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ON HOMOTOPY CONLEY INDEX FOR MULTIVALUED FLOWS IN HILBERT SPACES

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Abstract. An approximation approach is applied to obtain a homotopy version of the Conley type index in Hilbert spaces considered by the first author and W. Kryszewski. The definition given in the paper is more elementary and, as a by-product, gives a natural connection between indices of Kunze and Mrozek in a finite-dimensional case. Some geometric properties of the index from a paper of the second author are discussed in an infinite dimensional situation. Some additional properties for gradient differential inclusions are also presented.

1. Introduction

In [5] C. Conley developed the theory of a homotopy index for invariant sets of a dynamical system in a locally compact metric space. This invariant, now called the Conley index, has become a very useful tool in the studies of the behaviour of various dynamical systems. It has many versions and generalizations motivated by several applications. One of them is the Conley index in Hilbert spaces for flows generated by vector fields of special form called LS-vector fields, as defined by K. Gęba, M. Izydorek and A. Pruszko in [10]. This index and its cohomology version has numerous applications, e.g. to Hamiltonian systems

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and elliptic problems (comp. [12], [14], [15]). On the other hand, a cohomological version of the Conley index for multivalued flows has been introduced by M. Mrozek [20] and applied to differential inclusions in $\mathbb{R}^n$. An approximation approach to the homotopy index for inclusions in $\mathbb{R}^n$ was presented by M. Kunze in [18], [19], and developed in [9]. In [6], [8] the first author and W. Kryszewski constructed a cohomological index in Hilbert spaces for multivalued LS-vector fields. It has been applied then to the study of Hamiltonian inclusions which generalize the classical problems to the nonsmooth case of locally Lipschitz potentials (see also [7]).

The main purpose of the present paper is to develop a homotopy Conley index theory based on [10] by the use of a graph approximation approach, which seems to be much more elementary. Let us note that in [13] a situation without uniqueness of solutions was also examined by different methods. In the preliminary section we prove appropriate approximation results. After the brief part (Section 3) presenting main information on multivalued flows, the main Section 4 contains a construction of the homotopy index by the use of graph approximation techniques. Our index is (at least formally) stronger than the invariant used in [8]. As an immediate corollary of our definitions and properties of the index it follows that the index introduced in [8] is obtained by taking a cohomology functor on our homotopy index (see Proposition 4.14). As a by-product we obtain that the same relation holds between the Mrozek and Kunze indices (which is quite natural).

In Section 4 we also address the following question: can we describe the homotopy index by a behaviour of the L-vector field $L + F$ on the boundary of a prescribed set of constraints? The answer (Theorem 4.17) generalizes the results obtained by the second author in [9].

We finish our paper (Section 5) by a short study of finite dimensional gradient differential inclusions. In particular, we give a simple sufficient condition for the existence of heteroclinic orbits (Corollary 5.6).

2. Preliminaries

Let X, Y be metric spaces. By a *set-valued map* φ from X into Y (written $\varphi\colon X \multimap Y$) we mean a map that assigns to each $x \in X$ a *closed nonempty* subset $\varphi(x)$ of Y. If, for any closed (resp. open) set $U \subset Y$, the *preimage* $\varphi^{-1}(U) := \{x \in X \mid \varphi(x) \cap U \neq \emptyset\}$ is closed (resp. open), then we say that φ is *upper* (resp. *lower*) semicontinuous (written usc (resp. lsc)); a map φ is *continuous* if it is upper and lower semicontinuous simultaneously. The *graph* $\mathrm{Gr}\,(\varphi) := \{(x, y) \in X \times Y \mid y \in \varphi(x)\}$ of an upper semicontinuous map φ is closed. A map φ is upper semicontinuous and has compact values (i.e. for each $x \in X$, the set $\varphi(x)$ is compact) if and only if, for any sequence $(x_n, y_n) \in \mathrm{Gr}\,(\varphi)$

such that $x_n \to x \in X$, there is a subsequence (y_{n_k}) such that $y_{n_k} \to y \in \varphi(x)$ (in other words the projection $p_\varphi\colon \mathrm{Gr}\,(\varphi) \to X$ is proper ([1])); in this case the *image* $\varphi(K) := \{y \in Y \mid y \in \varphi(x) \text{ for some } x \in K\}$ of any compact $K \subset X$ is compact. We say that a map φ is *compact* if it is upper semicontinuous and $\mathrm{cl}\,\varphi(X)$ is compact; φ is *completely continuous* if the restriction $\varphi|_B$ of φ to any bounded subset $B \subset X$ is compact.

A proper surjection $p\colon X \to Y$ is a *Vietoris map* if, for each $y \in Y$, the fibre $p^{-1}(y)$ is acyclic in the sense of the Alexander–Spanier cohomology. A map $p\colon (X, X') \to (Y, Y')$ of pairs (X, X'), (Y, Y') (i.e. $p\colon X \to Y$ and $p(X') \subset Y'$) is a *Vietoris* map, if p is a Vietoris map and $p^{-1}(Y') = X'$ (observe that the restriction $p'\colon X' \to Y'$ of p is a Vietoris map, too).

A map $\varphi\colon X \multimap Y$ is *admissible* (in the sense of Górniewicz) if there exist a space Γ, a Vietoris map $p\colon \Gamma \to X$ and a continuous map $q\colon \Gamma \to Y$ such that, for every $x \in X$, $\varphi(x) = q(p^{-1}(x))$. It is clear that admissible maps are upper semicontinuous with nonempty compact values.

The class of admissible maps is rich: for example any acyclic map $\varphi\colon X \multimap Y$ is admissible (φ is *acyclic* if it is upper semicontinuous and, for any $x \in X$, $\varphi(x)$ is acyclic); it is determined by the pair (p_φ, q_φ) where $p_\varphi\colon \mathrm{Gr}\,(\varphi) \to X$ and $q_\varphi\colon \mathrm{Gr}\,(\varphi) \to Y$ are the restrictions of the projections $X \times Y \to X$ and $X \times Y \to Y$, respectively. Moreover, a superposition of acyclic maps is admissible. For more details concerning admissible maps see [11].

Let us prove the following elementary:

Proposition 2.1. *Let X be a metric space, E be a linear normed space, and $F\colon X \multimap E$ a map with nonempty values. Then, for each $\varepsilon > 0$, there exists a continuous map $f\colon X \to E$ such that $f(x) \in \mathrm{conv}(F(B_\varepsilon(x)))$.*

Proof. For each $x \in X$ we can choose a point $v_x \in F(x)$. Let $\{\lambda_s\}_{s \in S}$ be a continuous partition of unity subordinated to the covering $\{V_s\}$, which is a locally finite refinement of the covering $\{B_\varepsilon(x)\}_{x \in X}$. For $s \in S$ we fix a point x_s such that $\mathrm{supp}\,\lambda_s \subset V_s \subset B_\varepsilon(x_s)$. Define a continuous map

$$f(x) := \sum_{s \in S} \lambda_s(x) v_s,$$

where $v_s = v_{x_s}$. If $s \in S_x = \{s \mid \lambda_s(x) \neq 0\}$, then $x \in B_\varepsilon(x_s)$. Thus $x_s \in B_\epsilon(x)$, and hence $v_s \in F(B_\varepsilon(x))$. Therefore $f(x) \in \mathrm{conv} F(B_\epsilon(x))$. $\square$

([1]) Recall that a continuous map $f\colon X \to Y$ is *proper* if, for each compact $K \subset Y$, the preimage $f^{-1}(K)$ is compact; it is worth reminding that f is proper if and only if it is *perfect*, i.e. continuous, closed and such that, for any $y \in Y$, $f^{-1}(y)$ is compact. Observe that a continuous surjection $f\colon X \to Y$ is perfect if and only if the multivalued map $Y \ni y \multimap f^{-1}(y) \subset X$ is upper semicontinuous and has compact values.

Remark 2.2. Observe that E could be a topological vector space in the previous proposition. The map f is locally Lipschitz if we take a locally Lipschitz partition of unity.

We say that a continuous map $f: X \to Y$ is *a graph ε-approximation* of $\varphi: X \multimap Y$ if $f(x) \in B_\varepsilon(\varphi(B_\varepsilon(x)))$ for every $x \in X$. The following is a version of a classical result of A. Cellina [2] combined with the previous observation.

Theorem 2.3. *Let $\varphi: X \multimap E$ be usc with convex values, where X is a metric space and E is a Banach space. Then, for every $\varepsilon > 0$, there exists a locally Lipschitz graph ε-approximation f of φ such that $f(x) \in \operatorname{conv}\varphi(B_\varepsilon(x))$ for every $x \in X$.*

Proof. Let $\varepsilon > 0$. From upper semicontinuity of φ it follows that for every $x \in X$ there exists $0 < \delta(x) < \varepsilon/2$ such that $\varphi(B_{\delta(x)}(x)) \subset B_\varepsilon(\varphi(x))$. Consider a locally finite covering $\{V_s\}_{s\in S}$ of X which is a star-refinement of the covering $\{B_{\delta(x)}(x)\}_{x\in X}$, i.e. stars $\operatorname{st}(V_t) = \bigcup\{V_s : V_s \cap V_t \neq \emptyset\}$ refine the covering $\{B_{\delta(x)}(x)\}_{x\in X}$. Let $\{\lambda_s\}_{s\in S}$ be a locally Lipschitz partition of unity subordinated to the covering $\{V_s\}$. For each $s \in S$ we choose a point $x_s \in V_s$ and some $y_s \in \varphi(x_s)$.

Define
$$f(x) := \sum_{s\in S} \lambda_s(x) y_s.$$

Let $S_x = \{s \in S \mid \lambda_s(x) \neq 0\}$ and let $s \in S_x$. Then $x \in V_s$. It implies that $d(x_s, x) < \delta(x_s) < \varepsilon$ and hence $x_s \in B_\varepsilon(x)$. Therefore
$$f(x) = \sum_{s\in S_x} \lambda_s(x) y_s \in \operatorname{conv}\varphi(B_\varepsilon(x)).$$

Moreover, since $x \in \bigcap_{s\in S_x} V_s$, there exists x' such that $\bigcup_{s\in S_x} V_s \subset B_{\delta(x')}(x')$. Thus both $x, x_s \in V_s$, and thus $d(x, x_s) < 2\delta(x') < \varepsilon$. By our choice of $\delta(x')$ we have $y_s \in B_\varepsilon(\varphi(x'))$. But the latter set is convex, thus $f(x) = \sum \lambda_s(x) y_s \in B_\varepsilon(\varphi(x')) \subset B_\varepsilon(\varphi(B_\varepsilon(x))$ and the proof is complete. □

Corollary 2.4. *If φ is completely continuous, then the approximation f in Theorem* 2.3 *is also completely continuous.*

Proof. For every bounded set $U \subset X$ we have $f(U) \subset \overline{\operatorname{conv}}\varphi(B_\varepsilon(U))$, and the latter set is relatively compact. □

3. Multivalued flows

Let X be a metric space.

Definition 3.1. By a *multivalued flow* on X we mean an upper semicontinuous mapping $\varphi\colon X \times \mathbb{R} \multimap X$ with nonempty and compact values such that, for every $s,t \in \mathbb{R}$ and $x,y \in X$,

(a) $\varphi(x,0) = \{x\}$;
(b) if $s \cdot t \geq 0$, then $\varphi(x,t+s) = \varphi(\varphi(x,t) \times \{s\})$;
(c) $y \in \varphi(x,t)$ if and only if $x \in \varphi(y,-t)$.

Let $\Delta \subseteq \mathbb{R}$. A map $\sigma\colon \Delta \to X$ is a Δ-*trajectory of* φ if, for every $t,s \in \Delta$, $\sigma(t) \in \varphi(\sigma(s), t-s)$. It is an easy exercise to prove that every trajectory is continuous. Indeed, let us consider a sequence t_n converging to t_0. Let $U \ni \sigma(t_0)$ be open. Since φ is upper semicontinuous, $\varphi^{-1}(U)$ is open and $(\sigma(t_0),0) \in \varphi^{-1}(U)$, because $\sigma(t_0) \in \varphi(\sigma(t_0),0)$. There exist $\delta > 0$ and an open set $V \subset X$ such that $(\sigma(t_0),0) \in V \times (-\delta,\delta) \subset \varphi^{-1}(U)$. Therefore, for a large n, $|t_n - t_0| < \delta$ and then $\sigma(t_n) \in \varphi(\sigma(t_0), t_n - t_0) \subset U$.

Let $x \in N \subseteq X$. The set of all Δ-trajectories *in* N *originating in* x (i.e. such that $0 \in \Delta$, $\sigma(0) = x$ and $\sigma(t) \in N$ for $t \in \Delta$) is denoted by $\mathrm{Tr}_N(\varphi;\Delta,x)$.

Define the *invariant, right-invariant, left-invariant* (with respect to φ) part of N by:

$$\begin{aligned}
\mathrm{Inv}(N,\varphi) &:= \{x \in N \mid Tr_N(\varphi;\mathbb{R},x) \neq \emptyset\},\\
\mathrm{Inv}^+(N,\varphi) &:= \{x \in N \mid Tr_N(\varphi;\mathbb{R}_+,x) \neq \emptyset\},\\
\mathrm{Inv}^-(N,\varphi) &:= \{x \in N \mid Tr_N(\varphi;\mathbb{R}_-,x) \neq \emptyset\},
\end{aligned}$$

respectively.

Definition 3.2. A subset $K \subset X$ is *invariant* (resp. *positively* (*negatively*) *invariant*) with respect to φ if $\mathrm{Inv}(K,\varphi) = K$ (resp. $\mathrm{Inv}^+(K,\varphi) = K$ ($\mathrm{Inv}^-(K,\varphi) = K$)).

Note that, given $N \subset X$, the set $K := \mathrm{Inv}(N,\varphi)$ is invariant with respect to φ and it is the maximal invariant subset of N.

Proposition 3.3 ([8, Proposition 3.9]). *Let Λ be a metric space, $N \subset X$ be closed and let $\eta\colon X \times \mathbb{R} \times \Lambda \multimap X$ be a family of multivalued flows* (*i.e. η is upper semicontinuous and, for each $\lambda \in \Lambda$, $\eta(\,\cdot\,,\lambda)\colon X \times \mathbb{R} \multimap X$ is a multivalued flow*). *Then the graph of the set-valued map*

$$\Lambda \ni \lambda \mapsto \mathrm{Inv}(N, \eta(\,\cdot\,,\lambda))$$

is closed, i.e. for any sequence $(x_n,\lambda_n) \in N \times \Lambda$ such that $x_n \in \mathrm{Inv}(N,\eta(\,\cdot\,,\lambda_n))$, if $(x,\lambda) = \lim_{n\to\infty}(x_n,\lambda_n)$, then $x \in \mathrm{Inv}(N,\eta(\,\cdot\,,\lambda))$.

Definition 3.4. A closed and bounded set $N \subset X$ is an *isolating neighbourhood* for φ if $\mathrm{Inv}(N,\varphi) \subset \mathrm{int}\, N$. We say that a set K invariant with respect to φ is *isolated* if there is an isolating neighbourhood N for φ such that $K = \mathrm{Inv}(N,\varphi)$.

In particular, if $X = \mathbb{R}^n$, then each isolating neighbourhood N for φ is compact and, by Proposition 3.3, $K = \mathrm{Inv}(N,\varphi)$ is closed in N, hence compact.

4. Conley index in Hilbert spaces

We shall assume the following:

Let $\mathbb{H} = (\mathbb{H}, \langle\,\cdot\,,\,\cdot\,\rangle)$ be a real Hilbert space and $L\colon \mathbb{H} \to \mathbb{H}$ a linear bounded operator with spectrum $\sigma(L)$. We assume the following

- $\mathbb{H} = \bigoplus_{k=0}^{\infty} \mathbb{H}_k$ with all subspaces $\mathbb{H}_k$ being mutually orthogonal and of finite dimension;
- $L(\mathbb{H}_0) \subset \mathbb{H}_0$ where $\mathbb{H}_0$ is the invariant subspace of L corresponding to the part of spectrum $\sigma_0(L) = i\mathbb{R} \cap \sigma(L)$ lying on the imaginary axis,
- $L(\mathbb{H}_k) = \mathbb{H}_k$ for all $k > 0$,
- $\sigma_0(L)$ is isolated in $\sigma(L)$, i.e. $\sigma_0(L) \cap \mathrm{cl}\,(\sigma(L) \setminus \sigma_0(L)) = \emptyset$.

Definition 4.1. A multivalued flow $\varphi\colon \mathbb{H} \times \mathbb{R} \multimap \mathbb{H}$ is called an *L-flow*, if it has the form

$$\varphi(x,t) = e^{tL}x + U(t,x),$$

where $U\colon \mathbb{H} \times \mathbb{R} \multimap \mathbb{H}$ is an admissible map which is completely continuous. Let Λ be a metric space. By a *family of L-flows* we understand a set-valued map $\eta\colon \mathbb{H}\times\mathbb{R}\times\Lambda \multimap \mathbb{H}$ of the form $\eta(x,t,\lambda) = e^{tL}x + U(x,t,\lambda)$, where $U\colon \mathbb{H}\times\mathbb{R}\times\Lambda \multimap \mathbb{H}$ is an admissible completely continuous mapping, such that, for each $\lambda \in \Lambda$, $\eta(\,\cdot\,,\lambda)\colon \mathbb{H} \times \mathbb{R} \multimap \mathbb{H}$ is a multivalued flow.

It is clear that, if $\eta\colon \mathbb{H} \times \mathbb{R} \times \Lambda \multimap \mathbb{H}$ is a family of L-flows, then, for each $\lambda \in \Lambda$, $\eta(\,\cdot\,,\lambda)\colon \mathbb{H} \times \mathbb{R} \multimap \mathbb{H}$ is an L-flow. Moreover, each L-flow is an admissible flow.

Proposition 4.2 ([8, Proposition 3.15]). *If $X \subset \mathbb{H}$ is closed and bounded, then the set-valued map $\Lambda \ni \lambda \mapsto \mathrm{Inv}(X, \eta(\,\cdot\,,\lambda)) \subset X$ is usc and it has compact (possibly empty) values.*

Definition 4.3. An usc mapping $f\colon \mathbb{H} \multimap \mathbb{H}$ is an *L-vector field* if it is of the form $f(x) = Lx + K(x)$, where $K\colon \mathbb{H} \multimap \mathbb{H}$ is completely continuous with compact convex values, and if f induces an L-flow π on H.

Given an L-vector field $f := L + F : \mathbb{H} \multimap \mathbb{H}$, F having a sublinear growth (i.e. there is a constant $C > 0$ such that, for each $u \in \mathbb{H}$ and $y \in F(u)$, $\|y\| \le C(1 + \|u\|)$), the standard fixed point argument (see, e.g. [16, Theorem 5.2.2]) implies that, for each $x \in \mathbb{H}$, there is a *mild solution* to the Cauchy problem

$$\begin{cases} u' \in f(u) & \text{a.e. on } \mathbb{R}; \\ u(0) = x, \end{cases} \tag{4.1}$$

i.e. a continuous function $u\colon \mathbb{R} \to \mathbb{H}$ and a locally (Bochner) integrable function $w\colon \mathbb{R} \to \mathbb{H}$ such that $w(t) \in F(u(t))$ and $u(t) = e^{tL}x + \int_0^t e^{(t-s)L} w(s)\,ds$ for all $t \in \mathbb{R}$.

Let $S(x) \subset C(\mathbb{R}, \mathbb{H})$ (2) be the set of all solutions to (4.1), $x \in \mathbb{H}$.

Consider a map $\varphi\colon \mathbb{H} \times \mathbb{R} \multimap \mathbb{H}$ given by the formula

$$\varphi(x,t) := \{u(t) \mid u \in S(x)\}, \quad x \in \mathbb{H},\ t \in \mathbb{R}. \tag{4.2}$$

It is shown in [8, Example 3.3]) that φ is an admissible multivalued flow on $\mathbb{H}$ (we say that φ is *generated by* f).

We consider here only flows generated by L-vector fields. In particular, if F is single-valued and locally Lipschitz, then f generates a usual (single-valued) flow.

Recall that a *suspension* of a pointed space (X, x_0) is the quotient space $(SX, *) := (S^1 \times X)/(S^1 \times \{x_0\} \cup \{s_0\} \times X)$, where S^1 denotes a circle.

Let $\nu\colon \mathbb{N} \to \mathbb{N}$ be a given map.

Definition 4.4. A pair of sequences $X = ((X_n, x_n)_{n=n(X)}^{\infty}, (\gamma_n))$ is *a spectrum* provided the maps $\gamma_n\colon S^{\nu(n)}X_n \to X_{n+1}$ are homotopy equivalences for some $n_1 \geq n(X)$ and each $n \geq n_1$.

We can define a natural notion of a *map of spectra* $f\colon X \to X'$ as a sequence of maps $f_n\colon X_n \to X'_n; n \geq n_0 = \max\{n(X), n(X')\}$ such that the diagrams

$$\begin{tikzcd} S^{\nu(n)}X_n \arrow[r, "S^{\nu(n)}f_n"] \arrow[d, "\gamma_n"] \arrow[dr, phantom, "\curvearrowright"] & S^{\nu(n)}X'_n \arrow[d, "\gamma'_n"] \\ X^{n+1} \arrow[r, "f_{n+1}"'] & X'_{n+1} \end{tikzcd}$$

are homotopy commutative for all $n \geq n_0$.

Two spectra are *homotopy equivalent* if there is $n_1 \geq n_0$ such that f_n are homotopy equivalences for $n \geq n_1$. The equivalence class of this relation is called *the homotopy type of a spectrum* X and is denoted by $[X]$. One observes that the homotopy type of a spectrum X is determined by the homotopy type of the pointed space (X_n, x_n) with n sufficiently large.

(2) $C(\mathbb{R}, \mathbb{H})$ stands for the Fréchet space (i.e. locally convex metrizable and complete) of all continuous maps $\mathbb{R} \to \mathbb{H}$ with the topology of the almost uniform convergence.

We denote by $\underline{0}$ the spectrum such that for each $n \geq 0$ the space X_n consists only of a base point with the only maps $\epsilon_n: X_n \to X_{n+1}$. This is called a *trivial spectrum*.

One can also define usual topological operations like a "wedge sum" and smash product of spectra and on their homotopy types (see [12, Section 2] for details).

Let now $f = L + K: \mathbb{H} \to \mathbb{H}$ be a single-valued L-vector field and let $\varphi: \mathbb{H} \times \mathbb{R} \to \mathbb{H}$ be an L-flow generated by f.

Denote by $\mathbb{H}^n := \bigoplus_{i=0}^n \mathbb{H}_i$ and by $P_n: \mathbb{H} \to \mathbb{H}$ an orthogonal projection onto $\mathbb{H}^n$.

Let $\mathbb{H}_n^{\pm} := \mathbb{H}_n \cap \mathbb{H}^{\pm}, n \geq 1$, where $\mathbb{H}^+$ and $\mathbb{H}^-$ denote L-invariant subspaces of H corresponding to parts of the spectrum of L with positive and negative real parts, respectively. Define $\nu: \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ by $\nu(n) = \dim \mathbb{H}_{n+1}^+$.

Define $f_n: \mathbb{H}^n \to \mathbb{H}^n$ by $f_n(x) := Lx + P_n(K(x))$ and let $\varphi_n: \mathbb{H}^n \times \mathbb{R} \to \mathbb{H}^n$ be a flow generated by f_n.

Lemma 4.5 ([10, Lemma 4.1]). *Let $N \subset \mathbb{H}$ be an isolating neighbourhood for φ. Then there exists n_0 such that, for all $n \geq n_0$, the set $N^n = N \cap \mathbb{H}^n$ is an isolating neighbourhood for φ_n.*

Thus the isolated invariant set $S_n = \mathrm{Inv}(N^n, \varphi_n)$ admits *an index pair* (P_1, P_2), (see [21]), i.e. a compact pair (P_1, P_2) such that

(i) the set $\overline{P_1 \setminus P_2}$ is an isolating neighbourhood for S_n in N^n;
(ii) (positive invariance of P_2 in P_1) if $x \in P_2$ with $\varphi_n(x,t) \in P_1$ for every $t \in [0, t_0]$, then $\varphi_n(x,t) \in P_2$ for every $t \in [0, t_0]$;
(iii) if $x \in P_1$ and there is $t \geq 0$ with $\varphi_n(x,t) \notin P_1$, then there exists $0 \leq t_0 < t$ such that $\varphi_n(x, t_0) \in P_2$.

The classical homotopy Conley index of S_n is the homotopy type of the pointed space $[P_1/P_2, *]$. By the use of the continuation property of the classical Conley index it was proved in [10], that the family of such index pairs (P_1^n, P_2^n) for $n \geq n_0$ forms a spectrum in the above sense. A homotopy type of this spectrum is called *an $\mathcal{LS}$-index* of the isolating neighbourhood N.

Let us denote this index by $h_{\mathcal{LS}}(N, \varphi)$. The following two basic properties have been proved in [10].

Proposition 4.6 (Nontriviality). *Let φ be an single-valued L-flow and $N \subset \mathbb{H}$ an isolating neighbourhood. If $h_{\mathcal{LS}}(N, \varphi) \neq \underline{0}$, then* $\mathrm{Inv}(N, \varphi) \neq \emptyset$.

Proposition 4.7 (Continuation). *Let Λ be a compact, connected and locally contractible metric space. Assume that φ_λ is a family of single-valued L-flows and let $N \subset \mathbb{H}$ be an isolating neighbourhood for the flow φ_λ for some $\lambda \in \Lambda$.*

Then there is a compact neighbourhood $C \subset \Lambda$ of λ such that

$$h_{\mathcal{LS}}(N, \varphi_\mu) = h_{\mathcal{LS}}(N, \varphi_\nu) \quad \text{for all } \mu, \nu \in C.$$

Let us now consider a multivalued L-vector field $L + F: \mathbb{H} \multimap \mathbb{H}$. Denote by $a(F, \varepsilon)$ the set of all ε-approximations of F in the sense of Theorem 2.3.

Proposition 4.8. *Let $N = \overline{U} \subset \mathbb{H}$ be an isolating neighbourhood for a multivalued flow generated by $L + F$. There exists an $\varepsilon > 0$ such that for arbitrary $f_0, f_1 \in a(F, \varepsilon)$ N is an isolating neighbourhood for a family of L-flows η_λ generated by the family of L-vector fields $\Psi_\lambda = L + (1 - \lambda) f_0 + \lambda f_1$.*

Proof. Let $r > 0$ be such that $N \subset B_r(0)$, and find the Urysohn function $u: \mathbb{H} \to [0, 1]$ such that $u(x) = 0$ for $x \in \overline{B_r(0)}$ and $u(x) = 1$ for any $x \in \mathbb{H} \setminus B_{2r}(0)$.

Consider a homotopy $h: \mathbb{H} \times [0, 1] \multimap \mathbb{H}$,

$$h(x, s) := Lx + (1 - u(x))(\overline{\operatorname{conv}} F(B_s(x)) + \overline{B_s(0)}) \cap \overline{\operatorname{conv}} F(\overline{B_{2r}(0)}).$$

Since $\overline{\operatorname{conv}} F(\overline{B_{2r}(0)})$ is compact, the map h generates a family η of multivalued L-flows on $\mathbb{H}$. Notice that $h(\,\cdot\,, 0) = L + F$ on $\overline{B_r(0)}$. From Proposition 4.2 it follows that the map $s \mapsto \operatorname{Inv}(N, \eta(\,\cdot\,, s))$ is usc with compact values.

Now, suppose the contrary to our claim. Then, for a sequence $\varepsilon_n = 1/n$, there are approximations $f_0^n, f_1^n \in a(F, 1/n)$ and numbers $\lambda_n \in [0, 1]$ with $\operatorname{Inv}(N, \gamma_{f_{\lambda_n}}) \not\subset U$, where $\gamma_{f_{\lambda_n}}$ is the flow generated by $L + (1 - \lambda_n) f_0 + \lambda_n f_1$. This implies that there are points $y_n \in \operatorname{Inv}(N, \gamma_{f_{\lambda_n}}) \cap (N \setminus U)$. Note that $f_{\lambda_n}(\,\cdot\,) \subset \overline{\operatorname{conv}} F(B_{1/n}(\,\cdot\,)) + \overline{B_{1/n}(0)}$ for every $n \geq 1$. Since the map $s \mapsto \operatorname{Inv}(N, \eta(\,\cdot\,, s))$ is usc with compact values, there exists a subsequence (f_k), where $f_k := f_{\lambda_{n_k}}$, such that $\operatorname{Inv}(N, \gamma_{f_k}) \subset \operatorname{Inv}(N, \varphi) + B_{1/k}(0)$ for every $k \geq 1$. Indeed, it is sufficient to notice that $\operatorname{Inv}(N, \gamma_{f_k}) \subset \operatorname{Inv}(N, \eta(\,\cdot\,, 1/n_k))$.

Now, we can choose a sequence $(z_k) \subset \operatorname{Inv}(N, \varphi)$ such that $|z_k - y_{n_k}| < 1/k$. Since the set $\operatorname{Inv}(N, \varphi)$ is compact, we can assume that $z_k \to z_0 \in \operatorname{Inv}(N, \varphi)$. So, $y_{n_k} \to z_0$. But then $z_0 \in \operatorname{Inv}(N, \varphi) \cap (N \setminus U)$; a contradiction. $\square$

The above proposition proves that the following crucial notion of this note does not depend on the approximation f.

Definition 4.9. If N is an isolating neighbourhood for an L-flow φ generated by $L + F$, then we define *a homotopy index*

$$h(N, \varphi) := h_{\mathcal{LS}}(N, \varphi_f),$$

where φ_f is the flow generated by $L + f$; $f \in a(F, \varepsilon)$ with $\varepsilon > 0$ sufficiently small.

Now we establish some properties of the index. The first one is an obvious consequence of Proposition 4.6.

Proposition 4.10. *If N is an isolating neighbourhood for a multivalued L-flow φ and the homotopy index is nontrivial $h(N,\varphi) \neq \underline{0}$, then* $\operatorname{Inv}(N,\varphi) \neq \emptyset$.

Proposition 4.11. *If N_0, N_1 are two isolating neighbourhoods for an L-flow φ such that* $\operatorname{Inv}(N_0,\varphi) \subset \operatorname{int} N_1$, $\operatorname{Inv}(N_1,\varphi) \subset \operatorname{int} N_0$, *then* $h(N_0,\varphi) = h(N_1,\varphi)$.

Proposition 4.12. *Let $\varphi\colon \mathbb{H} \times [0,1] \times \mathbb{R} \multimap \mathbb{H}$ be a family of multivalued L-flows and let $N \subset \mathbb{H}$ be an isolating neighbourhood for all $\varphi(\,\cdot\,,\lambda)$, $\lambda \in [0,1]$. Then $h(N,\varphi(\,\cdot\,,0)) = h(N,\varphi(\,\cdot\,,1))$.*

Proof. Consider the family of vector fields $\widetilde{F}\colon \mathbb{H} \times [0,1] \multimap \mathbb{H}$ such that for every $\lambda \in [0,1]$ the multivalued flow $\varphi(\,\cdot\,,\lambda,\,\cdot\,)$ is generated by the L-vector field $L + \widetilde{F}(\,\cdot\,,\lambda)\colon \mathbb{H} \multimap \mathbb{H}$. Applying Theorem 2.3 to the map $\widetilde{F}$ we obtain, for every $\varepsilon > 0$, a locally Lipschitz compact single-valued map $\widetilde{f}\colon \mathbb{H} \times [0,1] \to \mathbb{H}$ such that

$$(*)\qquad \widetilde{f}(x,\lambda) \in \overline{\operatorname{conv}}\widetilde{F}(\overline{B_\varepsilon}(x) \times \overline{B_\varepsilon}(\lambda)) + B_\varepsilon(0)) \quad \text{for all } x \in \mathbb{H},\ \lambda \in [0,1].$$

Let us fix $\lambda \in [0,1]$. We shall show that N is an isolating neighbourhood for the flows generated by $\widetilde{f}(\,\cdot\,,\lambda')$, where $\lambda' \in (\lambda - \varepsilon, \lambda + \varepsilon)$, if ε is small enough. Assume that $\overline{N} \subset B_r(0)$.

Let us define a homotopy $h\colon \mathbb{H} \times [0,1] \multimap \mathbb{H}$ by the formula

$$h(x,s) = Lx + u(x)[(\overline{\operatorname{conv}}\widetilde{F}(\overline{B_s}(x) \times \overline{B_s}(\lambda)) + \overline{B_s}(0)] \cap \overline{\operatorname{conv}}\widetilde{F}(\overline{B_{2r}}(0) \times [0,1]),$$

where $u\colon \mathbb{H} \to \mathbb{R}$ is an Urysohn function such that $u(x) = 1$ for $|x| \leq r$ and $u(x) = 0$ for $|x| \geq 2r$.

Since h is a family of multivalued L-vector fields, it generates a family of multivalued L-flows $\eta(\,\cdot\,,s)$. Moreover, $h(\,\cdot\,,0) = L + \widetilde{F}(\,\cdot\,,\lambda)$. By Proposition 4.2 the mapping $s \multimapdotbothA \operatorname{Inv}(N,\eta(\,\cdot\,,s))$ is usc and $\operatorname{Inv}(N,\eta(\,\cdot\,,0)) \subset \operatorname{int} N$. Therefore there exists $s > 0$ such that for all $s' \leq s$ we have $\operatorname{Inv}(N,\eta(\,\cdot\,,s')) \subset \operatorname{int} N$. If we choose $0 < \varepsilon_\lambda < s/2$, then for all $\lambda' \in [\lambda - \varepsilon_\lambda, \lambda + \varepsilon_\lambda]$ we obtain by Theorem 2.3 that for $\varepsilon < \varepsilon_\lambda$

$$\widetilde{f}(x,\lambda') \in \overline{\operatorname{conv}}\widetilde{F}(\overline{B}_\varepsilon(x) \times \overline{B}_\varepsilon(\lambda')) + \overline{B}_\varepsilon(0) \subset \overline{\operatorname{conv}}\widetilde{F}(\overline{B}_s(x) \times \overline{B}_s(\lambda)) + \overline{B}_s(0).$$

We can assume that also $B_s(N) \subset B_r(N) \subset B_{2r}(0)$. It follows that the map $L + \widetilde{f}(\,\cdot\,,\lambda')$ is a selection of $h(\,\cdot\,,s)$. Therefore for the L-flow $\psi(\,\cdot\,,\lambda')$ generated by the vector field $L + \widetilde{f}(\,\cdot\,,\lambda')$ we have the inclusion $\operatorname{Inv}(N,\psi(\,\cdot\,,\lambda')) \subset \operatorname{int} N$, i.e. N is an isolating neighbourhood.

Intervals $I_\lambda = (\lambda - \varepsilon_\lambda, \lambda + \varepsilon_\lambda) \cap [0,1]$ form an open covering of $[0,1]$. Choosing a finite subcovering $I_{\lambda_1}, \ldots, I_{\lambda_k}$ we find $\overline{\varepsilon} < \min\{\varepsilon_{\lambda_i}\}$ such that, for $\widetilde{f}$ satisfying $(*)$ with $\varepsilon = \overline{\varepsilon}$, the set N is an isolating neighbourhood for flows generated by $L + \widetilde{f}(\,\cdot\,,\lambda)$ for all $\lambda \in [0,1]$. Thus by Proposition 4.7 the homotopy index $h(N,\psi(\,\cdot\,,\lambda))$ does not depend on λ.

On the other hand, the approximation $\widetilde{f}$ can be taken with an additional condition satisfied:

$$\widetilde{f}(\,\cdot\,,i) \in a(\widetilde{F}(\,\cdot\,,i),\varepsilon), \quad \text{for } i=0,1.$$

In order to assure this condition is satisfied, one repeats the proof of Theorem 2.3 with the following modification: For (x,λ) with $\lambda \notin \{0,1\}$ we take $B_{\delta(x,\lambda)}(x,\lambda)$ such that $B_{\delta(x,\lambda)}(x,\lambda)\cap(\mathbb{H}\times\{0,1\})=\emptyset$, $B_{\delta(x,0)}(x,0)\cap(\mathbb{H}\times\{1\})=\emptyset$, $B_{\delta(x,1)}(x,1)\cap(\mathbb{H}\times\{0\})=\emptyset$ and for a locally finite covering $\{V_s\}$ of $\mathbb{H}\times[0,1]$ we choose $(x_s,\lambda_s)\in V_s$ such that $\lambda_s=i$, $i\in\{0,1\}$, if $V_s\cap(\mathbb{H}\times\{i\})\neq\emptyset$. □

PROPOSITION 4.13. *Let $\varphi\colon\mathbb{H}\times\mathbb{R}\multimap\mathbb{H}$ be an L-flow and let N_1,N_2,N be isolating neighbourhoods for φ such that $N_1\cap N_2=\emptyset$, $N_1\cup N_2\subset N$ and $\operatorname{Inv}(N,\varphi)\subset N_1\cup N_2$. Then $h(N,\varphi)=h(N_1,\varphi)\vee h(N_2,\varphi)$.*

PROOF. The property follows from the obvious observation that for each n $\mathbb{H}^n\cap N_1\cap N_2=\emptyset$ and thus the appropriate index pairs (P,Q) defining the classical Conley index for the isolating neighbourhood $N\cap\mathbb{H}^n$ can be chosen in the form of disjoint sums $(P_1\cup P_2,Q_1\cup Q_2)$, where (P_1,Q_1), (P_2,Q_2) are index pairs for N_1, N_2, respectively. The rest is the definition of the wedge sum of spectra (see [12] for details). □

In [8] a cohomological version of the Conley index for multivalued L-flows in a Hilbert space was established starting from the finite-dimensional case given in [20]. Instead of the homotopy type of index pairs the authors consider the Alexander–Spanier cohomology groups of these pairs. Since all the maps in the spectra are homotopy equivalences for n large enough, the inverse limit of the groups is well-defined

$$CH^q(N,\varphi)=\varprojlim\{H^{q+\rho(n)}(Y_n,Z_n),\widetilde{\gamma_n}\}.$$

Similarly as in the single-valued case (see [12]) we obtain

PROPOSITION 4.14. *Let N be an isolating neighbouurhood for a multivalued L-flow φ. Then the cohomology index from* [8] *is equal to the cohomology of our spectrum: $CH^q(N,\varphi)=H^q(h(N,\varphi))$ for all $q\in\mathbb{Z}$.*

As a by-product we obtain that the cohomology index of Mrozek (see [20]) for a multvalued flow φ in $\mathbb{R}^n$ generated by a differential inclusion is just a cohomology of the homotopy index considered by Kunze in [18].

An interesting question appears: can the homotopy index $h(N,\varphi)$ be described using a behavior of an L-vector field $L+F$ on the boundary of a prescribed set of constraints? In [9] the author gave a positive answer for differential inclusions in a finite dimensional space. We show that an infinite-dimensional version of this result is possible.

We will need the following extension result on graph approximations. Recall that $P_n: \mathbb{H} \to \mathbb{H}^n$ denotes the ortogonal projection.

Lemma 4.15. *Let $B = \overline{B(0,r)} \subset \mathbb{H}$ be a closed ball in $\mathbb{H}$ and $B^n := P_n(B) \subset B$. Let $F: B \multimap \mathbb{H}$ be a compact upper semicontinuous map with convex values and $F_n := P_n \circ F$. Then, for every $\varepsilon > 0$ there exists $n_0 \geq 1$ such that for any $n \geq n_0$ there exists a $\delta_n > 0$ such that any continuous* (*locally Lipschitz*) *δ_n-approximation $f: B^n \to \mathbb{H}^n$ of F_n over B^n may be extended to a continuous* (*locally Lipschitz*) *ε-approximation $g: B \to E$ of F, i.e. $g|_{B^n} = f$.*

Proof. Let $\varepsilon > 0$ be arbitrary. We will proceed in several steps.

Step 1. There exists a locally Lipschitz function $\eta: B \to (0, \infty)$ such that, for every $x \in \mathbb{H}$, there is $x' \in B(x, \varepsilon)$ such that $B_{\varepsilon/2}(F(B_{\eta(x)}(x))) \subset B_\varepsilon(F(x'))$.

Indeed, for each $x \in B$ we choose $0 < r_x < \varepsilon$ such that $F(B_{2r_x}(x)) \subset B_{\varepsilon/2}(F(x))$, since F is usc, and take a locally finite and locally Lipschitz partition of unity $\{\lambda\}_{s\in S}$ subordinated to the covering $\{B(x, r_x)\}_{x\in B}$. For each $s \in S$ denote $r_s := r_{x_s}$, where $\operatorname{supp} \lambda_s \subset B(x_s, r_{x_s})$ for some $x_s \in B$.

Define $\eta: B \to (0, \infty)$,

$$\eta(x) := \sum_{s\in S} \lambda_s(x) r_s, \quad x \in B.$$

Obviously, η is locally Lipschitz. Let $x \in B$, and let $S_x := \{s \in S : \lambda_s(x) > 0\}$. Since the partition of unity is locally finite, we can find $s \in S_x$ such that $\eta(x) \leq r_s$. Hence, $||x - x_s|| < r_s < \varepsilon$ and, for any $y \in B_{\eta(x)}(x)$, $||y - x_s|| \leq ||y - x|| + ||x - x_s|| < 2r_s$. Therefore $B_{\eta(x)}(x) \subset B_{2r_s}(x_s)$ and

$$F(B_{\eta(x)}(x)) \subset F(B_{2r_s}(x_s)) \subset B_{\varepsilon/2}(F(x_s)).$$

Hence, putting $x' := x_s$, we obtain

$$B_{\varepsilon/2}(F(B_{\eta(x)}(x))) \subset B_\varepsilon(F(x')).$$

Step 2. For any $(x, y) \in B \times B$ we define

$$U(x, y) := [\eta^{-1}((\eta(x)/2, \infty)) \cap B_{\eta(x)/2}(x)] \times B_{\varepsilon/2}(y)$$

and an open neighbourhood of the graph of F

$$\mathcal{U} := \bigcup_{(x,y)\in \operatorname{Gr}(F)} U(x, y).$$

Notice that, if $W \subset B$ is any subset, and a continuous map $f: W \to \mathbb{H}$ satisfies $\operatorname{Gr}(f) \subset \mathcal{U}$, then, for each $x \in W$, there exists $(x', y') \in \operatorname{Gr}(F)$ such that $(x, f(x)) \in U(x, y)$. Hence, $f(x) \in B_{\varepsilon/2}(y')$ and $||x - x'|| < \eta(x')/2 < \eta(x)$. This implies that $f(x) \in B_{\varepsilon/2}(F(B_{\eta(x)}(x))) \subset B_\varepsilon(F(B_\varepsilon(x)))$.

Step 3. There is $n_0 \geq 1$ such that $||P_n(y) - y|| < \varepsilon/4$ for every $n \geq n_0$ and $y \in F(B)$. Fix $n \geq n_0$, and define

$$\widetilde{U}(x,y) := [\eta^{-1}((\eta(x)/2,\infty)) \cap B_{\eta(x)/2}(x)] \times B_{\varepsilon/4}(y)$$

and an open neighbourhood of the graph of F_n in $B \times \mathbb{H}$

$$\mathcal{U}_n := \bigcup_{(x,y)\in \mathrm{Gr}\,(F_n)} \widetilde{U}(x,y).$$

Notice that, if $(u,v) \in \widetilde{U}(x,y)$, then $v \in B_{\varepsilon/4}(y)$ and $y = P_n(y')$ for some $y' \in F(x)$. Hence, $||v - y'|| \leq ||v - y|| + ||y - y'|| < \varepsilon/2$. It implies that $(u,v) \in U(x,y')$ and, consequently, $\mathcal{U}_n \subset \mathcal{U}$.

Using the partition of unity technique, as in Step 1, it is easy to find a continuous function $\rho' \colon \mathbb{H}^n \to (0,\varepsilon)$ such that, any $\rho'(\,\cdot\,)$-approximation $f\colon B^n \to \mathbb{H}^n$ of F_n, i.e. $f(x) \in B_{\rho'(x)}(F_n(B_{\rho'(x)}(x)))$ for any $x \in B^n$, satisfies $\mathrm{Gr}\,(f) \subset \mathcal{U}_n$. Analogously, let $\theta\colon \mathbb{H} \to (0,\varepsilon)$ be a continuous function such that any $\theta(\,\cdot\,)$-approximation $f\colon B \to \mathbb{H}$ of F, satisfies $\mathrm{Gr}\,(f) \subset \mathcal{U}$ (comp. [17, Proposition 1.2]).

Since B^n is compact, there exists $0 < \delta = \delta_n < \min\{\rho'(x); x \in B^n\}$.

Step 4. Now, let $f\colon B^n \to \mathbb{H}^n$ be any locally Lipschitz δ-approximation of F_n over B^n. Then $\mathrm{Gr}\,(f) \subset \mathcal{U}_n$. Since B^n is a Lipschitz retract of B, there exists a locally Lipschitz extension $k\colon B \to \mathbb{H}$ of f. Since $\mathcal{U}_n \subset \mathcal{U}$ and $\mathcal{U}$ is open in $B \times \mathbb{H}$, there is an open neighbourhood Ω of B^n in B such that $(x, k(x)) \in \mathcal{U}$ for every $x \in \Omega$. Hence,

$$k(x) \in B_{\varepsilon/2}(F(B_{\eta(x)}(x))) \quad \text{for every } x \in \Omega.$$

Take an open set $\Omega_0 \subset B$ with $B^n \subset \Omega_0 \subset \overline{\Omega_0} \subset \Omega$ and a locally Lipschitz Urysohn function $\beta\colon \mathbb{H} \to [0,1]$ with $\beta(\overline{\Omega_0}) = \{1\}$ and $\beta(\mathbb{H} \setminus \Omega) = \{0\}$. Take any locally Lipschitz $\theta(\,\cdot\,)$-approximation $h\colon B \to \mathbb{H}$ of F, where $\theta(\,\cdot\,)$ is from Step 3. Then $\mathrm{Gr}\,(h) \subset \mathcal{U}$. Define $g\colon B \to \mathbb{H}$, $g(x) := \beta(x)k(x) + (1 - \beta(x))h(x)$ for every $x \in B$. Obviously, $g|_{B^n} = f$.

Take any x with $\beta(x) > 0$. Then $x \in \Omega$, and

$$\{k(x), h(x)\} \subset B_{\varepsilon/2}(F(B_{\eta(x)}(x))).$$

By Step 1, $\{k(x), h(x)\} \subset B_\varepsilon(F(x'))$ for some x' with $||x - x'|| < \varepsilon$. By the convexity of values of F,

$$g(x) \in B_\varepsilon(F(x')) \subset B_\varepsilon(F(B_\varepsilon(x))).$$

If $\beta(x) = 0$, then, since $\theta(x) < \varepsilon$ for every $x \in B$, $g(x) = h(x) \in B_\varepsilon(F(B_\varepsilon(x)))$, too. Hence, g is the required approximation. □

In the sequel we will use the following

Theorem 4.16 (comp. [9, Theorem 4.1]). *Let $K = \overline{\operatorname{int} K}$ be a subset of a finite dimensional space E and $F: E \multimap E$ be an usc map with compact convex values and a sublinear growth and such that $K^-(F)$ is a closed strong deformation retract of some open neighbourhood $V \subset K$ of $K^-(F)$ in K. Assume that* $\operatorname{int} T_K(x) \neq \emptyset$ *for every $x \in K \setminus K^-(F)$, and $T_K(\,\cdot\,)$ is lsc outside $K^-(F)$. Then*

$$h(\operatorname{Inv}(K,\varphi),\varphi) = [K/K^-(F),[K^-(F)]],$$

where φ is a multivalued flow generated by F, and $h(\operatorname{Inv}(K,\varphi),\varphi)$ is defined, if K is an isolating neighbourhood, as the Conley index for any flow generated by a sufficiently close Lipschitz approximation of F.

Here $T_K(x)$ denotes the Bouligand tangent cone:

$$T_K(x) := \bigg\{ v \in \mathbb{R}^n \;\bigg|\; \liminf_{h \to 0^+} \frac{\operatorname{dist}(x + hv, K)}{h} = 0 \bigg\}.$$

Let $L + F: \mathbb{H} \multimap \mathbb{H}$ be a multivalued L-vector field, and let φ be an L-flow generated by $L+F$. On the boundary of a set $K \subset \mathbb{H}$ of constraints we consider the following *exit set*:

$$K^-(L+F) := \{x_0 \in \partial K \mid \text{for all } x \in S(x_0) \text{ for all } t > 0 : x([0,t]) \not\subset K\}.$$

It means that all trajectories starting at points in $K^-(L+F)$ immediately leave the set K. Assume that K is an isolating neighbourhood for φ.

Suppose that the pair $(K, K^-(L+F))$ generates a spectrum (K_n/K_n^-), where K_n^- is the exit set for $K_n = K \cap \mathbb{H}^n$ with respect to $L + P_n F$. Moreover, for some $N \geq 1$ and each $n \geq N$, let the following regularity conditions be satisfied:

- (H1) Each K_n is *epi-Lipschitz* outside K_n^-, i.e. $\operatorname{int} T_{K_n}(x) \neq \emptyset$ for every $x \in K_n \setminus K_n^-$.
- (H2) K_n is *sleek* outside K_n^-, i.e. $T_{K_n}(\,\cdot\,)$ is lsc on $K_n \setminus K_n^-$.
- (H3) K_n^- is a strong deformation retract of some open neighbourhood V_n of K_n^- in K_n.

Denote by $[K, L+F]$ the homotopy type of the spectrum $(K_n/K_n^-))$.

Theorem 4.17. *Under the above assumptions*

$$h(K,\varphi) = [K, L+F].$$

Proof. Let $\varepsilon > 0$ be such that $h(K,\varphi) := h_{\mathcal{LS}}(K,\varphi_f)$, for any $f \in a(F,\varepsilon)$. Let $B = \overline{B(0,r)}$ be a ball in $\mathbb{H}$ such that $K \subset \operatorname{int} B$, and let $n_0 \geq N$ be such that K_n is an isolating neighbourhood for each $n \geq n_0$, and n_0 is as in Lemma 4.15. We want to find $f \in a(F,\varepsilon)$ such that $h_{\mathcal{LS}}(K,\varphi_f)$ is a homotopy type of a spectrum $(Y_n/Z_n)_{n \geq n_0}$, and $([Y_n/Z_n,[Z_n]]) = ([K_n/K_n^-,[K_n]])$.

From Theorem 4.16 it follows that there exists a δ_{n_0}-approximation g_{n_0}: $B^{n_0} \to \mathbb{H}^{n_0}$ of $P_{n_0} F|_{B^{n_0}}$ such that its Conley index $[Y_{n_0}/Z_{n_0},[Z_{n_0}]]$ equals

$[K_{n_0}/K_{n_0}^-, [K_{n_0}]]$. We extend g_{n_0} to an ε-approximation $f: B \to \mathbb{H}$ of F. Since the spectrum $(Y_n/Z_n)_{n \geq n_0}$ for $L + f$ is uniquely determined up to a homotopy type by $(Y_{n_0}/Z_{n_0}, [Z_{n_0}])$, and $(Y_{n_0}/Z_{n_0}, [Z_{n_0}])$ is homotopy equivalent to $(K_{n_0}/K_{n_0}^-, [K_{n_0}])$, we obtain $h(K, \varphi) = [K, L + F]$. $\square$

5. Conley index for finite dimensional gradient differential inclusions

Let $L: \mathbb{R}^d \to \mathbb{R}^d$ be a linear operator, and let $f: \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz function satisfying

$$\sup_{y \in \partial f(u)} |y| \leq c(1 + |u|) \quad \text{for some } c \geq 0 \text{ and every } u \in \mathbb{R}^d.$$

Then the function $\Phi: \mathbb{R}^d \to \mathbb{R}$,

$$\Phi(u) = \frac{1}{2}\langle Lu, u\rangle + f(u), \quad \text{for } u \in \mathbb{R}^d, \tag{5.1}$$

is locally Lipschitz, and the Clarke generalized gradient

$$F(u) := \partial\Phi(u) = Lu + \partial f(u)$$

is well defined (see [1], [4] for definitions and properties of the gradient). Moreover, $F: \mathbb{R}^d \multimap \mathbb{R}^d$ is usc with compact convex values and a sublinear growth. Hence, the differential inclusion $\dot{x} \in F(x)$ generates a multivalued L-flow (see Section 3). We say that $\dot{x} \in \partial\Phi(x)$ is a *gradient differential inclusion*.

Assume that $F: \mathbb{R}^d \multimap \mathbb{R}^d$ is of the form $F(u) = Lu + \varphi(u)$ for some usc map φ with compact convex values and sublinear growth. We say that F has a *variational structure*, if there exists a locally Lipschitz function $f: \mathbb{R}^d \to \mathbb{R}$ such that $\partial\Phi(u) \subset F(u)$, where Φ is defined in (5.1). As we will see in the sequel, multivalued maps with a variational structure play an important role in our investigations. If $\mathbb{H}$ is a Hilbert space, $P^d: \mathbb{H} \to \mathbb{H}^d$ is the ortogonal finite-dimensional projection, and $F: \mathbb{H} \multimap \mathbb{H}$ is of the form $F(u) = Lu + \partial f(u)$ for some linear bounded operator $L: \mathbb{H} \to \mathbb{H}$ with $L(\mathbb{H}^d) \subset \mathbb{H}^d$, and a locally Lipschitz map $f: \mathbb{H} \to \mathbb{R}$, then the map $F_d: \mathbb{H}^d \multimap \mathbb{H}^d$, $F_d(x) := Lx + P_d(\partial f(i_d(x)))$, where $i_d: \mathbb{H}^d \hookrightarrow \mathbb{H}$ is the inclusion map, has a variational structure, since $Lx + \partial(f \circ i_d)(x)) \subset Lx + P_d(\partial f(i_d(x)))$ (see [1], [4]). Moreover, F_d need not be a generalized gradient of any locally Lipschitz function.

Example 5.1. Consider a 1-Lipschitz function $f: \mathbb{R}^2 \to \mathbb{R}$,

$$f(x, y) = \begin{cases} |x| & \text{for } |x| \leq |y|, \\ |y| & \text{otherwise.} \end{cases}$$

Obviously, $\partial(f \circ i)(x) = \{0\}$, where $i(x) := (x, 0)$, while $F_1(x) = P_1(\partial f(x, 0))) = \{0\}$ for $x \neq 0$ and $F_1(0) = P_1(\partial f(0, 0))) = [-1, 1]$. Hence, F_1 is not a generalized gradient of any locally Lipschitz function.

In the previous section we constructed a homotopy index which is equal to the index defined by Kunze in [18] in a special case of a differential inclusion $\dot{x} \in F(x)$ with compact convex values in a finite-dimensional space. If K is an isolated invariant set, and N is its isolating neighbourhood, i.e. $K = \mathrm{Inv}(N, F) := \mathrm{Inv}(N, \varphi)$, where φ is a multivalued flow generated by the inclusion $\dot{x} \in F(x)$, then the index can be written as

$$H(K, F) := h(N, \varphi_g)$$

for sufficiently near smooth approximation g of F, where φ_g is a flow generated by the equation $\dot{x} = g(x)$.

For $F = \partial\Phi$, where Φ is of the form (5.1), the index can be described using smooth approximations of the given locally Lipschitz map f, as we can see below.

We say that $\widetilde{f}\colon U \to \mathbb{R}$ is a C^∞_ε*-approximation* of a locally Lipschitz function $f\colon U \to \mathbb{R}$, where $U \subset \mathbb{R}^d$ is open and $\varepsilon\colon U \to (0, \infty)$ is a continuous map, if

(i) $|f(x) - \widetilde{f}(x)| < \varepsilon(x)$, for every $x \in U$,
(ii) $\nabla\widetilde{f}$ is an ε-approximation of ∂f.

In the sequel we will apply only a simplified version of the following result with a constant function $\varepsilon(x) = \varepsilon > 0$.

Proposition 5.2 ([3, Theorem 3.7]). *Suppose that $f\colon U \to \mathbb{R}$, where $U \subset \mathbb{R}^d$ is open, is a locally Lipschitz function. Then for any continuous map $\varepsilon\colon U \to (0, +\infty)$ there exists a C^∞_ε-approximation of f.*

Assume that $F\colon \mathbb{R}^d \multimap \mathbb{R}^d$, $F(u) = Lu + \varphi(u)$ has a variational structure with a multivalued selection $\partial\Phi = L + \partial f$. If $K = \mathrm{Inv}(N, F)$, then

$$H(K, F) = H(\mathrm{Inv}(N, \partial\Phi), \partial\Phi) = h(N, \varphi_{\widetilde{f}})$$

for every sufficiently near C^∞_ε-approximation $\widetilde{f}$ of f, since near approximations of $\partial\Phi$ are near approximations of F.

Propositions 4.10 and 4.12 give the existence and additivity properties of the index, respectively. Moreover, the continuation property, which is a consequence of Proposition 4.12, obtains in a finite dimensional case the form:

Proposition 5.3 (Continuation). *Let $F\colon [0,1] \times \mathbb{R}^d \multimap \mathbb{R}^d$ be a compact convex valued usc map with $\sup_{y \in F(\lambda, u)} |y| \le c(1 + |u|)$ for some $c > 0$ and every $(\lambda, u) \in [0,1] \times \mathbb{R}^d$. If $K_\lambda = \mathrm{Inv}(N, F(\lambda, \cdot\,)) \subset \mathrm{int}\, N$ for every $\lambda \in [0,1]$, then $H(K_\lambda, F(\lambda, \cdot\,))$ is independent of $\lambda \in [0,1]$.*

Remark 5.4. In the continuation property Theorem 5.3.4 in [18] the author assume that $F(\lambda, \cdot\,)$ is usc, and $F(\,\cdot\,, u)$ is continuous for every $u \in \mathbb{R}^d$ and usc uniformly on bounded subsets of $\mathbb{R}^d$. As the author proves in Lemma 5.3.3, under these assumptions the map F is jointly usc in every (λ, u), so the assumption in Proposition 5.3 is weaker than in Theorem 5.3.4 in [18]. Our formulation is suitable for gradient inclusions. Indeed, if $\Phi\colon [0,1]\times\mathbb{R}^d\to\mathbb{R}$ is of the form

$$\Phi(\lambda,u)=\frac{1}{2}\langle L_\lambda u,u\rangle+f(\lambda,u),\quad \text{for } u\in\mathbb{R}^d,$$

where $\lambda\mapsto L_\lambda(\,\cdot\,)$ and f are locally Lipschitz, then the generalized gradient of Φ with respect to the second variable satisfies

$$\partial_u\Phi(\lambda,u)=L_\lambda u+\partial_u f(\lambda,u),$$

and it is jointly usc in every $(\lambda,u)\in[0,1]\times\mathbb{R}^d$. Note also that the continuation property is true if an isolating neighbourhood ranges during a homotopy.

EXAMPLE 5.5. One can check that for a function $\Phi\colon[0,1]\times\mathbb{R}$, $\Phi(\lambda,u):=f(\lambda,u)=|u|^{1+\lambda}$, i.e. with $L\equiv 0$, the generalized gradient $\partial_u\Phi$ is not continuous with respect to the first variable.

From the additivity property of the index we immediately obtain:

COROLLARY 5.6. *If K_1,K_2 are disjoint isolated invariant sets, $K_1\cup K_2\subset K$, and $K=\mathrm{Inv}(N,F)$. If $H(K,F)\neq H(K_1,F)\vee H(K_2,F)$, then there exists a full trajectory in K which is not contained in $K_1\cup K_2$. In particular, it is the case if $H(K,F)=\bar 0$ and $H(K_i,F)\neq\bar 0$, for some $i\in\{1,2\}$, where $\bar 0$ is the trivial homotopy type.*

This result can be used for maps with a variational structure to obtain the following

COROLLARY 5.7. *Assume that $F\colon\mathbb{R}^d\multimap\mathbb{R}^d$, $F(u)=Lu+\varphi(u)$ has a variational structure with a multivalued selection $\partial\Phi=L+\partial f$. Let p_1,p_2 be critical points of Φ in an isolating neighbourhood N for F, which are isolated invariant sets for F. Assume that $\langle\partial\Phi(u),\partial\Phi(u)\rangle^->0$ for every $u\in N\setminus\{p_1,p_2\}$, where $\langle\partial\Phi(u),\partial\Phi(u)\rangle^-:=\min\{\langle y,y'\rangle\mid y,y'\in\partial\Phi(u)\}$. If $H(\mathrm{Inv}(N,F),F)\neq H(\{p_1\},F)\vee H(\{p_2\},F)$, then there exists a heteroclinic or homoclinic nontrivial orbit in N. In particular, if $(\{p_1\},\{p_2\})$ is an attractor-repeller pair, then there is a trajectory joining the equilibria.*

PROOF. Notice that p_1, p_2 are the only critical points of Φ in N. From Lemma 4.5 in [9] it follows that $\{p_1\}$ and $\{p_2\}$ are isolated invariant sets for $\partial\Phi$. Therefore $H(\{p_i\},F)=H(\{p_i\},\partial\Phi)$ for $i\in\{1,2\}$. Since $\partial\Phi$ is a selection of F, we have

$$\begin{aligned}H(\mathrm{Inv}(N,\partial\Phi),\partial\Phi)&=H(\mathrm{Inv}(N,F),F)\\&\neq H(\{p_1\},F)\vee H(\{p_2\},F)=H(\{p_1\},\partial\Phi)\vee H(\{p_2\},\partial\Phi).\end{aligned}$$

Now, Corollary 5.6 applies, and there is a full trajectory $x(\,\cdot\,)$ for $\partial\Phi$ (so, for F) in N with $x(0) = x_0 \in N$. Lemma 4.4 in [9] shows that $\omega(x_0) \cup \alpha(x_0) \subset \{p_1, p_2\}$. The proof is finished. $\square$

Remark 5.8. Note that the only critical points p_1, p_2 of Φ need not be isolated invariant sets for F. For example, we can examine the map $F\colon \mathbb{R} \multimap \mathbb{R}$,

$$F(x) = \begin{cases} 1 & \text{for } |x| > 1, \\ [-1,1] & \text{for } |x| \leq 1, \end{cases}$$

with the selection $\partial\Phi$, where

$$\Phi(x) = \begin{cases} x+2 & \text{for } x < -1, \\ -x & \text{for } |x| \leq 1, \\ x-2 & \text{for } x > 1. \end{cases}$$

Obviously, Φ has two critical points -1 and 1 which are not isolated invariant sets for F. Notice that $H(\mathrm{Inv}([-2,2],F),F) = \overline{0}$ and $H(\{-1\},\partial\Phi) \vee H(\{1\},\partial\Phi) = \Sigma^0 \vee \Sigma^1 \neq \overline{0}$. Moreover, $\langle \partial\Phi(x), \partial\Phi(x)\rangle^- = 1 > 0$ for $x \notin \{-1,1\}$. One can easily find a trajectory joining 1 with -1.

We leave to further investigation a challenge of constructing a homotopy index for gradient differential inclusions in infinite dimensional spaces via smooth approximations. We hope this will allow to get more information on dynamics of multivalued Hamiltonian systems (comp. [8]).

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