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BIFURCATION OF FREDHOLM MAPS II. THE DIMENSION OF THE SET OF BIFURCATION POINTS

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Abstract. We obtain an estimate for the covering dimension of the set of bifurcation points for solutions of nonlinear elliptic boundary value problems from the principal symbol of the linearization along the trivial branch of solutions.

1. Introduction

In [12] we defined an index of bifurcation points $\beta(f)$ of a parametrized family f of C^1-Fredholm maps. Nonvanishing of $\beta(f)$ entails the existence of at least one point of bifurcation from a trivial branch of zeroes of the family f. Linearization of f along the trivial branch produces a parametrized family of linear Fredholm operators L. The index $\beta(f)$ depends only on the stably fiberwise homotopy equivalence class of the index bundle $\operatorname{Ind} L$ of L. The nonvanishing of $\beta(f)$ can be checked through the Stiefel–Whitney and Wu characteristic classes of $\operatorname{Ind} L$ since they are invariant under stably fiberwise homotopy equivalence.

If the parameter space is a manifold, the nonvanishing of the Stiefel–Whitney and Wu classes of $\operatorname{Ind} L$ not only implies that the set $B(f)$ of all bifurcation points of f is nonempty, but also provides some further information about the covering dimension of this set and its position in the parameter space.

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In this paper, using the above observation (Theorem 3.1 in Section 3) together with our results from [12], we obtain an estimate on the covering dimension of the set of bifurcation points of solutions of nonlinear elliptic boundary value problems parametrized by a smooth manifold.

Roughly speaking, the approach is as follows: assuming that the coefficients of the leading terms of L are independent from the parameter near the boundary, an extension to families of the Agranovich reduction identifies the complexification of $\operatorname{Ind} L$ with the index bundle of a family of pseudo-differential operators S on $\mathbb{R}^n$. Applying to S a cohomological form of the Atiyah–Singer family index theorem, due to Fedosov, we determine the Chern character of the complexification $c(\operatorname{Ind} L)$ as an integral along the fiber of a differential form associated to the symbol of the family.

In principle, the above leads to the computation of Wu classes of the index bundle of L, since they are polynomials in Pontriagin classes "homotopu class" of $\operatorname{Ind} L$ with Z_p coefficients. However, the general expression is messy and can hardly be used in practice. It becomes much simpler by evaluating Wu classes of $\operatorname{Ind} L$ on spherical homology classes. Restricting the family f to spheres embedded in the parameter space and using our approach in [12], we obtain explicit conditions for non vanishing of Wu classes and hence estimates for the dimension of the set of bifurcation points. A similar use of the Stiefel–Whitney classes gives some complementing results.

The topological dimension of the set of solutions of nonlinear equations and the set of bifurcation points has been discussed in various places, mainly in the case of compact vector fields and semi-linear Fredholm maps [5], [3], [11], [9], [1], [6]. However, it should be remarked, that our estimates are obtained directly from the leading coefficients of linearized equations without the need to solve them. This is the main reason for using elliptic invariants in a topological approach to bifurcation which complements the classical Lyapunov–Schmidt method.

The paper is organized as follows: in Section 2 we state our main result, Theorem 2.1. In section 3 we relate Wu classes of $\operatorname{Ind} L$ to the dimension of the set of bifurcation points of f. In Section 4, after discussing the Agranovich reduction, we carry out the computations of the relevant characteristic classes, completing in this way the proof of Theorem 2.1. In Section 5, assuming that the linearization along the trivial branch is a (real) lower-order perturbation of a family of elliptic boundary value problems with complex coefficients, we obtain sufficient conditions for bifurcation in dimensions not covered by Theorem 2.1, using Stiefel–Whitney classes.

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2. The main theorem

We consider boundary value problems of the form

$$(2.1)\qquad \begin{cases} \mathcal{F}(\lambda, x, u, \dots, D^k u) = 0 & \text{for } x \in \Omega, \\ \mathcal{G}^i(\lambda, x, u, \dots, D^{k_i} u) = 0 & \text{for } x \in \partial\Omega,\ 1 \le i \le r, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $u\colon \overline{\Omega} \to \mathbb{R}^m$ is a vector function, λ is a parameter belonging to a smooth compact connected d-dimensional manifold Λ and, denoting with k^* the number of multindices α's such that $|\alpha| \le k$,

$$\mathcal{F}\colon \Lambda \times \overline{\Omega} \times \mathbb{R}^{mk^*} \to \mathbb{R}^m \quad \text{and} \quad \mathcal{G}^i\colon \Lambda \times \overline{\Omega} \times \mathbb{R}^{mk_i^*} \to \mathbb{R}$$

are smooth with $\mathcal{F}(\lambda, x, 0) = 0$, $\mathcal{G}^i(\lambda, x, 0) = 0$, $1 \le i \le r$.

Here and below we will freely use the notation from [12].

We will denote with $(\mathcal{F}, \mathcal{G})$ the family of nonlinear differential operators

$$(2.2)\qquad (\mathcal{F}, \mathcal{G})\colon \mathbb{R}^q \times C^\infty(\overline{\Omega}; \mathbb{R}^m) \to C^\infty(\overline{\Omega}; \mathbb{R}^m) \times C^\infty(\partial\Omega; \mathbb{R}^r)$$

defined by

$$\begin{aligned}(\mathcal{F}, \mathcal{G})(\lambda, u) = (\mathcal{F}(\lambda, x, u, \dots, D^k u), \tau\mathcal{G}^1(\lambda, x, u, \dots, D^{k_1} u), \dots, \\ \tau\mathcal{G}^r(\lambda, x, u, \dots, D^{k_r} u),\end{aligned}$$

where τ is the restriction to the boundary.

We assume:

(H$_1$) For all $\lambda \in \Lambda$, the linearization $(\mathcal{L}_\lambda(x, D), \mathcal{B}_\lambda(x, D))$ of $(\mathcal{F}_\lambda, \mathcal{G}_\lambda)$ at $u \equiv 0$, is an elliptic boundary value problem. Namely, $\mathcal{L}_\lambda(x, D)$ is elliptic, properly elliptic at the boundary, and the boundary operator

$$\mathcal{B}_\lambda(x, D) = (\mathcal{B}^1_\lambda(x, D), \dots, \mathcal{B}^r_\lambda(x, D))^t$$

verifies the Shapiro–Lopatinskiĭ condition with respect to $\mathcal{L}_\lambda(x, D)$.

(H$_2$) There exists a point $\nu \in \Lambda$ such that, for every $f \in C^\infty(\overline{\Omega}; \mathbb{C}^m)$ and $g \in C^\infty(\partial\Omega; \mathbb{C}^r)$, the problem:

$$\begin{cases} \mathcal{L}_\nu(x, D)u(x) = f(x) & \text{for } x \in \Omega, \\ \mathcal{B}_\nu(x, D)u(x) = g(x) & \text{for } x \in \partial\Omega, \end{cases}$$

has a unique smooth solution.

(H$_3$) (i) The coefficients $b^i_\alpha(\lambda, x)$, $|\alpha| = k_i$, $1 \le i \le r$, of the leading terms of boundary operators $\mathcal{B}^1_\lambda(x, D), \dots, \mathcal{B}^r_\lambda(x, D)$ are independent of λ.

(ii) There exist a compact set $K \subset \Omega$ such that the coefficients $a_\alpha(\lambda, x)$; $|\alpha| = k$ of the leading terms of $\mathcal{L}_\lambda$, are independent of λ for x in $\overline{\Omega} - K$.

Let $p(\lambda, x, \xi) \equiv \sum_{|\alpha|=k} a_\alpha(\lambda, x)\xi^\alpha$ be the principal symbol of $\mathcal{L}_\lambda$.

By ellipticity, $p(\lambda, x, \xi) \in \mathrm{GL}(m; \mathbb{C})$ if $\xi \neq 0$. On the other hand, by (H_3),

$$p(\lambda, x, \xi) = p(\nu, x, \xi) \quad \text{for } x \in \overline{\Omega} - K.$$

Therefore, putting

$$\sigma(\lambda, x, \xi) = \mathrm{id} \quad \text{for any } (\lambda, x, \xi) \text{ with } x \notin K,$$

the map $\sigma(\lambda, x, \xi) = p(\lambda, x, \xi)p(\nu, x, \xi)^{-1}$ extends to a smooth map

$$\sigma: \Lambda \times (\mathbb{R}^{2n} - K \times \{0\}) \to \mathrm{GL}(m; \mathbb{C}). \tag{2.3}$$

Assuming, without loss of generality, that $K \times \{0\}$ is contained in the unit ball $B^{2n} \subset \mathbb{R}^{2n}$, we associate to σ the restriction (pullback) of the matrix one-form $\sigma^{-1}d\sigma$ to $\Lambda \times \partial B^{2n} \simeq \Lambda \times S^{2n-1}$, which will be denoted in the same way.

Taking the trace of the $(q + 2n - 1)$-th power of the matrix $\sigma^{-1}d\sigma$ we obtain an ordinary $(q + 2n - 1)$-form $\mathrm{tr}(\sigma^{-1}d\sigma)^{q+2n-1}$ on $\Lambda \times S^{2n-1}$.

Let $\Sigma \simeq S^q \subset \Lambda$ be an embedded sphere of even dimension q. We define the *degree* $\deg(\sigma; \Sigma)$ of σ on Σ by

$$\deg(\sigma; \Sigma) = \frac{(q/2 + n - 1)!}{(2\pi i)^{(q/2+n)}(q + 2n - 1)!} \int_{\Sigma \times S^{2n-1}} \mathrm{tr}(\sigma^{-1}d\sigma)^{q+2n-1}. \tag{2.4}$$

We will see later that, for any embedded sphere Σ, $\deg(\sigma; \Sigma)$ is an integral number.

Let us recall that a *bifurcation point from the trivial branch* for solutions of (2.1) is a point $\lambda_* \in \Lambda$ such that there exist a sequence $(\lambda_n, u_n) \in \Lambda \times \mathbb{C}^\infty(\overline{\Omega})$ of solutions of (2.1) with $u_n \neq 0$, $\lambda_n \to \lambda_*$ and $u_n \to 0$ uniformly with all of its derivatives.

Theorem 2.1. *Let the boundary value problem* (2.1) *verify* (H_1)–(H_3) *and let p be an odd prime such that $p \leq d/2 + 1$. If, Λ is orientable and, for some embedded sphere $\Sigma \subset \Lambda$, of dimension $q = 2(p - 1)$,* $\deg(\sigma; \Sigma)$ *is not divisible by p, then*

- (a) *the Lebesgue covering dimension of the set B of all bifurcation points of* (2.1) *is at least $d - q$,*
- (b) *the set B either disconnects Λ or is not contractible in Λ to a point.*

3. Characteristic classes and bifurcation of Fredholm maps

We begin with a short recapitulation of [12]. From now on, Fredholm means Fredholm of index 0.

Let O be an open subset of a Banach space X and let Λ be a finite connected CW-complex. A family of C^1-Fredholm maps continuously parametrized by Λ is a continuous map $f: \Lambda \times O \to X$ such that the map $f_\lambda: O \to Y$ defined by $f_\lambda(x) = f(\lambda, x)$ is C^1, for each $\lambda \in \Lambda$. Moreover, $Df_\lambda(x)$ is a Fredholm operator of index 0 which depends continuously on (λ, x) with respect to the norm topology in the space of $\mathcal{L}(X, Y)$.

We will assume everywhere in this paper that O is a neighbourhood of the origin $0 \in X$ and that $f(\lambda, 0) = 0$ for all λ in Λ. Solutions $(\lambda, 0)$ of the equation $f(\lambda, x) = 0$ form the *trivial branch*, which we will identify with the parameter space Λ.

A point λ_* in Λ is called *bifurcation point from the trivial branch* for solutions of the equation $f(\lambda, x) = 0$ if every neighbourhood of $(\lambda_*, 0)$ contains nontrivial solutions of this equation.

The *linearization of the family f along the trivial branch* is the family of operators $L\colon \Lambda \to \Phi_0(X, Y)$ defined by $L_\lambda = Df_\lambda(0)$, where the right hand side denotes the Frechet derivative of f_λ at 0.

Bifurcation only occurs at points $\lambda \in \Lambda$ where L_λ is singular but, in general, the set $B(f)$ of all bifurcation points of a family f is only a proper closed subset of the set $\Sigma(L)$ of all singular points of L.

Given a compact space Λ, let $\mathrm{KO}(\Lambda)$ (resp. $K(\Lambda)$) be the Grothendieck group of all real (resp. complex) virtual vector bundles over Λ, and let $\widetilde{\mathrm{KO}}(\Lambda)$ (resp. $\widetilde{K}(\Lambda)$ be the corresponding reduced group, i.e. the kernel of the rank homomorphism. Recalling that two vector bundles are *stably equivalent* if they become isomorphic after addition of trivial bundles to both sides, in this paper, we will identify $\widetilde{\mathrm{KO}}(\Lambda)$ with the group of all stable equivalence classes of vector bundles over Λ.

With the above identification the *index bundle* $\operatorname{Ind} L$ of a family L of Fredholm operators is defined as follows: using compactness of Λ, one can find a finite dimensional subspace V of Y such that

$$\operatorname{Im} L_\lambda + V = Y \quad \text{for all } \lambda \in \Lambda. \tag{3.1}$$

Because of (3.1) the family $E_\lambda = L_\lambda^{-1}(V)$, $\lambda \in \Lambda$ is a vector bundle E. By definition, $\operatorname{Ind} L = [E] \in \widetilde{\mathrm{KO}}(\Lambda)$, where $[E]$ denotes the stable equivalence class of E. For families of Fredholm operators between complex Banach spaces the same construction produces an element $\operatorname{Ind} L \in \widetilde{K}(\Lambda)$.

If $f(\lambda, x) = L_\lambda x$, where $\{L_\lambda\}_{\lambda \in \Lambda}$ is a family of linear Fredholm operators, then the set of singular points $\Sigma(L)$ coincides with the set of bifurcation points of f. By definition of the index bundle, $\Sigma(L) = B(f)$ is nonempty whenever $\operatorname{Ind} L \neq 0$ in $\widetilde{\mathrm{KO}}(\Lambda)$. Hence, in the case of linear families, bifurcation is caused by the nonvanishing of the index bundle. However, in order to detect the presence of bifurcation for a family of nonlinear Fredholm maps $\operatorname{Ind} L$ is not sufficient, and we have to resort to the image of $\operatorname{Ind} L$ by the generalized J-homomorphism $J\colon \widetilde{\mathrm{KO}}(\Lambda) \to J(\Lambda)$ [2].

Let us recall that two vector bundles E, F are fiberwise homotopy equivalent if there is a fiber preserving homotopy equivalence between the corresponding unit sphere bundles $S(E)$ and $S(F)$. Moreover, E and F are said to be *stably fiberwise homotopy equivalent* (shortly sfh-equivalent) if they become fiberwise homotopy equivalent after addition of trivial bundles to both sides.

Let $J(\Lambda)$ be the quotient group of $\widetilde{\mathrm{KO}}(\Lambda)$ by the subgroup generated by elements of the form $[E]-[F]$ with E sfh-equivalent to F. The *generalized J-homomorphism* $J\colon\widetilde{\mathrm{KO}}(\Lambda)\to J(\Lambda)$ is the projection to the quotient. By definition, $J([E])$ vanishes in $J(\Lambda)$ if and only if E is sfh-trivial. The groups $J(\Lambda)$ were introduced by Atiyah in [2] who showed that, if Λ is a finite CW-complex, the group $J(\Lambda)$ is finite.

Let $f\colon\Lambda\times O\to Y$ be a continuous family of C^1-Fredholm maps (of index 0) parametrized by a finite connected CW-complex Λ, such that $f(\lambda,0)=0$. The *index of bifurcation points* $\beta(f)\in J(\Lambda)$ of the family f is defined by

$$\beta(f)=J(\operatorname{Ind}L).$$

Theorem 1.2.1 in [12] states that, if $\beta(f)\neq 0$ in $J(\Lambda)$ and $\Sigma(L)$ is a proper subset of Λ, then the family f possesses at least one bifurcation point from the trivial branch.

An n-dimensional vector bundle (n-plane bundle) is a vector bundle $\pi\colon E\to\Lambda$ such that $\dim E_\lambda=n$ for all $\lambda\in\Lambda$. Let $\mathrm{Vect}^n(\Lambda)$ be the set of all isomorphism classes of n-plane bundles over Λ.

If R is a ring, a characteristic class $c\colon\mathrm{Vect}^n(\,\cdot\,)\to H^*(\,\cdot\,;R)$ is said to be of sfh-type (or spherical) if it depends only on the sfh-equivalence class of the vector bundle.

Spherical characteristic classes detect elements with nontrivial J-image. Here we will consider only Wu classes with values in the singular cohomology theory $H^{(2p-1)*}(\,\cdot\,,\mathbb{Z}_p)$, if p is an odd prime, and Stiefel-Whitney classes, for $p=2$.

Below we collect the needed properties of Wu classes. We will denote with X^* the Alexander one-point compactification of a locally compact space X. X^* is naturally a pointed space. A proper map $f\colon X\to Y$ extends uniquely to a map $\overline{f}\colon X^*\to Y^*$ preserving base points. Moreover, $(X\times Y)^*$ is homeomorphic to the wedge

$$X^*\wedge Y^*=X^*\times Y^*/(X^*\times\{\infty\}\cup\{\infty\}\times Y^*).$$

This makes the one point compactification into a product preserving functor from the category of locally compact spaces to the category of pointed compact spaces.

Thom space of an n-plane bundle $\pi\colon E\to\Lambda$ is the one point compactification E^* of its total space E. Using the above homeomorphism with $X=E$ and Y

a trivial m-plane bundle over a point we conclude that the Thom space of $E\oplus\theta^m$ is the m-th suspension of the Thom space of E.

Let $r_\lambda\colon E_\lambda^* \to E^*$ be the extension of the inclusion of the fiber E_λ into E. An *orientation* (*Thom*) *class* for the vector bundle E is an element $u \in \widetilde{H}^n(E^*;\mathbb{Z}_p)$ such that, for all λ, $r_\lambda^*(u)$ is a generator of $\widetilde{H}^n(E_\lambda^*;\mathbb{Z}_p) \simeq \mathbb{Z}_p$.

It is easy to see that every n-plane bundle admits an orientation over $\mathbb{Z}_2$, and that a bundle is orientable over Z_p with $p > 2$ if and only if it is orientable, i.e. it admits a reduction of its structure group to $SO(n)$.

The map $d\colon E \to \Lambda \times E$ defined by $d(v) = (\pi(v), v)$, being proper, extends to a map $\delta\colon E^* \to \Lambda^* \wedge E^*$. Composing the wedge product

$$\wedge\colon \widetilde{H}^*(\Lambda^*;\mathbb{Z}_p) \times \widetilde{H}^*(E^*;\mathbb{Z}_p) \to \widetilde{H}^*(\Lambda^* \wedge E^*;\mathbb{Z}_p)$$

with $\delta^*\colon \widetilde{H}^*(\Lambda^* \wedge E^*;\mathbb{Z}_p) \to \widetilde{H}^*(E^*;\mathbb{Z}_p)$ and using $H^*(\Lambda;\mathbb{Z}_p) \simeq \widetilde{H}^*(\Lambda^*;\mathbb{Z}_p)$ we obtain a cup product

$$\cup\colon H^*(\Lambda;\mathbb{Z}_p) \times \widetilde{H}^*(E^*;\mathbb{Z}_p) \to \widetilde{H}^*(E^*;\mathbb{Z}_p).$$

Thom's isomorphism theorem states that, if $u \in \widetilde{H}^n(E^*;\mathbb{Z}_p)$ is a Thom class for E, the homomorphism

$$\Psi_u\colon H^*(\Lambda;\mathbb{Z}_p) \to \widetilde{H}^{*+n}(E^*;\mathbb{Z}_p)$$

defined by $\Psi_u(a) = a \cup u$ is an isomorphism. Let p be an odd prime. The *k-th Wu characteristic class* $q_k(E) \in H^{2(p-1)k}\ (\Lambda;\mathbb{Z}_p)$ of an n-plane bundle E orientable over $\mathbb{Z}_p$ is defined by

$$q_k(E) = \Psi_u^{-1} P^k u = \Psi_u^{-1} P^k \Psi_u(1), \tag{3.2}$$

where

$$P^k\colon \widetilde{H}^n(E^*;\mathbb{Z}_p) \to \widetilde{H}^{n+2(p-1)k}(E^*;\mathbb{Z}_p)$$

is the k-th Steenrod reduced power [13].

One of the consequences of the Thom isomorphism theorem is that a module $\widetilde{H}^*(E^*;\mathbb{Z}_p)$ is free over the ring $H^*(\Lambda;\mathbb{Z}_p)$ generated by u via the cup product defined above.

Any two Thom classes $u, u' \in \widetilde{H}^n(E^*;\mathbb{Z}_p)$ are related by $u = a \cup u'$ with $a \in H^0(\Lambda;\mathbb{Z}_p)$ invertible. Since P^k are module homomorphisms substituting $u = a \cup u'$ in (3.2) we obtain that the classes $q_k(E)$ are independent from the choice of the Thom class u. Since the suspension homomorphism commutes with r_λ^*, from the characterizing property of Thom's class it follows that the m-th suspension $u' = \sigma^m(u)$ of a Thom class u of E is a Thom class for $E \oplus \theta^m$. Moreover, since the cup product verifies $a \cup \sigma u = \sigma(a \cup u)$, we have $\Psi_{u'} = \sigma^m \Psi_u$. On the other hand also P^k commute with the suspension. Hence, we get

$$q_k(E \oplus \theta^m) = \Psi_{u'}^{-1} P^k u' = \Psi_u^{-1} \sigma^{-m} P^k \sigma^m u = \Psi_u^{-1} P^k u = q_k(E).$$

Thus q_k depends only on the stable equivalence class of E, and hence we have a well defined natural transformation $q_k\colon \widetilde{\mathrm{KSO}}(\,\cdot\,) \to H^{2(p-1)k}(\,\cdot\,;\mathbb{Z}_p)$, where $\widetilde{\mathrm{KSO}}(\,\cdot\,)$ is the ring of stable equivalence classes of orientable bundles. As a matter of fact, the classes q_k can be defined for all elements of $\widetilde{\mathrm{KO}}(\,\cdot\,)$, but we will not use this here.

On the other hand, a fiberwise homotopy equivalence $h\colon S(E) \to S(F)$ by radial extension produces a proper homotopy equivalence between the total spaces of E and F and hence a base point preserving homotopy equivalence $\overline{h}\colon E^* \to F^*$. This later restricts to a homotopy equivalence $\overline{h}_\lambda\colon E^*_\lambda \to F^*_\lambda$. It follows from this that, if v is an orientation for F, then $u = \overline{h}^*(v)$ is an orientation for E and moreover $\Psi_u = \overline{h}^*\Psi_v$.

If v is an orientation for F, then $u = \overline{h}^*(v)$ is an orientation for E and moreover $\Psi_u = \overline{h}^*\Psi_v$. Substituting in (3.2) we get $q_k(E) = q_k(F)$. Thus q_k depends only on the fiber preserving homotopy class of the sphere bundle $S(E)$ and hence $q_k\colon \widetilde{\mathrm{KSO}}(\,\cdot\,) \to H^{2(p-1)k}(\,\cdot\,;\mathbb{Z}_p)$ factorizes through the functor $J(\,\cdot\,)$.

The same holds for the Stiefel–Whitney classes $\omega_k \in H^k(\,\cdot\,;\mathbb{Z}_2)$ since they are constructed from the Thom class of E using Steenrod squares Sq^k in (3.2).

With this at hand we can state the main result of this section.

Let us first recall that the covering dimension of a topological space X is defined to be the minimum value of n such that every open cover of X has an open refinement for which no point is included in more than $n+1$ elements. By a well known characterization due to Hurewicz, the topological dimension of a compact space X is at least n if, for some closed subset C of X, the Alexander–Spanier cohomology $H^n(X,C) \neq 0$.

Theorem 3.1. *Let Λ be a compact connected topological manifold and let $f\colon \Lambda \times O \to Y$ be a continuous family of C^1-Fredholm maps verifying $f(\lambda,0)=0$ and such that $\Sigma(L)$ is a proper subset of Λ.*

(a) *If Λ and $\operatorname{Ind} L$ are orientable and, for some odd prime p, there is a $k \geq 1$ such that $q_k(\operatorname{Ind} L) \neq 0$ in $H^{2(p-1)k}(\Lambda;\mathbb{Z}_p)$, then the Lebesgue covering dimension of the set $B(f)$ is at least $\dim \Lambda - 2(p-1)k$.*

(b) *If $\omega_k(\operatorname{Ind} L) \neq 0$ in $H^k(\Lambda;\mathbb{Z}_2)$ for some $k \geq 1$, then the dimension of $B(f)$ is at least $\dim \Lambda - k$.*

Moreover, either the set $B(f)$ disconnects Λ or it cannot be deformed in Λ into a point.

Proof (see also [6], [3]). We will denote with $\overline{H}^*(X;\mathbb{Z}_p)$ the Alexander–Spanier cohomology of X with $\mathbb{Z}_p$ coefficients. It is well known that $\overline{H}^*(X;\mathbb{Z}_p)$ coincides with the singular cohomology of X when X is a manifold.

Let $B = B(f)$. If $\dim \Lambda = m$ and $\Lambda - B$ is not connected then the covering dimension of B must be at least $m - 1$, since sets of smaller dimension cannot disconnect Λ. Hence, in this case, the theorem is proved.

From now we assume that $\Lambda - B$ is connected. Let $\alpha \in H_{2(p-1)k}(\Lambda; \mathbb{Z}_p)$ be any homology class such that the Kroenecker pairing $\langle q_k(\operatorname{Ind} L); \alpha\rangle \neq 0$ and let $\eta \in H^{m-2(p-1)k}(\Lambda; \mathbb{Z}_p)$ be the Poincaré dual of α.

Let $i \colon B \to \Lambda$ be the inclusion and let $\zeta = i^*(\eta) \in \overline{H}^*(B; \mathbb{Z}_p)$ be the restriction of η to B. If we can show that $\zeta \neq 0$ in $\overline{H}^*(B; \mathbb{Z}_p)$, then the theorem is proved. Indeed, $\zeta = i^*(\eta)$ is an obstruction to the deformation of the subspace B to a point and, by cohomological characterization of the covering dimension, $\dim B$ must be at least $m - 2(p-1)k$.

In order to show that $\zeta \neq 0$ let us consider the following commutative diagram

$$\begin{tikzcd}
& H^{m-2(p-1)k}(\Lambda; \mathbb{Z}_p) \arrow[r, "i^*"] & \overline{H}^{m-2(p-1)k}(B; \mathbb{Z}_p) \\
H_{2(p-1)k}(\Lambda - B; \mathbb{Z}_p) \arrow[r, "j^*"'] & H_{2(p-1)k}(\Lambda; \mathbb{Z}_p) \arrow[u] \arrow[r, "\pi_*"'] & H_{2(p-1)k}(\Lambda, \Lambda - B; \mathbb{Z}_p) \arrow[u]
\end{tikzcd}$$

where the vertical arrows are the Poincaré duality isomorphisms and the bottom sequence is the homology sequence of a pair.

By commutativity, ζ is dual to $\pi_*(\alpha)$. Hence, it is enough to show that the homology class $\pi_*(\alpha)$ does not vanish.

If so, by exactness, $\alpha = j_*(\beta)$ for some $\beta \in H_{2(p-1)k}(\Lambda - B; \mathbb{Z}_p)$. Since singular homology has compact supports, there exists a finite connected CW-complex P and a map $h \colon P \to \Lambda - B$ such that $\beta = h_*(\delta)$ for some $\delta \in H_{2(p-1)k}(P; \mathbb{Z}_p)$ (for this it is enough to take as P any closed connected polyhedral neighbourhood of the support of a singular cochain representing β).

Since $\Lambda - B$ is connected, we can assume without loss of generality that some point $\lambda_0 \in \Lambda - \Sigma(L)$ belongs to the image of h. Let us consider now $\overline{h} = jh$ and the family $\overline{f} \colon P \times X \to Y$ defined by $\overline{f}(p, x) = f(\overline{h}(p), x)$.

The linearization at the trivial branch of $\overline{f}$ is $\overline{L} = L\overline{h}$. Since $\lambda_0 = h(p_0)$ is not a singular of L, the set $\Sigma(\overline{L})$ is a proper subset of P. On the other hand, by construction, $\overline{h}$ sends bifurcation points of $\overline{f}$ into bifurcation points of f, and since $\overline{h}(P) \cap B = \emptyset$, the family $\overline{f}$ has no bifurcation points.

By [12, Theorem 1.2.1], $J(\operatorname{Ind} \overline{L}) = 0$ and hence all characteristic classes q_k of $\operatorname{Ind} \overline{L}$ must vanish. But $\operatorname{Ind} \overline{L} = \overline{h}^*(\operatorname{Ind} L)$, and by naturality of characteristic classes $q_k(\operatorname{Ind}(\overline{L})) = h^* j^* q_k(\operatorname{Ind} L)$. Hence,

$$0 = \langle h^* j^* [q_k(\operatorname{Ind} L)]; \delta\rangle = \langle q_k(\operatorname{Ind} L); j_*(\beta)\rangle = \langle q_k(\operatorname{Ind} L); \alpha\rangle,$$

which contradicts the choice of α.

The proof of (b) is similar. $\square$

4. Proof of the main theorem

Denoting by H^s the Sobolev–Hardy spaces, the map

$$(\mathcal{F},\mathcal{G})\colon \Lambda\times C^\infty(\Omega;\mathbb{R}^m)\to C^\infty(\Omega;\mathbb{R}^m)\times\prod_{i=1}^{r}C^\infty(\partial\Omega;\mathbb{R})$$

defined by (2.2) extends to a smooth map

$$h=(f,g)\colon \Lambda\times H^{k+s}(\Omega;\mathbb{R}^m)\to H^s(\Omega;\mathbb{R}^m)\times H^+(\partial\Omega;\mathbb{R}^r),$$

where, by definition, $H^+(\partial\Omega;\mathbb{R}^r)=\prod_{i=1}^{r}H^{k+s-k_i-1/2}(\partial\Omega;\mathbb{R})$.

By our assumptions, $u\equiv 0$ is a solution of $h_\lambda(u)=0$ for all $\lambda\in\Lambda$. Hence $\Lambda\times\{0\}$ is a trivial branch for h. The Frechet derivative Dh_λ at $u=0$ is the operator (L_λ,B_λ) induced on Hardy–Sobolev spaces by $(\mathcal{L}_\lambda,\mathcal{B}_\lambda)$.

It is shown in [12, Proposition 5.2.1] that, under the hypothesis of Theorem 2.1, we can find a neighbourhood O of 0 in $H^{k+s}(\Omega;\mathbb{C}^m)$ such that

$$h\colon \Lambda\times O\to H^s(\Omega;\mathbb{R}^m)\times H^+(\partial\Omega;\mathbb{R}^r)$$

is a smooth parametrized family of Fredholm maps of index 0. By [12, Proposition 5.2.2], the set of bifurcation points of (2.1) coincides with the set $\mathrm{Bif}(h)$ of bifurcation points of the map h. Moreover, denoting with (L,B) the linearization of h along the trivial branch, we have that $\nu\notin\Sigma(L,B)$.

If $\mathrm{Ind}\,(L,B)$ is nonorientable, $\omega_1\mathrm{Ind}\,(L,B)\neq 0$. By assertion (b) of Theorem 3.1, $\dim B\geq d-1$ and $\mathrm{Bif}(h)$ carries a nontrivial class of positive degree in cohomology with Z_2 coefficients. Hence, in this case, the conclusions of Theorem 2.1 hold regardless any condition on $d(\sigma,\Sigma)$.

If $\mathrm{Ind}\,(L,B)$ is orientable, the proof of Theorem 2.1 is obtained by relating the degree $\deg(\sigma;\Sigma)$ defined in (2.4) with the evaluation of the first Wu class of $\mathrm{Ind}\,(L,B)$ on the spherical homology class $[\Sigma]$. Only the first Wu class is of interest for us because, as we will see, all characteristic numbers obtained in this way from higher Wu classes vanish.

The relation between characteristic classes of the index bundle and the degree $\deg(\sigma;\Sigma)$ comes from the family version of the Agranovich reduction and the Atiyah–Singer theorem.

Let $\sigma\colon\Lambda\times(\mathbb{R}^{2n}-K\times\{0\})\to\mathrm{GL}(m;\mathbb{C})$ be the map defined by (2.3). In [12] we have constructed a smooth family $\mathcal{S}\colon\Lambda\to\mathrm{Ell}(\mathbb{R}^n)$ of elliptic pseudo-differential operators of order zero on $\mathbb{R}^n$ such that the principal symbol of $\mathcal{S}_\lambda$ coincides with σ_λ. The Agranovich reduction relates $\mathcal{S}$ with the family $(\mathcal{L},\mathcal{B})$ considered as a family of differential operators with complex coefficients.

More precisely, denoting by $S_\lambda\colon H^s(\Omega;\mathbb{C}^m)\to H^s(\Omega;\mathbb{C}^m)$ the operator induced by $\mathcal{S}_\lambda$ on Hardy–Sobolev spaces and with (L^c,B^c) the complexification of

(L, B), Theorem 4.1.1 of [12] states that in $\widetilde{K}(\Lambda)$

$$\operatorname{Ind}(L^c, B^c) = \operatorname{Ind} S.$$

Since the index bundle of the family (L^c, B^c) is the complexification of the (real) index bundle $\operatorname{Ind}(L, B)$, we have:

$$c(\operatorname{Ind}(L, B)) = \operatorname{Ind} S, \tag{4.1}$$

where $c\colon \widetilde{\mathrm{KO}}(\Lambda) \to \widetilde{K}(\Lambda)$ is the complexification homomorphism.

Using the Chern–Weil theory of characteristic classes of smooth vector bundles over smooth manifolds, in [4] Fedosov obtained an explicit expression for the differential form representing Chern character $\operatorname{ch}(\operatorname{Ind} S)$ in de Rham cohomology with complex coefficients. He showed that $\operatorname{ch}(\operatorname{Ind} S)$ is the cohomology class of the form

$$-\sum_{j=n}^{\infty} \frac{(j-1)!}{(2\pi i)^j (2j-1)!} \oint_{S^{2n-1}} \operatorname{tr}(\sigma^{-1}\, d\sigma)^{2j-1}, \tag{4.2}$$

where $\oint$ denotes the *integration along the fiber* (see [12, Appendix C]) and S^{2n-1} is the boundary of a ball in $\mathbb{R}^{2n}$ containing the set $\{(x, \eta)/ \det \sigma(\lambda, x, \eta) = 0\}$.

Using Fedosov's formula we will show that, under the hypothesis of Theorem 2.1, $q_1(\operatorname{Ind}(L, B)) \neq 0$ in $H^{2(p-1)}(\Lambda; \mathbb{Z}_p)$. Then Theorem 2.1 will follow immediately from Theorem 3.1(a) and [12, Proposition 5.2.1], which shows that the set $B(h)$ of bifurcation points of the map h coincides with the set B of bifurcation points for classical solutions of the system (2.1).

The rest of this section is devoted to show that $q_1(\operatorname{Ind}(L, B)) \neq 0$. For this, we will consider the restriction of the family h to $\Sigma \times H^{k+s}(\Omega; \mathbb{R}^m)$.

More precisely, if $q = 2(p-1)$ and $e\colon S^q \to \Lambda$ is an orientation preserving embedding with $\operatorname{Im} e = \Sigma$, let

$$\overline{h}\colon S^q \times H^{k+s}(\Omega; \mathbb{R}^m) \to H^s(\Omega; \mathbb{R}^m) \times H^+(\partial\Omega; \mathbb{R}^r)$$

be defined by $\overline{h}(\alpha, u) = h(e(\alpha), u)$.

The family $\overline{h}$ is the nonlinear Fredholm map induced in functional spaces by the pullback of the problem (2.1) to the sphere S^q. The linearization of $\overline{h}$ along the trivial branch is the family $(\overline{L}, \overline{B}) = (L, B) \circ e$.

Let $\overline{\sigma}(\alpha, x, \eta) = \sigma(e(\alpha), x, \eta)$ and $\overline{S} = S \circ e$. Then $\overline{S}$ is induced in functional spaces by the family of pseudo-differential operators $\overline{\mathcal{S}}$ with principal symbol $\overline{\sigma}$.

By the previous discussion, from (4.1) we get

$$c(\operatorname{Ind}(\overline{L}, \overline{B})) = \operatorname{Ind} \overline{S} \tag{4.3}$$

Let us show that, for $q = 2(p-1)$, the Kroenecker pairing

$$\langle q_1(\operatorname{Ind}(L, B)); e_*([S^q]\rangle \neq 0.$$

By naturality of characteristic classes and the index bundle we have

$$\langle q_1(\mathrm{Ind}\,(L,B)); e_*([S^q])\rangle = \langle q_1(\mathrm{Ind}\,(\overline{L},\overline{B})); [S^q]\rangle \tag{4.4}$$

In order to compute the right hand side let us recall that, putting $r = (p-1)/2$, the Wu class $q_k(E)$ of an n-plane bundle over X can be written as a polynomial $K_{rk}(p_1, \dots, p_{rk})$ in Pontriagin classes reduced $\mathrm{mod}\, p$. The polynomials K_{rk} are with $\mathbb{Z}_p$ coefficients and belong to a multiplicative sequence associated to the function $\phi(t) = 1 + t^r$ [10].

To shorten notations, let $\eta = \mathrm{Ind}\,(\overline{L},\overline{B})$ and $\eta^c = c(\mathrm{Ind}\,(\overline{L},\overline{B}))$.

For vector bundles over S^{4r}, we have $p_i(\eta) = 0$ for $i < r$ and $q_1(\eta) = \pm r p_r(\eta)$ reduced $\mathrm{mod}\, p$. Indeed, it follows from Lemma 1.4.1 in [7] and Newton's identity relating power sums to elementary symmetric functions, that, over the integers, the coefficient of the integral Pontriagin class p_r in K_r is given by $s_r(0, \dots, 0, 1) = \pm r$.

By Bott's integrality theorem [8, Section 18.9], the Chern character $\mathrm{ch}(\eta^c) = \mathrm{ch}_{2r}(\eta^c)$ is an integral class. Moreover, using (4.3), for the integral Pontriagin class p_r it holds that

$$p_r(\eta) = (-1)^r c_{2r}(\eta^c) = \pm(2r-1)!\mathrm{ch}_{2r}(\mathrm{Ind}\,\overline{S}),$$

which gives

$$\langle q_1(\mathrm{Ind}\,(\overline{L},\overline{B})); [S^{4r}]\rangle = \pm r(2r-1)!\langle \mathrm{ch}_{2r}(\mathrm{Ind}\,\overline{S}); [S^{4r}]\rangle \mathrm{mod}\, p. \tag{4.5}$$

Here we denote in the same way the fundamental class in homology with coefficients in $\mathbb{Z}$ and $\mathbb{Z}_p$.

By Fedosov's formula (4.2), the differential form representing $\mathrm{ch}_{2r}(\mathrm{Ind}\,\overline{S})$ in de Rham cohomology is

$$\Omega = \frac{(n+2r-1)!}{(2\pi i)^{(n+2r)}(2n+4r-1)!}\oint_{S^{2n-1}} \mathrm{tr}(\overline{\sigma}^{-1} d\overline{\sigma})^{2n+4r-1}.$$

Since the cohomology class of Ω belongs to $H^{4r}(S^{4r};\mathbb{Z})$, we have that

$$\begin{aligned}\langle \mathrm{ch}_{2r}(S); [S^{4r}]\rangle &= \frac{(n+2r-1)!}{(2\pi i)^{(n+2r)}(2n+4r-1)!}\int_{S^{4r}}\oint_{S^{2n-1}} \mathrm{tr}(\overline{\sigma}^{-1} d\overline{\sigma})^{2n+4r-1} \\ &= \frac{(n+2r-1)!}{(2\pi i)^{(n+2r)}(2n+4r-1)!}\int_{S^{4r}\times S^{2n-1}} \mathrm{tr}(\overline{\sigma}^{-1}\, d\overline{\sigma})^{2n+4r-1}\end{aligned}$$

is an integer. The last term of the above expression coincides with $\deg(\sigma;\Sigma)$ defined in (2.4). Thus, from (4.5) we obtain

$$\langle q_1(\mathrm{Ind}\,(\overline{L},\overline{B})); [S^{4r}]\rangle = \pm r(2r-1)!\deg(\sigma;\Sigma) \quad \mathrm{mod}\, p.$$

Since $r(2r-1)!$ is not divisible by p, by (4.4) $q_1(\mathrm{Ind}\,(L,B)) \neq 0$ whenever $\deg(\sigma;\Sigma)$ is not divisible by p. This concludes the proof of the theorem. $\square$

Notice that the above calculation gives

$$\langle q_k(\operatorname{Ind}(\overline{L},\overline{B}));[S^{4r}]\rangle = 0, \quad \text{for } k>1.$$

5. Other results

In this section, we will obtain sufficient conditions for bifurcation of solutions for a particular class of nonlinear elliptic boundary value problems (2.1) on dimensions not covered by Theorem 2.1.

More precisely, we substitute m with $m'=2m$ and r with $r'=2r$ in (2.1) and we denote with $(\mathcal{L}',\mathcal{B}')$ the linearization of (2.1) along the trivial branch in both hypotheses (H_2) and (H_3). But instead of (H_1) we assume:

(H_1') The principal part of the linearization $(\mathcal{L}',\mathcal{B}')$ at the trivial branch is obtained from a family of elliptic boundary value problems for linear partial differential operators with complex coefficients

$$(\mathcal{L},\mathcal{B})\colon \mathbb{R}^q\times C^\infty(\Omega;\mathbb{C}^m)\to C^\infty(\Omega;\mathbb{C}^m)\times\ C^\infty(\partial\Omega;\mathbb{C}^r)$$

by forgetting the complex structure.

Notice that (H_1') is verified by semilinear equations arising as real lower-order perturbations of families of linear elliptic boundary value problems with complex coefficients. It also holds in the case of quasilinear elliptic problems for functions with values in $\mathbb{C}^m$ of the form

$$\mathcal{F}(\lambda,x,u,D^ku)=\sum_{|\alpha|=k}a_\alpha(\lambda,x,u,D^{k-1}u)D^\alpha u+\text{l.o.t.}, \tag{5.1}$$

where $a_\alpha\in\mathbb{C}^{m\times m}$, and the boundary map g defined in the same way. In this case the assumption (H_1') is verified because, for maps $\mathcal{F}_\lambda$ defined by (5.1), the principal part of the linearization at 0 is given by

$$\mathcal{L}_\lambda v=\sum_{|\alpha|=k}a_\alpha(\lambda,x,0)D^\alpha v,$$

and similarly for $\mathcal{G}$.

Assuming H_1', we will use the principal symbol $p(\lambda,\xi)$ of the complex differential operator $\mathcal{L}$ to define

$$\sigma\colon\Lambda\times(\mathbb{R}^{2n}-K\times\{0\})\to\mathrm{GL}(m;\mathbb{C})$$

in the same way as in (2.3). Moreover, given a q-sphere Σ embedded in Λ, we define $\deg(\sigma;\Sigma)$ by the equation (2.4).

Theorem 5.1. *Let the boundary value problem* (2.1) *verify* (H_1'), (H_2) *and* (H_3). *If, for some sphere* Σ *of dimension* $q=2,4$ *embedded in* Λ, *the number* $\deg(\sigma;\Sigma)$ *defined by* (2.4) *is odd, then the Lebesgue covering dimension of the* *set B of all bifurcation points of* (2.1) *is at least $d - q$ and the set $B(f)$ either disconnects Λ or is not contractible in Λ to a point.*

Proof. A complex Fredholm operator of index 0 is still Fredholm, with the same index, when viewed as a real operator. Hence, from (H'_1) it follows that the family $(L'B')$ induced by $(\mathcal{L}', \mathcal{B}')$ is a family of Fredholm operators of index 0 and moreover (L'_ν, B'_ν) is invertible.

In the same way as in the previous section, we can find a neighbourhood O of $u \equiv 0$ such that $h = (f, g)\colon \Lambda \times O \to H^s(\Omega; \mathbb{R}^{m'}) \times H^+(\partial\Omega; \mathbb{R}^{r'})$ is a smooth parametrized family of Fredholm maps of index 0 whose linearization at the trivial branch is the family (L', B').

If $\mathcal{S}$ is the family of pseudo-differential operators associated to σ, then, by Theorem 4.1.1 of [12], $\operatorname{Ind}(L, B) = \operatorname{Ind} S$.

Let $r\colon \widetilde{K}(\Lambda) \to \widetilde{\mathrm{KO}}(\Lambda)$ be the homomorphism obtained by forgetting the complex structure. We will compute the Stiefel–Whitney classes of $r(\operatorname{Ind}(S))$ by restricting them to Σ. As before, let e be an embedding of the sphere S^q, $q = 2s$, whose image is Σ.

For any element $\eta \in \widetilde{K}(S^{2s})$, the Chern character $\mathrm{ch}(\eta) = \mathrm{ch}_s(\eta)$ is an integral class. Moreover, the Chern class $c_s(\eta) = \pm(s-1)!\mathrm{ch}_s(\eta)$.

The Stiefel–Whitney class $\omega_q(\eta)$ is the image of c_s under the change of coefficients $\rho\colon H^q(S^q; \mathbb{Z}) \to H^q(S^q; \mathbb{Z}_2)$ and therefore, $\langle \omega_q r(\eta); [S^q] \rangle$ is the mod 2 reduction of $(s-1)!\langle \mathrm{ch}_s(\eta); [S^q] \rangle$. Thus, if $s = 1, 2$ and $\langle \mathrm{ch}_s(\eta); [S^q] \rangle$ is odd, $\omega_q r(\eta) \neq 0$ in $H^q(S^q; \mathbb{Z}_2)$.

Using Fedosov's formula for the Chern character of the index bundle as before, we obtain $\langle \mathrm{ch}_s(e^*(\operatorname{Ind} S)); [S^q] \rangle = \deg(\sigma; \Sigma)$. Taking $\eta = e^*(\operatorname{Ind} S)$ in the above discussion, if $q = 2, 4$ and $d(\sigma; \Sigma)$ is odd, then $e^* \omega_q r(\operatorname{Ind} S) \neq 0$ in $H^q(S^q; \mathbb{Z}_2)$ and hence $\omega_q r(\operatorname{Ind}(L, B)) = \omega_q r(\operatorname{Ind} S) \neq 0$ in $H^q(\Lambda; \mathbb{Z}_2)$ as well. Thus $B(h)$ verifies the conclusion (b) of Theorem 3.1 and the bootstrap [12, Proposition 5.2.1] concludes the proof. □

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