $\widehat{\psi}\colon B'/B'^-\to B/B^-$ by formulas:

$$\widehat{\varphi}(x):=\begin{cases}\varphi(x) & \text{if } x\in B\setminus B^-,\ \Phi([x,1])\in U(B')\setminus s_{B'}(X),\\ [B'^-] & \text{otherwise.}\end{cases}$$
$$\widehat{\psi}(x):=\begin{cases}\psi(x) & \text{if } x\in B'\setminus B'^-,\ \Psi([x,1])\in U(B)\setminus s_{B}(X),\\ [B^-] & \text{otherwise.}\end{cases}$$

The composition $\widehat{\psi}\circ\widehat{\varphi}$ is given by formula

$$\widehat{\psi}\circ\widehat{\varphi}(x):=\begin{cases}(\psi\circ\varphi)(x) & \text{if } x\in B\setminus B^-,\ \Psi(\Phi([x,1]))\in U(B)\setminus s_{B}(X),\\ [B^-] & \text{otherwise.}\end{cases}$$

Let (H,h) be a homotopy joining $(\Psi\circ\Phi,\psi\circ\varphi)$ with $(\mathrm{id}_{U(B)},\mathrm{id}_X)$. Define a map $\widehat{h}\colon B/B^-\times[0,1]\to B/B^-$ by formula:

$$\widehat{h}(x,t):=\begin{cases}h(x,t) & \text{if } x\in B\setminus B^-,\ H([x,1]),t)\in U(B)\setminus s_{B}(X),\\ [B^-] & \text{otherwise.}\end{cases}$$

One can easily verify that $\widehat{h}$ is a well-defined continuous homotopy joining $\widehat{\psi} \circ \widehat{\varphi}$ with id_{B/B^-}.

Similarly one can define a homotopy joining $\widehat{\varphi} \circ \widehat{\psi}$ with id_{B'/B'^-}. Thus, $\widehat{\varphi}$ and $\widehat{\psi}$ are mutually inverse homotopy equivalences, so B/B^- and B'/B'^- have the same homotopy type. □

The following very simple example shows that the inverse implication in Theorem 4.12 is false.

Example 4.13. Consider the space $X = \mathbb{R} \setminus \{0\}$ and two flows $\rho, \rho' \colon X \times \mathbb{R} \to X$ induced by differential equations

$$x' = x \cdot (1 - x) \quad \text{and} \quad x' = x \cdot (1 + x).$$

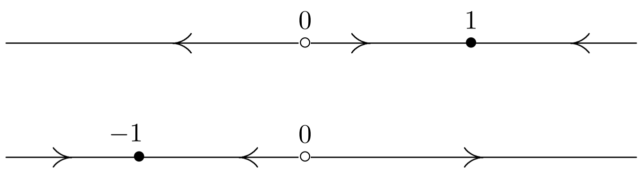

Figure 5. Phase spaces of the flows ρ and ρ'.

One can easily notice that $S = \{1\}$ and $S' = \{-1\}$ are attracting fixed points which are isolated invariant sets respectively for ρ and ρ'. There is no continuation between (ρ, S) and (ρ', S'). The classical Conley index for both sets is the same because ρ and ρ' behave the same way in the neighbourhood of their fixed points. If we choose $B = [1/2, 2]$ an isolating block for (ρ, S), then $B' = \{x \in X : -x \in B\} = [-2, -1/2]$, will be an isolating block for (ρ', S'). In both cases the exit set is empty.

We define continuous maps

$$\Phi \colon B/\emptyset \ni [x] \mapsto [-x] \in B'/\emptyset, \qquad \Psi \colon B'/\emptyset \ni [x] \mapsto [-x] \in B/\emptyset,$$

which are mutually inverse homotopy equivalences.

We will show that $(U(B), r_B, s_B)$ and $(U(B'), r_{B'}, s_{B'})$ have different fiber-wise moving homotopy types over X. Assume the opposite. Then there would exist morphisms (Φ, φ) and (Ψ, ψ) satisfying (2.4) and (2.5). One can easily see that $\Phi(U(B) \setminus s_B(X))$ would have to be contained in $U(B') \setminus s_{B'}(X)$, so $(r_{B'} \circ \Phi)(U(B) \setminus s_B(X)) \subseteq (-\infty; 0)$. On the other hand, $\varphi \simeq \mathrm{id}_X$, so $(\varphi \circ r_B)(U(B) \setminus s_B(X)) \subseteq (0; +\infty)$. This would be, however, impossible because $r_{B'} \circ \Phi = \varphi \circ r_B$. Thus, $\widehat{h}(S, \rho) \neq \widehat{h}(S', \rho')$.

Example 4.20 illustrates a nontrivial application of the Conley mh-index over a phase space to prove the lack of continuation between two isolated invariant sets, whose classical Conley indices coincide.

Now we are going to compare The Conley mh-index over a phase space for a flow and the Conley M-index for its discretization.

Let $\rho\colon X \times \mathbb{R} \to X$ be a flow with an isolated invariant set S. From Theorem 3.24 we know that S is also an isolated invariant set for each discretization $\rho_t = \rho(\,\cdot\,, t)$, where $t > 0$. By Theorems 3.13 and 3.26 we can find an isolating block B and a regular index pair P for ρ, which is also an index pair for ρ_T, if $T > 0$ is small. The block B and the pair P satisfy $B \setminus B^- = P_1 \setminus P_2$, so, up to natural identifications, $U(B) = U(P)$. We also have the following lemma:

Lemma 4.14.

$$\exists\, T_0 > 0 \ \forall T \in (0, T_0] \ (f_P, f) \simeq_* (\mathrm{id}_{U(P)}, \mathrm{id}_X) = (\mathrm{id}_{U(B)}, \mathrm{id}_X),$$

where $f = \rho_T$ and f_P denotes its index map.

Proof. The desired homotopies $H\colon U(P) \times \mathbb{I} \to U(P)$ and $h\colon X \times \mathbb{I} \to X$ are given by formulas:

$$H([u, q]_P, t) = \begin{cases} [u \cdot tT, 1]_P & \text{if } q = 1,\ u \cdot [0, tT] \subset P_1 \setminus P_2, \\ [u \cdot tT, 0]_P & \text{otherwise,} \end{cases}$$

$$h(u, t) = u \cdot tT.$$

P is an index pair for all maps ρ_{tT}, so H is well-defined and continuous and (H, h) is a homotopy joining (f_P, f) with $(\mathrm{id}_{U(P)}, \mathrm{id}_X)$. □

The above lemma and Lemma 4.8 imply

Corollary 4.15. *Let $\rho, \rho'\colon X \times \mathbb{R} \to X$ be two flows, S, S' – isolated invariant sets for ρ and ρ' (thus, for their discretizations as well). Then there exists $T_0 > 0$ such that for all $T, T' \in (0, T_0]$*

$$\widehat{h}(S, \rho) = \widehat{h}(S', \rho') \ \Leftrightarrow\ \widehat{h}_d(S, \rho_T) = \widehat{h}_d(S', \rho'_{T'}).$$

For a given flow ρ with an isolated invariant set S and positive numbers T_1 and T_2, there is a continuation between (ρ_{T_1}, S) and (ρ_{T_2}, S) (homotopy along trajectories), hence, from the continuation property of a discrete Conley M-index over a phase space, in the above corollary we can easily ignore the assumption that T and T' should be small.

Theorem 4.16. *Let $\rho, \rho'\colon X \times \mathbb{R} \to X$ be two flows, S, S' – isolated invariant sets for ρ and ρ' (thus, for their discretizations as well). Then, for all $T, T' > 0$,*

$$\widehat{h}(S, \rho) = \widehat{h}(S', \rho') \ \Leftrightarrow\ \widehat{h}_d(S, \rho_T) = \widehat{h}_d(S', \rho'_{T'}).$$

A natural question that one can ask is: what is the relation between the Conley mh-index over a phase space and the Conley index over a base defined in [5]?

Examples 2.12 and 2.13 show that there is no relation between fiberwise deforming and moving homotopy types over a phase space, even if we consider objects $(U(P), r_P, s_P)$, where $P=(P_1,P_2)$ is a pair of compact sets $P_2\subset P_1\subset X$.

These examples, however, do not answer our question, as the pairs P and Q are neither isolating blocks nor index pairs for any flow. It appears that the answer in general is not that easy. Assuming that the exit sets for isolating blocks are empty and taking a phase space as a base we can easily prove a relation between the Conley mh-index and the Conley index over a base:

REMARK 4.17. Let $\rho,\rho'\colon X\times\mathbb{R}\to X$ be two flows with isolated invariant sets S and S'. If there exist $B\in \mathrm{IB}(S)$ and $B'\in\mathrm{IB}(S')$ such that $B^-=B'^-=\emptyset$, then

$$\widehat{h}(S,\rho)=\widehat{h}(S',\rho')\Rightarrow h_{\mathrm{id}_X}(S,\rho)=h_{\mathrm{id}_X}(S',\rho').$$

PROOF. $\widehat{h}(S,\rho)=\widehat{h}(S',\rho')$, so there exist continuous maps

$$\Phi\colon U(B)\to U(B'),\qquad \Psi\colon U(B')\to U(B),\qquad \varphi,\psi\colon X\to X$$

such that $\varphi\simeq \mathrm{id}_X\simeq\psi$ and

$$\begin{aligned} r_{B'}\circ\Phi &= \varphi\circ r_B\simeq r_B, & r_B\circ\Psi &=\psi\circ r_{B'}\simeq r_{B'},\\ \Psi\circ\Phi &\simeq \mathrm{id}_{U(B)}, & \Phi\circ\Psi &\simeq \mathrm{id}_{U(B')}. \end{aligned}$$

Define maps $\widetilde{\Phi}\colon U(B)\to U(B')$ and $\widetilde{\Psi}\colon U(B')\to U(B)$ by formulas:

$$\widetilde{\Phi}([u,q]_B)=\begin{cases}\Phi([u,q]_B) & \text{if } q=1,\\ [u,0]_{B'} & \text{if } q=0,\end{cases}$$

$$\widetilde{\Psi}([u,q]_{B'})=\begin{cases}\Psi([u,q]_{B'}) & \text{if } q=1,\\ [u,0]_{B} & \text{if } q=0.\end{cases}$$

$B^-=B'^-=\emptyset$, so the above maps are well-defined and continuous and satisfy conditions (2.1)–(2.3), hence $h_{\mathrm{id}_X}(S,\rho)=h_{\mathrm{id}_X}(S',\rho')$. □

It is quite surprising that even under so restrictive assumption the reverse implication is not obvious.

The main problem which appears when one wants to compare these indices is the lack of relation between homotopies in (2.2) and (2.3) and homotopies in (2.4) and (2.5). Anyway, it is an interesting problem to solve.

4.4. Properties of the index. The fiberwise moving homotopy type of $(X,\mathrm{id}_X,\mathrm{id}_X)$ over X will be called a trivial Conley mh-index over a phase space and will be denoted by $[0]_X$.

It is obvious that $\emptyset$ is an isolating block for an empty set, so

THEOREM 4.18 (The Ważewski property). *If $\widehat{h}(S,\rho)$ is nontrivial, then $S \neq \emptyset$.*

THEOREM 4.19 (The continuation property). *If $\Lambda \subseteq \mathbb{R}$ is a compact interval and S is an isolated invariant set for a flow $\rho\colon X \times \Lambda \times \mathbb{R} \to X \times \Lambda$ such that $\rho(x,\lambda,t) = (\rho_\lambda(x,t),\lambda) \subseteq X \times \Lambda$, for all $x \in X$, $t \in \mathbb{R}$ and $\lambda \in \Lambda$, then for any $\lambda,\nu \in \Lambda$,*

$$\widehat{h}(S_\lambda,\rho_\lambda) = \widehat{h}(S_\nu,\rho_\nu).$$

PROOF. S_λ, S_ν are isolated invariant sets respectively for ρ_λ and ρ_ν. There is a continuation between (S_λ,ρ_λ) and (S_ν,ρ_ν), so from Theorem 3.24 and Lemma 3.25 we know that S_λ, S_ν are also isolated invariant sets respectively for $\rho_{\lambda T}$ and $\rho_{\nu T}$, for any $T \in \mathbb{R}$. Moreover, there is a continuation between $(S_\lambda,\rho_{\lambda T})$ and $(S_\nu,\rho_{\nu T})$. From the continuation property of the discrete index we get

$$\widehat{h}_d(S_\lambda,\rho_{\lambda T}) = \widehat{h}_d(S_\nu,\rho_{\nu T}).$$

Thus, from Theorem 4.16 we get the thesis. □

EXAMPLE 4.20. Consider the space $X = \mathbb{R}^3 \setminus Oz$, where $Oz = \{(0,0,z) : z \in \mathbb{R}\}$ and two flows $\rho,\rho'\colon X \times \mathbb{R} \to X$, for which there exists a hyperbolic periodic orbit winding once and twice around Oz. Periodic orbits S, S' in both cases are isolated invariant sets, while thin cylinders B and B' containing these orbits are isolating blocks for S and S'.

The homotopy types of B/B^- and B'/B'^- do not depend on the number of rotations the periodic orbits make, so $h(S,\rho) = h(S',\rho')$.

Example 5.17 and Theorem 5.6 in [9] show that $\widehat{h}_d(S,\rho_T) \neq \widehat{h}_d(S',\rho'_{T'})$ for any $T,T' > 0$. Theorems 4.16 and 4.19 imply the lack of continuation between (ρ,S) and (ρ',S').

The example may be easily generalized by considering two hyperbolic periodic orbits winding p and q times ($p \neq q$) around any line.

The same example was considered in [5], where the above result was obtained by means of the Conley index over a base.

The following example illustrates potential applications of the mh-index in studying dynamical systems on noncontractible manifolds.

EXAMPLE 4.21. Consider a torus $X = [-1,1] \times [-1,1]/\equiv$, where $(x,-1) \equiv (x,1)$, for all $x \in [-1,1]$ and $(-1,y) \equiv (1,y)$, for all $y \in [-1,1]$. Define two flows on X:

$$\rho_1 \text{ given by } \begin{cases} x' = 0, \\ y' = y^3 - y, \end{cases} \qquad \rho_2 \text{ given by } \begin{cases} x' = x^3 - x, \\ y' = 0. \end{cases}$$

As one can easily notice $S_1 := \{[x,0] \in X\}$ is an isolated invariant set of fixed points for ρ_1 and $S_2 := \{[0,y] \in X\}$ is an isolated invariant set of fixed points for ρ_2.

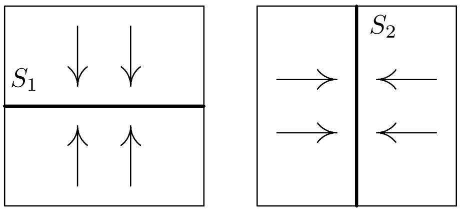

Figure 6. Two flows on torus

Take $B_1 := \{[x,y] \in X : x \in [-1,1],\ y \in [-1/2,1/2]\}$ and $B_2 := \{[x,y] \in X : x \in [-1/2,1/2],\ y \in [-1,1]\}$ as the isolating blocks for S_1 and S_2. Exit sets in both cases are empty. Obviously, the classical Conley indices in both cases are the same.

If it were the case, as far as the mh-indices are concerned, there would exist morphisms (Φ,φ) and (Ψ,ψ) satisfying (2.4) and (2.5). This would imply that

$$\Phi(U(B_1) \setminus s_{B_1}(X)) \cong U(B_2) \setminus s_{B_2}(X),$$

so

$$(r_{B_2} \circ \Phi)(U(B_1) \setminus s_{B_1}(X)) \cong B_2$$

($C \cong D$ denotes C and D have the same homotopy type in X). On the other hand, $\varphi \simeq \mathrm{id}_X$, so

$$(\varphi \circ r_{B_1})(U(B_1) \setminus s_{B_1}(X)) \cong B_1.$$

This would contradict the fact that (Φ,φ) is a morphism. Thus, $\widehat{h}(S_1,\rho_1) \neq \widehat{h}(S_2,\rho_2)$ and there is no continuation between these sets.

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