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SYSTEMS OF FIRST ORDER INCLUSIONS ON TIME SCALES

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Abstract. This paper presents existence results for systems of first order inclusions on time scales with an initial or a periodic boundary value condition. The method of solution-tube is developed for this system.

1. Introduction

In 1990, S. Hilger [18] introduced the concept of dynamic equations on time scales. This concept provides a unified approach to continuous and discrete calculus with the introduction of the notion of delta-derivative $x^{\Delta}(t)$. This notion coincides with $x'(t)$ (resp. $\Delta x(t)$) in the case where the time scale $\mathbb{T}$ is an interval (resp. a discrete set $\{0,\ldots,n\}$).

In this paper, we establish an existence result for the following system of first order inclusions on time scales:

$$
\begin{aligned}
x^{\Delta}(t) &\in F(t,x(\sigma(t))), \quad \Delta\text{-a.e. } t\in\mathbb{T}_0,\\
x &\in (\mathrm{BC}).
\end{aligned}
\tag{1.1}
$$

Here, $\mathbb{T}$ is an arbitrary compact time scale, where we note $a=\min\mathbb{T}$, $b=\max\mathbb{T}$ and $\mathbb{T}_0=\mathbb{T}\setminus\{b\}$. The multivalued map $F\colon\mathbb{T}_0\times\mathbb{R}^n\to\mathbb{R}^n$ satisfies some

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hypothesis that will be stated later, and (BC) denotes the initial or the periodic boundary conditions:

$$x(a) = x_0, \tag{1.2}$$

$$x(a) = x(b). \tag{1.3}$$

In the literature, this kind of problem was mainly treated for $n = 1$ in the particular case where the time scale is a discrete set (difference equation). Some existence results were obtained with the method of lower and upper solutions for one difference equation as in [5] and [12], and for one difference inclusion as in [2]. As far as we know, F. M. Atici and D. C. Biles [4] are the only ones who considered a first order inclusion on an arbitrary compact time scale. Their results were also established with the method of lower and upper solutions.

Systems of first-order equations on time scales were treated by Q. Dai and C. C. Tisdell [11] and, by the second author, [16].

To our knowledge, this paper is the first one in which systems of first order inclusions on time scales are studied. In order to get existence results, we introduce a notion which extends to systems of first order inclusions on time scales, the notions of lower and upper solutions, see [1]. This notion is called solution-tube of system (1.1). A notion of solution-tube was introduced for first order systems of differential inclusions by B. Mirandette [20] (see also [14], [15]).

2. Preliminaries and notations

2.1. Multivalued maps. We recall some definitions and classical results for multivalued maps. They can be found in more generality in [19], see also [8].

Let X, Y be metric spaces and $G: X \to Y$ a multivalued map. The map G is *upper semi-continuous* (u.s.c.) if $\{x \in X : G(x) \cap C \neq \emptyset\}$ is closed for every closed set $C \subset Y$ and it is *compact* if $G(X) = \bigcup_{x \in X} G(x)$ is relatively compact. Let Ω be a measurable space, we say that a multivalued map $G: \Omega \to X$ is *measurable* (resp. *weakly measurable*) if $\{t \in \Omega : G(t) \cap C \neq \emptyset\}$ is measurable for every closed (resp. open) set $C \subset X$.

Proposition 2.1. *Let $G: \Omega \to X$ be a multivalued map.*

(a) *If G is measurable then it is weakly measurable.*
(b) *If G is weakly measurable and has compact values, then it is measurable.*
(c) *The map G is weakly measurable if and only if the multivalued map $\overline{G}: \Omega \to X$ defined by $\overline{G}(t) = \overline{G(t)}$ is weakly measurable.*

Proposition 2.2. *For $n \in \mathbb{N}$, let $G_n: \Omega \to X$ be measurable multivalued maps.*

(a) *The map $G: \Omega \to X$ defined by $G(t) = \bigcup_{n \in \mathbb{N}} G_n(t)$ is measurable.*

(b) *If X is separable, G_n has closed values, and for each t, at least one $G_{n_t}(t)$ is compact, then $G\colon\Omega \to X$ defined by $G(t) = \bigcap_{n\in\mathbb{N}} G_n(t)$ is measurable.*

Theorem 2.3 (Kuratowski, Ryll, Nardzewski). *Let X be a separable Banach space and let $G\colon\Omega \to X$ be a measurable multivalued map. Then G has a measurable selection, i.e. there exists a single-valued measurable map $g\colon\Omega \to X$ such that $g(t) \in G(t)$ for almost every $t \in \Omega$.*

2.2. Functions on time scales. For sake of completeness, we recall some notations, definitions and results concerning functions defined on time scales. The interested reader may consult [6], [7], [18] and the references therein to find the proofs and to get a complete introduction to this subject.

Let $\mathbb{T}$ be a compact time scale with $a = \min\mathbb{T} < b = \max\mathbb{T}$. The *forward jump operator* $\sigma\colon\mathbb{T} \to \mathbb{T}$ (resp. the *backward jump operator* $\rho\colon\mathbb{T} \to \mathbb{T}$) is defined by

$$\sigma(t) = \begin{cases} \inf\{s \in \mathbb{T} : s > t\} & \text{if } t < b, \\ b & \text{if } t = b, \end{cases}$$

$$\left(\text{resp.} \quad \rho(t) = \begin{cases} \sup\{s \in \mathbb{T} : s < t\} & \text{if } t > a, \\ a & \text{if } t = a. \end{cases}\right)$$

We say that $t < b$ is *right-scattered* (resp. $t > a$ is *left-scattered*) if $\sigma(t) > t$ (resp. $\rho(t) < t$), otherwise, we say that t is *right-dense* (resp. *left-dense*). The set of right-scattered points of $\mathbb{T}$ is at most countable, see [10]. We denote it by

$$R_{\mathbb{T}} := \{t \in \mathbb{T} : t < \sigma(t)\} = \{t_i : i \in I\}$$

for some $I \subset \mathbb{N}$. The *graininess function* $\mu\colon\mathbb{T} \to [0,\infty)$ is defined by $\mu(t) = \sigma(t) - t$. We denote

$$\mathbb{T}^\kappa = \mathbb{T} \setminus (\rho(b), b] \quad \text{and} \quad \mathbb{T}_0 = \mathbb{T} \setminus \{b\}.$$

So, $\mathbb{T}^\kappa = \mathbb{T}$ if b is left-dense, otherwise $\mathbb{T}^\kappa = \mathbb{T}_0$.

Definition 2.4. A map $f\colon\mathbb{T} \to \mathbb{R}^n$ is Δ-*differentiable* at $t \in \mathbb{T}^\kappa$ if there exists $f^\Delta(t) \in \mathbb{R}^n$ (called the Δ-*derivative* of f at t) such that for all $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\|(f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s))\| \le \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

We say that f is Δ-*differentiable* if $f^\Delta(t)$ exists for every $t \in \mathbb{T}^\kappa$.

Proposition 2.5. *Let* $f\colon \mathbb{T} \to \mathbb{R}^n$ *and* $t \in \mathbb{T}^\kappa$.

(a) *If* f *is* Δ*-differentiable at* t*, then* f *is continuous at* t.
(b) *If* f *is continuous at* $t \in R_{\mathbb{T}}$*, then*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(c) *The map* f *is* Δ*-differentiable at* $t \in \mathbb{T}^\kappa \setminus R_{\mathbb{T}}$ *if and only if*

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

Proposition 2.6. *If* $f\colon \mathbb{T} \to \mathbb{R}^n$ *and* $g\colon \mathbb{T} \to \mathbb{R}^m$ *are* Δ*-differentiable at* $t \in \mathbb{T}^\kappa$*, then:*

(a) *if* $n = m$, $(\alpha f + g)^\Delta(t) = \alpha f^\Delta(t) + g^\Delta(t)$ *for every* $\alpha \in \mathbb{R}$*;*
(b) *if* $m = 1$,

$$(fg)^\Delta(t) = g(t)f^\Delta(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + g(\sigma(t))f^\Delta(t);$$

(c) *if* $m = 1$ *and* $g(t)g(\sigma(t)) \neq 0$*, then*

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{g(t)f^\Delta(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))};$$

(d) *if* $W \subset \mathbb{R}^n$ *is open and* $h\colon W \to \mathbb{R}$ *is differentiable at* $f(t) \in W$ *and* $t \notin R_{\mathbb{T}}$*, then* $(h \circ f)^\Delta(t) = \langle h'(f(t)), f^\Delta(t)\rangle$.

We recall some notions and results related to the theory of Δ-*measure*.

Definition 2.7 ([6]). A set $A \subset \mathbb{T}$ is said to be Δ-*measurable* if for every set $E \subset \mathbb{T}$,

$$m_1^*(E) = m_1^*(E \cap A) + m_1^*(E \cap (\mathbb{T} \setminus A)),$$

where

$$m_1^*(E) = \begin{cases} \inf\left\{\sum_{k=1}^{m}(d_k - c_k) : E \subset \bigcup_{k=1}^{m}[c_k, d_k) \text{ with } c_k, d_k \in \mathbb{T}\right\} & \text{if } b \notin E, \\ \infty & \text{if } b \in E. \end{cases}$$

The Δ-*measure* on $\mathcal{M}(m_1^*) := \{A \subset \mathbb{T} : A \text{ is } \Delta\text{-measurable}\}$, denoted by μ_Δ, is the restriction of m_1^* to $\mathcal{M}(m_1^*)$. So, $(\mathbb{T}, \mathcal{M}(m_1^*), \mu_\Delta)$ is a complete measurable space.

The notions of Δ-measurable and Δ-integrable functions $f\colon \mathbb{T} \to \mathbb{R}^n$ can be defined similarly to the theory of Lebesgue integral. We omit here these definitions which can be found in [10].

Let $E \subset \mathbb{T}$ be a Δ-measurable set and $f\colon \mathbb{T} \to \mathbb{R}^n$ be a Δ-measurable function. We say that $f \in L^1_\Delta(E, \mathbb{R}^n)$ provided

$$\int_E \|f(s)\|\Delta s < \infty.$$

The set $L^1_\Delta(\mathbb{T}_0, \mathbb{R}^n)$ is a Banach space endowed with the norm

$$\|f\|_{L^1_\Delta} := \int_{\mathbb{T}_0} \|f(s)\|\Delta s.$$

Here is an analog of the Lebesgue dominated convergence theorem.

Theorem 2.8. *Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of functions in $L^1_\Delta(\mathbb{T}_0, \mathbb{R}^n)$. Assume there exists a function $f\colon \mathbb{T}_0 \to \mathbb{R}^n$ such that $f_k(t) \to f(t)$ Δ-a.e. $t \in \mathbb{T}_0$, and there exists a function $g \in L^1_\Delta(\mathbb{T}_0)$ such that $\|f_k(t)\| \le g(t)$ Δ-a.e. $t \in \mathbb{T}_0$ and for every $k \in \mathbb{N}$. Then $f_k \to f$ in $L^1_\Delta(\mathbb{T}_0, \mathbb{R}^n)$.*

In order to compare the Δ-integral on $\mathbb{T}$ and the Lebesgue integral on $[a, b]$, A. Cabada and D. R. Vivero [10] considered the following extension of a function $f\colon \mathbb{T} \to \mathbb{R}^n$ on $[a, b]$:

$$\widehat{f}(t) := \begin{cases} f(t) & \text{if } t \in \mathbb{T}, \\ f(t_i) & \text{if } t \in (t_i, \sigma(t_i)) \text{ and } t_i \in R_\mathbb{T}. \end{cases} \tag{2.1}$$

Theorem 2.9. *Let $E \subset \mathbb{T}_0$ be a Δ-measurable set and let*

$$\widehat{E} = E \cup \bigcup_{t_i \in E\cap R_\mathbb{T}} (t_i, \sigma(t_i)).$$

Let $f\colon \mathbb{T} \to \mathbb{R}^n$ be a Δ-measurable function and $\widehat{f}\colon [a, b] \to \mathbb{R}^n$ its extension on $[a, b]$. Then, f is Δ-integrable on E if and only if $\widehat{f}$ is Lebesgue integrable on $\widehat{E}$. In this case we have,

$$\int_E f(s)\Delta s = \int_{\widehat{E}} \widehat{f}(s)\, ds.$$

Using the previous theorem, we obtain the following result.

Theorem 2.10. *Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of functions in $L^1_\Delta(\mathbb{T}_0, \mathbb{R}^n)$. If $\{\widehat{f_k}\}$ converges weakly to γ in $L^1([a, b], \mathbb{R}^n)$, then γ is the extension $\widehat{f}$ of a function f defined on $\mathbb{T}_0$ in the sense of definition* (2.1)*. Moreover, for every Δ-measurable set $E \subset \mathbb{T}_0$ and every continuous function $g\colon \mathbb{T} \to \mathbb{R}$, we have*

$$\lim_{k\to\infty} \int_E g(s)f_k(s)\Delta s = \int_E g(s)f(s)\Delta s.$$

Proof. Since $\{\widehat{f_k}\}$ converges weakly to γ in $L^1([a, b], \mathbb{R}^n)$, we have for every continuous function $g\colon \mathbb{T} \to \mathbb{R}$,

$$\int_A \widehat{g}(s)\widehat{f_k}(s)\, ds \to \int_A \widehat{g}(s)\gamma(s)\, ds \quad \text{for every measurable set } A \subset [a, b].$$

Thus, for $t_i \in R_{\mathbb{T}}$,

$$\begin{aligned}\int_{(t_i,\sigma(t_i))} \widehat{g}(s)\widehat{f_k}(s)\,ds &= \int_{(t_i,\sigma(t_i))} g(t_i)f_k(t_i)\,ds \\ &= g(t_i)f_k(t_i)\mu(t_i) \to \int_{(t_i,\sigma(t_i))} \widehat{g}(s)\gamma(s)\,ds.\end{aligned}$$

So, $\{f_k(t_i)\}_{k\in\mathbb{N}}$ converges to some $f(t_i) \in \mathbb{R}^n$. Thus, $\{\widehat{f_k}\}$ converges strongly to the constant function $f(t_i)$ in $L^1((t_i,\sigma(t_i)),\mathbb{R}^n)$, and we can assume that $\gamma \equiv f(t_i)$ on $[t_i,\sigma(t_i))$. The first part of the proposition is proved if we define $f = \gamma|_{\mathbb{T}}$. Finally, by Theorem 2.9,

$$\begin{aligned}\int_E g(s)f_k(s)\Delta s &= \int_{\widehat{E}} \widehat{g}(s)\widehat{f_k}(s)\,ds \\ &\to \int_{\widehat{E}} \widehat{g}(s)\gamma(s)\,ds = \int_{\widehat{E}} \widehat{g}(s)\widehat{f}(s)\,ds = \int_E g(s)f(s)\Delta s.\end{aligned}$$

□

In this context, there is also a notion of absolute continuity, see [9].

Definition 2.11. A function $f\colon \mathbb{T} \to \mathbb{R}^n$ is said to be *absolutely continuous on* $\mathbb{T}$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\{[a_k,b_k)\}_{k=1}^m$ with $a_k, b_k \in \mathbb{T}$ is a finite pairwise disjoint family of subintervals satisfying

$$\sum_{k=1}^m (b_k - a_k) < \delta, \quad \text{then} \quad \sum_{k=1}^m \|f(b_k) - f(a_k)\| < \varepsilon.$$

The two following results were obtained in [16].

Proposition 2.12. *If* $g \in L^1_\Delta(\mathbb{T}_0,\mathbb{R}^n)$ *and* $f\colon \mathbb{T} \to \mathbb{R}^n$ *is the function defined by*

$$f(t) := \int_{[a,t)\cap\mathbb{T}} g(s)\Delta s,$$

then f *is absolutely continuous and* $f^\Delta(t) = g(t)$ Δ*-almost everywhere on* $\mathbb{T}_0$.

Proposition 2.13. *If* $f\colon \mathbb{T} \to \mathbb{R}^n$ *is an absolutely continuous function then the* Δ*-measure of the set* $\{t \in \mathbb{T}_0 \setminus R_{\mathbb{T}} : f(t) = 0$ *and* $f^\Delta(t) \neq 0\}$ *is zero.*

We also recall a notion of Sobolev space, see [3],

$$\begin{aligned}W^{1,1}_\Delta(\mathbb{T},\mathbb{R}^n) =&\{x : \mathbb{T} \to \mathbb{R}^n : x \text{ is absolutely continuous and} \\ &\quad x^\Delta \in L^1_\Delta(\mathbb{T}_0,\mathbb{R}^n)\} \\ =&\bigg\{x \in L^1_\Delta(\mathbb{T}_0,\mathbb{R}^n) : \text{ there exists } g \in L^1_\Delta(\mathbb{T}_0,\mathbb{R}^n) \text{ such that} \\ &\quad \int_{\mathbb{T}_0} x(s)\phi^\Delta(s)\Delta s = -\int_{\mathbb{T}_0} g(s)\phi(\sigma(s))\Delta s \text{ for all } \phi \in C^1_{0,rd}(\mathbb{T})\bigg\},\end{aligned}$$

where

$$C^1_{0,rd}(\mathbb{T}) = \{\phi\colon \mathbb{T}\to\mathbb{R} : \phi(a)=0=\phi(b), \phi \text{ is } \Delta\text{-differentiable and } \phi^\Delta \text{ is continuous at right-dense points of } \mathbb{T} \text{ and its left-sided limits exist at left-dense points of } \mathbb{T}\}.$$

The following maximum principle is obtained in [16].

Lemma 2.14. *Let $r \in W^{1,1}_\Delta(\mathbb{T})$ such that $r^\Delta(t) < 0$ Δ-a.e. $t \in \{t \in \mathbb{T}_0 : r(\sigma(t)) > 0\}$. If one of the following conditions holds:*

(a) $r(a) \le 0$,
(b) $r(a) \le r(b)$,

then $r(t) \le 0$ for every $t \in \mathbb{T}$.

Let the exponential function $e_1(\,\cdot\,, t_0)$ be defined by

$$e_1(t,t_0) = \exp\bigg(\int_{t_0}^{t} \xi_1(\mu(s))\Delta s\bigg), \tag{2.2}$$

where

$$\xi_1(h) = \begin{cases} 1 & \text{if } h = 0,\\ \dfrac{\log(1+h)}{h} & \text{if } h > 0.\end{cases}$$

This function permits us to write the solution of equations on time scales. The following results are direct consequences of Propositions 2.6 and 2.12.

Proposition 2.15. *Let $g \in L^1_\Delta(\mathbb{T}_0,\mathbb{R}^n)$. The function $x\colon\mathbb{T}\to\mathbb{R}^n$ defined by*

$$x(t) = e_1(a,t)\bigg(x_0 + \int_{[a,t)\cap\mathbb{T}} e_1(s,a)g(s)\Delta s\bigg)$$

is in $W^{1,1}_\Delta(\mathbb{T},\mathbb{R}^n)$ and is a solution of the problem

$$\begin{aligned} x^\Delta(t) + x(\sigma(t)) &= g(t), \quad \Delta\text{-a.e. } t\in\mathbb{T}_0,\\ x(a) &= x_0.\end{aligned}$$

Proposition 2.16. *Let $g \in L^1_\Delta(\mathbb{T}_0,\mathbb{R}^n)$. The function $x\colon\mathbb{T}\to\mathbb{R}^n$ defined by*

$$x(t) = \frac{1}{e_1(t,a)}\bigg(\frac{1}{e_1(b,a)-1}\int_{[a,b)\cap\mathbb{T}} g(s)e_1(s,a)\Delta s + \int_{[a,t)\cap\mathbb{T}} g(s)e_1(s,a)\Delta s\bigg)$$

is in $W^{1,1}_\Delta(\mathbb{T},\mathbb{R}^n)$ and is a solution of the problem

$$\begin{aligned} x^\Delta(t) + x(\sigma(t)) &= g(t), \quad \Delta\text{-a.e. } t\in\mathbb{T}_0,\\ x(a) &= x(b).\end{aligned}$$

3. Existence theorem

In this section, we establish an existence result for the problem (1.1) with an initial condition or a periodic boundary value condition. To obtain a solution to our problem, that is a function $x \in W^{1,1}_{\Delta}(\mathbb{T},\mathbb{R}^n)$ satisfying (1.1), we introduce the notion of solution-tube of this problem.

Definition 3.1. Let $(v,M) \in W^{1,1}_{\Delta}(\mathbb{T},\mathbb{R}^n) \times W^{1,1}_{\Delta}(\mathbb{T},[0,\infty))$. We say that (v,M) is a *solution-tube* of (1.1) if

(a) Δ-a.e. $t \in \mathbb{T}_0$ and for every $x \in \mathbb{R}^n$ such that $\|x - v(\sigma(t))\| = M(\sigma(t))$, there exists $\delta > 0$ such that, for every $u \in \mathbb{R}^n$ such that $\|u - x\| < \delta$ and $\|u - v(\sigma(t))\| \geq M(\sigma(t))$, there exists $y \in F(t,u)$ such that

$$\langle u - v(\sigma(t)), y - v^{\Delta}(t)\rangle \leq M^{\Delta}(t)\|u - v(\sigma(t))\|;$$

(b) $v^{\Delta}(t) \in F(t, v(\sigma(t)))$ Δ-a.e. $t \in \mathbb{T}_0$ such that $M(\sigma(t)) = 0$;
(c) $M(t) = 0$ for every $t \in \mathbb{T}_0$ such that $M(\sigma(t)) = 0$;
(d) if (BC) denotes (1.2), $\|x_0 - v(a)\| \leq M(a)$;
if (BC) denotes (1.3), then $\|v(b) - v(a)\| \leq M(a) - M(b)$.

We denote

$$T(v,M) = \{x \in W^{1,1}_{\Delta}(\mathbb{T},\mathbb{R}^n) : \|x(t) - v(t)\| \leq M(t) \text{ for every } t \in \mathbb{T}\}.$$

We assume the following hypothesis:

(F1) $F\colon \mathbb{T}_0 \times \mathbb{R}^n \to \mathbb{R}^n$ is a multivalued map with compact and convex values such that $t \mapsto F(t,x)$ is Δ-measurable for every $x \in \mathbb{R}^n$, and $x \mapsto F(t,x)$ is u.s.c. Δ-a.e. $t \in \mathbb{T}_0$.

(F2) For every $r > 0$, there exists a function $h_r \in L^1_{\Delta}(\mathbb{T}_0,[0,\infty))$ such that

$$\max\{\|y\| : y \in F(t,x),\ \|x\| \leq r\} \leq h_r(t) \quad \Delta\text{-a.e. } t \in \mathbb{T}_0.$$

(ST) There exists $(v,M) \in W^{1,1}_{\Delta}(\mathbb{T},\mathbb{R}^n) \times W^{1,1}_{\Delta}(\mathbb{T},[0,\infty))$ a solution-tube of (1.1).

To prove our existence theorem, we consider the following modified problem:

$$\begin{aligned} x^{\Delta}(t) + x(\sigma(t)) &\in F_0(t, x(\sigma(t))) + \overline{x}(\sigma(t)), \quad \Delta\text{-a.e. } t \in \mathbb{T}_0, \\ x &\in (BC); \end{aligned} \tag{3.1}$$

with $\overline{x}(\sigma(t)) = x^-(\sigma(t), x(\sigma(t)))$, where for $t \in \mathbb{T}_0$ and $x \in \mathbb{R}^n$,

$$F_0(t,x) = F(t, x^-(\sigma(t),x)) \cap G(t,x); \tag{3.2}$$

with

$$x^-(t,x) = \begin{cases} \dfrac{M(t)}{\|x - v(t)\|}(x - v(t)) + v(t) & \text{if } \|x - v(t)\| > M(t), \\ x & \text{otherwise;} \end{cases}$$

and

$$G(t,x)=\begin{cases} v^{\Delta}(t) & \text{if } M(\sigma(t))=0,\\ \mathbb{R}^n & \text{if } M(\sigma(t))>0\\ & \text{and } \|x-v(\sigma(t))\|\le M(\sigma(t)),\\ \{z:\langle x-v(\sigma(t)),z-v^{\Delta}(t)\rangle & \\ \qquad \le M^{\Delta}(t)\|x-v(\sigma(t))\|\}, & \text{otherwise.}\end{cases}$$

Remark 3.2. Remark that, for every (t,x) such that

$$\|x-v(\sigma(t))\|>M(\sigma(t))>0,$$

(a) $G(t,x)=G(t,x_\theta(\sigma(t)))$ for all $\theta\in[0,1[$, where

$$x_\theta(\sigma(t))=\theta x^-(\sigma(t),x)+(1-\theta)x.$$

(b) $G(t,x)=\{z:\langle x^-(\sigma(t),x)-v(\sigma(t)),z-v^{\Delta}(t)\rangle\le M^{\Delta}(t)M(\sigma(t))\}$.

Indeed, for $\theta\in[0,1]$,

$$x_\theta(\sigma(t))-v(\sigma(t))=\bigg(1-\theta+\frac{\theta M(\sigma(t))}{\|x-v(\sigma(t))\|}\bigg)(x-v(\sigma(t))).$$

Thus,

$$\begin{aligned}G(t,x)&=\{z:\langle x-v(\sigma(t)),z-v^{\Delta}(t)\rangle\le M^{\Delta}(t)\|x-v(\sigma(t))\|\}\\ &=\{z:\langle x_\theta(\sigma(t))-v(\sigma(t)),z-v^{\Delta}(t)\rangle\le M^{\Delta}(t)\|x_\theta(\sigma(t))-v(\sigma(t))\|\}.\end{aligned}$$

So, for $\theta\in[0,1[$, $G(t,x)=G(t,x_\theta(\sigma(t)))$ since $\|x_\theta(\sigma(t))-v(\sigma(t))\|>M(\sigma(t))$.

We first study the properties of the map G.

Proposition 3.3. *The multivalued map $G\colon\mathbb{T}_0\times\mathbb{R}^n\to\mathbb{R}^n$ satisfies the following properties:*

(a) *$G(t,x)$ has nonempty, closed, convex values for all $x\in\mathbb{R}^n$ and for Δ-almost every $t\in\mathbb{T}_0$;*
(b) *$x\mapsto G(t,x)$ has closed graph for Δ-almost every $t\in\mathbb{T}_0$;*
(c) *$t\mapsto G(t,x)$ is Δ-measurable for every $x\in\mathbb{R}^n$.*

Proof. (a) It is obvious that G has nonempty, closed, convex values.

(b) To show that

$$A_t=\{(x,y)\in\mathbb{R}^{2n}:y\in G(t,x)\}$$

is closed for Δ-a.e. $t\in\mathbb{T}_0$, we just have to check the case where $t\in\mathbb{T}_0$ is such that $M(\sigma(t))\ne0$. Let $\{(x_k,y_k)\}$ be in A_t such that $x_k\to x$ and $y_k\to y$. If $\|x-v(\sigma(t))\|\le M(\sigma(t))$ then $y\in G(t,x)=\mathbb{R}^n$. So, $(x,y)\in A_t$. Otherwise,

$\|x - v(\sigma(t))\| > M(\sigma(t))$ and for k sufficiently large $\|x_k - v(\sigma(t))\| > M(\sigma(t))$ and

$$\langle x_k - v(\sigma(t)), y_k - v^{\Delta}(t)\rangle \leq M^{\Delta}(t)\|x_k - v(\sigma(t))\|.$$

Therefore,

$$\langle x - v(\sigma(t)), y - v^{\Delta}(t)\rangle \leq M^{\Delta}(t)\|x - v(\sigma(t))\|, \quad \text{and hence } (x,y) \in A_t.$$

(c) Let C be a nonempty, closed subset of $\mathbb{R}^n$, and fix $x \in \mathbb{R}^n$. Let $\{y_m : m \in N\}$ be a countable, dense subset of C. Observe that

$$B_x = \{t \in \mathbb{T}_0 : G(t,x) \cap C \neq \emptyset\} = B_1 \cup B_2 \cup (B_3 \cap B_4),$$

where

$$\begin{aligned} B_1 &= \{t \in \mathbb{T}_0 : v^{\Delta}(t) \in C\} \cap \{t \in \mathbb{T}_0 : M(\sigma(t)) = 0\}, \\ B_2 &= \{t \in \mathbb{T}_0 : \|x - v(\sigma(t))\| - M(\sigma(t)) \leq 0\} \cap \{t \in \mathbb{T}_0 : M(\sigma(t)) > 0\}, \\ B_3 &= \{t \in \mathbb{T}_0 : \|x - v(\sigma(t))\| - M(\sigma(t)) > 0\} \cap \{t \in \mathbb{T}_0 : M(\sigma(t)) > 0\}, \\ B_4 &= \bigcap_{k\in\mathbb{N}} \bigcup_{m\in N} \bigg\{t \in \mathbb{T}_0 : \langle x - v(\sigma(t)), y_m - v^{\Delta}(t)\rangle \leq M^{\Delta}(t)\|x - v(\sigma(t))\| + \frac{1}{k}\bigg\}. \end{aligned}$$

The Δ-measurability of the maps $t \mapsto v(\sigma(t))$, $t \mapsto M(\sigma(t))$, $t \mapsto v^{\Delta}(t)$, and $t \mapsto M^{\Delta}(t)$ imply that B_x is Δ-measurable, and so is $t \mapsto G(t,x)$. □

We now define the multivalued map $\mathcal{F}\colon C(\mathbb{T},\mathbb{R}^n) \to L^1_{\Delta}(\mathbb{T}_0,\mathbb{R}^n)$ by

$$\mathcal{F}(x) = \{w \in L^1_{\Delta}(\mathbb{T}_0,\mathbb{R}^n) : w(t) \in F_0(t, x(\sigma(t)))\ \Delta\text{-a.e. } t \in \mathbb{T}_0\}.$$

Proposition 3.4. *Assume* (F1), (F2) *and* (ST). *Then, $\mathcal{F}$ has nonempty, convex values, and there exists $h \in L^1_{\Delta}(\mathbb{T}_0,[0,\infty))$ such that*

(3.3) $\|w(t)\| \leq h(t)$ *Δ-a.e. on $\mathbb{T}_0$ for all $w \in \mathcal{F}(x)$ and all $x \in C(\mathbb{T},\mathbb{R}^n)$.*

Proof. First of all, we want to show that $\mathcal{F}$ has nonempty values. Let $x \in C(\mathbb{T},\mathbb{R}^n)$. There exists a sequence of simple functions $\{x_m\}_{m\in\mathbb{N}}$ such that

$$\|x_m(\sigma(t)) - v(\sigma(t))\| > M(\sigma(t)) \qquad \Delta\text{-a.e. on } \{t : \|x(\sigma(t)) - v(\sigma(t))\| > M(\sigma(t))\},$$

and such that $x_m \to \overline{x}$ in $C(\mathbb{T},\mathbb{R}^n)$. Since the multivalued maps $t \mapsto F(t,y)$ and $t \mapsto G(t,y)$ are Δ-measurable for every $y \in \mathbb{R}^n$, the maps $t \mapsto F(t, x_m(\sigma(t)))$ and $t \mapsto G(t, x_m(\sigma(t)))$ are also Δ-measurable for every $m \in \mathbb{N}$.

Proposition 2.2 implies that, for every $m \in \mathbb{N}$,

$$t \mapsto F(t, x_m(\sigma(t))) \cap G(t, x_m(\sigma(t)))$$

is Δ-measurable, and for every $k \in \mathbb{N}$,

$$t \mapsto \bigcup_{m \geq k} (F(t, x_m(\sigma(t))) \cap G(t, x_m(\sigma(t))))$$

is Δ-measurable. Again, Propositions 2.1 and 2.2 imply that

$$t \mapsto \bigcap_{k \in \mathbb{N}} \overline{\bigcup_{m \geq k} (F(t, x_m(\sigma(t))) \cap G(t, x_m(\sigma(t))))}$$

is Δ-measurable.

Definition 3.1(a) guarantees that this map has nonempty values Δ-almost everywhere on $\{t : M(\sigma(t)) \neq 0\}$. Indeed, Δ-almost everywhere on

$$\{t : M(\sigma(t)) \neq 0 \text{ and } \|\overline{x}(\sigma(t)) - v(\sigma(t))\| < M(\sigma(t))\},$$

for $m \geq k$ sufficiently large, $\|x_m(\sigma(t)) - v(\sigma(t))\| < M(\sigma(t))$ and

$$F(t, x_m(\sigma(t))) \cap G(t, x_m(\sigma(t))) = F(t, x_m(\sigma(t))) \cap \mathbb{R}^n \neq \emptyset.$$

On the other hand, for Δ-almost every

$$t \in \{t : \|\overline{x}(\sigma(t)) - v(\sigma(t))\| = M(\sigma(t)) > 0\},$$

if there exists $m \geq k$ such that $\|x_m(\sigma(t)) - v(\sigma(t))\| \leq M(\sigma(t))$, then as before, $F(t, x_m(\sigma(t))) \cap G(t, x_m(\sigma(t))) \neq \emptyset$. Otherwise, there exists a $\delta > 0$ given by Definition 3.1(a) and $m \geq k$ sufficiently large such that

$$\|x_m(\sigma(t)) - \overline{x}(\sigma(t))\| < \delta, \quad \|x_m(\sigma(t)) - v(\sigma(t))\| > M(\sigma(t)),$$

and there exists $z \in F(t, x_m(\sigma(t)))$ such that

$$\langle x_m(\sigma(t)) - v(\sigma(t)), z - v^{\Delta}(t)\rangle \leq \|x_m(\sigma(t)) - v(\sigma(t))\| \, M^{\Delta}(t),$$

i.e. $z \in F(t, x_m(\sigma(t))) \cap G(t, x_m(\sigma(t)))$.

Thus, the multivalued map $\Gamma\colon \mathbb{T}_0 \to L^1_{\Delta}(\mathbb{T}_0, \mathbb{R}^n)$ defined by

$$\Gamma(t) = \begin{cases} \bigcap_{k \in \mathbb{N}} \overline{\bigcup_{m \geq k} (F(t, x_m(\sigma(t))) \cap G(t, x_m(\sigma(t))))} & \text{if } t \in \{t : M(\sigma(t)) \neq 0\}, \\ v^{\Delta}(t) & \text{if } t \in \{t : M(\sigma(t)) = 0\}, \end{cases}$$

is Δ-measurable and has nonempty and compact values. Finally, Theorem 2.3 guarantees the existence of a Δ-measurable selection w of Γ.

We must show that $w \in \mathcal{F}(x)$. Since $w(t) \in \Gamma(t)$ Δ-a.e., we have,

$$w(t) \in \overline{\bigcup_{m \geq k} (F(t, x_m(\sigma(t))) \cap G(t, x_m(\sigma(t))))} \quad \Delta\text{-a.e. in } \{t : M(\sigma(t)) \neq 0\}.$$

for every $k \in \mathbb{N}$. So, for Δ-almost every $t \in \{t : M(\sigma(t)) \neq 0\}$, there exists a subsequence

$$u_{m_l}(t) \in F(t, x_{m_l}(\sigma(t))) \cap G(t, x_{m_l}(\sigma(t)))$$

such that $u_{m_l}(t) \to w(t)$. If $\|x(\sigma(t)) - v(\sigma(t))\| \leq M(\sigma(t))$, since $y \mapsto F(t,y)$ and $y \mapsto G(t,y)$ have closed graph and since $x_{m_l}(\sigma(t)) \to \overline{x}(\sigma(t)) = x(\sigma(t))$, we deduce that

$$w(t) \in F(t, \overline{x}(\sigma(t))) \cap G(t, x(\sigma(t))) = F_0(t, x(\sigma(t))).$$

On the other hand, if $\|x(\sigma(t)) - v(\sigma(t))\| > M(\sigma(t))$, since $x_{m_l}(\sigma(t)) \to \overline{x}(\sigma(t))$, there exists a sequence $\{y_{m_l}\}$ such that

$$x^-(\sigma(t), y_{m_l}) = \overline{x}_{m_l}(\sigma(t)), \quad y_{m_l} \to x(\sigma(t))$$

and

$$x_{m_l}(\sigma(t)) = \theta_{m_l}\overline{x}_{m_l}(\sigma(t)) + (1-\theta_{m_l})y_{m_l} = (y_{m_l})_{\theta_{m_l}} \quad \text{for some } \theta_{m_l} \in [0,1[.$$

By Remark 3.2(a),

$$u_{m_l}(t) \in F(t, x_{m_l}(\sigma(t))) \cap G(t, x_{m_l}(\sigma(t))) = F(t, x_{m_l}(\sigma(t))) \cap G(t, y_{m_l}).$$

Again, since $y \mapsto F(t,y)$ and $y \mapsto G(t,y)$ have closed graph and since $x_{m_l}(\sigma(t)) \to \overline{x}(\sigma(t))$ and $y_{m_l} \to x(\sigma(t))$, we can deduce that

$$w(t) \in F(t, \overline{x}(\sigma(t))) \cap G(t, x(\sigma(t))) = F_0(t, x(\sigma(t))).$$

Moreover, Definition 3.1(b) implies that Δ-a.e. on $\{t : M(\sigma(t)) = 0\}$,

$$w(t) = v^\Delta(t) \in F(t, \overline{x}(\sigma(t))) \cap G(t, x(\sigma(t))) = F_0(t, x(\sigma(t))).$$

Hence, we can conclude that $w \in \mathcal{F}(x)$ since by hypothesis (F2), $w \in L^1_\Delta(\mathbb{T}_0, \mathbb{R}^n)$.

The convexity of $\mathcal{F}(x)$ follows from convexity of the values of F and G.

Finally, hypothesis (F2) guarantees the existence of $h := h_r \in L^1_\Delta(\mathbb{T}_0, [0,\infty))$ with $r = \max\{\|v(t)\| + M(t) : t \in \mathbb{T}\}$ such that for every $x \in C(\mathbb{T}, \mathbb{R}^n)$ and every $w \in \mathcal{F}(x)$,

$$\|w(t)\| \leq h(t) \quad \Delta\text{-a.e. } t \in \mathbb{T}_0. \qquad \square$$

Now, we define the multivalued operator $T_I : C(\mathbb{T}, \mathbb{R}^n) \to C(\mathbb{T}, \mathbb{R}^n)$ by

$$T_I(x) = \bigg\{ u \in C(\mathbb{T}, \mathbb{R}^n) : u(t) = e_1(a,t)\bigg(x_0 + \int_{[a,t)\cap\mathbb{T}} e_1(s,a)(w(s) + \overline{x}(\sigma(s)))\Delta(s)\bigg), \text{ where } w \in \mathcal{F}(x) \bigg\}.$$

We show that T_I has nice properties. Many arguments in the following proof are analogous to those used in the classical case (i.e. $\mathbb{T} = [a,b]$), see for instance [13], [17].

PROPOSITION 3.5. *Assume* (F1), (F2) *and* (ST). *The operator* T_I *is compact, u.s.c., with nonempty, convex and compact values.*

PROOF. The previous proposition insures that T_I has nonempty, convex values, and guarantees the existence of $h \in L^1_\Delta(\mathbb{T}_0, [0, \infty))$ satisfying (3.3).

Set $r = \max\{\|v(t)\| + M(t) : t \in \mathbb{T}\}$ and $c = \max\{|e_1(t, s)| : t, s \in \mathbb{T}\}$. To show that $T_I(C(\mathbb{T}, \mathbb{R}^n))$ is bounded, we just have to remark that for every $u \in T_I(C(\mathbb{T}, \mathbb{R}^n))$,

$$\|u(t)\| \leq c\bigg(\|x_0\| + \int_{[a,b)\cap\mathbb{T}} c(r + h(s))\Delta(s)\bigg) \quad \text{for all } t \in \mathbb{T}.$$

On the other hand, for every $t > \tau \in \mathbb{T}$,

$$\begin{aligned} \|u(t) - u(\tau)\| \leq{}& \|x_0\|\, |e_1(a, t) - e_1(a, \tau)| \\ &+ |e_1(a, t) - e_1(a, \tau)| \bigg| \int_{[a,\tau)\cap\mathbb{T}} e_1(s, a)(w(s) + \overline{x}(\sigma(s)))\Delta(s) \bigg| \\ &+ |e_1(a, t)| \bigg| \int_{[\tau,t)\cap\mathbb{T}} e_1(s, a)(w(s) + \overline{x}(\sigma(s)))\Delta(s) \bigg| \\ \leq{}& |e_1(a, t) - e_1(a, \tau)| \bigg(\|x_0\| + \int_{[a,b)\cap\mathbb{T}} c(h(s) + r)\Delta(s)\bigg) \\ &+ c^2 \int_{[\tau,t)\cap\mathbb{T}} (h(s) + r)\, \Delta(s). \end{aligned}$$

Thus, $T_I(C(\mathbb{T}, \mathbb{R}^n))$ is equicontinuous since

$$t \mapsto e_1(a, t) \quad \text{and} \quad t \mapsto \int_{[a,t)\cap\mathbb{T}} (h(s) + r)\, \Delta(s)$$

are continuous on $\mathbb{T}$. By an analogous version of the Arzelà–Ascoli Theorem adapted to our context, we conclude that $T_I(C(\mathbb{T}, \mathbb{R}^n))$ is relatively compact in $C(\mathbb{T}, \mathbb{R}^n)$.

We now prove that T_I has closed graph. Let $\{x_m\}$ and $\{u_m\}$ be convergent sequences in $C(\mathbb{T}, \mathbb{R}^n)$ such that $x_m \to x$, $u_m \to u$ and $u_m \in T_I(x_m)$. Let $w_m \in \mathcal{F}(x_m)$ be such that

$$u_m(t) = e_1(a, t)\bigg(x_0 + \int_{[a,t)\cap\mathbb{T}} e_1(s, a)(w_m(s) + \overline{x}_m(\sigma(s)))\Delta(s)\bigg).$$

Let h be the function given in (3.3). Considering the extensions $\widehat{w}_m$ and $\widehat{h}$ in $L^1([a, b])$, we have

$$\|\widehat{w}_m(t)\| \leq \widehat{h}(t) \quad \text{for almost every } t \in [a, b].$$

By Dunford–Pettis' Theorem, there exists $g \in L^1([a, b], \mathbb{R}^n)$ and a subsequence still denoted $\{\widehat{w}_m\}$ such that $\widehat{w}_m \rightharpoonup g$ in $L^1([a, b], \mathbb{R}^n)$. Since a closed convex set is weakly closed, there exist

$$\widehat{z}_m \in \mathrm{co}\{\widehat{w}_m, \widehat{w}_{m+1}, \dots\}$$

such that

$$\widehat{z}_m \to g \quad \text{in } L^1([a,b],\mathbb{R}^n).$$

Thus, there exists a subsequence again noted $\{\widehat{z}_m\}$ such that,

$$\widehat{z}_m(t) \to g(t) \quad \text{for almost every } t \in [a,b].$$

Therefore, for almost every $t \in [a,b]$,

$$\widehat{z}_m(t) \in \mathrm{co}\bigg\{\bigcup_{l\geq m} \widehat{w}_l(t)\bigg\} \subset \mathrm{co}\bigg\{\bigcup_{l\geq m} \widehat{F}(t,\overline{x}_l(\sigma(t))) \cap \widehat{G}(t,x_l(\sigma(t)))\bigg\}$$

where the multivalued maps $\widehat{F}$ and $\widehat{G}$ are respectively extensions of the multivalued maps F and G in the sense of (2.1). Taking the limit, we get

$$\begin{aligned} g(t) \in \bigcap_{m\in\mathbb{N}} \overline{\mathrm{co}}\bigg\{\bigcup_{l\geq m} \widehat{F}(t,\overline{x}_l(\sigma(t))) \cap \widehat{G}(t,x_l(\sigma(t)))\bigg\} \\ \subset \widehat{F}(t,\overline{x}(\sigma(t))) \cap \widehat{G}(t,x(\sigma(t))) = \widehat{F}_0(t,x(\sigma(t))), \end{aligned}$$

since $x_m \to x$ in $C(\mathbb{T},\mathbb{R}^n)$ and since $y \mapsto \widehat{F}(t,y)$ and $y \mapsto \widehat{G}(t,y)$ have closed graph and closed, convex values.

By Theorem 2.10, there exists a function $w\colon \mathbb{T}_0 \to \mathbb{R}^n$ such that $g = \widehat{w}$. So,

$$w(t) \in \widehat{F}_0(t,x(\sigma(t))) = F_0(t,x(\sigma(t))) \quad \Delta\text{-a.e. } t \in \mathbb{T}_0.$$

Thus, $w \in \mathcal{F}(x)$.

Finally, since $\widehat{w}_m \rightharpoonup \widehat{w}$ in $L^1([a,b],\mathbb{R}^n)$ and $x_m \to x$ in $C(\mathbb{T},\mathbb{R}^n)$, again by Theorem 2.10, we deduce that for every $t \in \mathbb{T}$,

$$\int_{[a,t)\cap\mathbb{T}} e_1(s,a)(w_m(s) + \overline{x}_m(\sigma(s)))\Delta s \to \int_{[a,t)\cap\mathbb{T}} e_1(s,a)(w(s) + \overline{x}(\sigma(s)))\Delta s.$$

Moreover, since $u_m \to u$ in $C(\mathbb{T},\mathbb{R}^n)$, we get that for every $t \in \mathbb{T}$,

$$u(t) = e_1(a,t)\bigg(x_0 + \int_{[a,t)\cap\mathbb{T}} e_1(s,a)(w(s) + \overline{x}(\sigma(s)))\Delta s\bigg).$$

Thus, $u \in T_I(x)$ and hence, T_I has closed graph.

Since T_I is compact and has closed graph, T_I has compact values.

We now prove that T_I is upper semi-continuous. Let $B \subset C(\mathbb{T},\mathbb{R}^n)$ be a closed set and

$$A = \{x \in C(\mathbb{T},\mathbb{R}^n) : T_I(x) \cap B \neq \emptyset\}.$$

Let $\{x_m\}$ be a sequence in A converging to x in $C(\mathbb{T},\mathbb{R}^n)$. There exists $u_m \in T_I(x_m) \cap B$. The compacity of T_I guarantees the existence of a subsequence still denoted $\{u_m\}$ converging to u in $C(\mathbb{T},\mathbb{R}^n)$. Since B is closed and T_I has closed graph, we deduce that $u \in T_I(x) \cap B$. Thus $x \in A$. □

Let $T_P\colon C(\mathbb{T},\mathbb{R}^n) \to C(\mathbb{T},\mathbb{R}^n)$ be the multivalued operator defined by

$$T_P(x)(t) = \bigg\{ v \in C(\mathbb{T},\mathbb{R}^n) : \\ v(t) = \frac{1}{e_1(t,a)}\bigg(\frac{1}{e_1(b,a)-1}\int_{[a,b)\cap\mathbb{T}} (w(s)+\overline{x}(\sigma(s)))e_1(s,a)\Delta s \\ + \int_{[a,t)\cap\mathbb{T}} (w(s)+\overline{x}(\sigma(s)))e_1(s,a)\Delta s\bigg),\ \text{where } w \in \mathcal{F}(x)\bigg\}.$$

The following result can be proved as the previous one.

PROPOSITION 3.6. *Assume* (F1)*,* (F2) *and* (ST)*. The operator* T_P *is compact and u.s.c. with nonempty, convex and compact values.*

Now, we can obtain our main theorem.

THEOREM 3.7. *Assume* (F1)*,* (F2) *and* (ST)*. The problem (1.1) has a solution* $x \in W^{1,1}_\Delta(\mathbb{T},\mathbb{R}^n) \cap T(v,M)$.

PROOF. By Proposition 3.5 (resp. Proposition 3.6), T_I (resp. T_P) is compact and upper semi-continuous with nonempty, convex, and compact values. It has a fixed point by the Kakutani fixed point Theorem. If (BC) denotes (1.2) (resp. (1.3)), Proposition 2.15 (resp. Proposition 2.16) implies that, x, this fixed point of T_I (resp. T_P) is a solution of Problem (3.1), (1.2) (resp. (3.1), (1.3)). To conclude, it suffices to show that $x \in T(v,M)$.

Consider the set $A = \{t \in \mathbb{T}_0 : \|x(\sigma(t)) - v(\sigma(t))\| > M(\sigma(t))\}$. By Proposition 2.6(d), Δ-a.e. on $A \setminus R_\mathbb{T}$, we have

$$\|x(t)-v(t)\|^\Delta = \frac{\langle x(t)-v(t), x^\Delta(t)-v^\Delta(t)\rangle}{\|x(t)-v(t)\|} = \frac{\langle x(\sigma(t))-v(\sigma(t)), x^\Delta(t)-v^\Delta(t)\rangle}{\|x(\sigma(t))-v(\sigma(t))\|}. \tag{3.4}$$

For $t \in A \cap R_\mathbb{T}$, $\mu_\Delta(\{t\}) > 0$ and,

$$\begin{aligned} \|x(t)-v(t)\|^\Delta &= \frac{\|x(\sigma(t))-v(\sigma(t))\| - \|x(t)-v(t)\|}{\mu(t)} \\ &\le \frac{\langle x(\sigma(t))-v(\sigma(t)), x(\sigma(t))-v(\sigma(t))-(x(t)-v(t))\rangle}{\mu(t)\|x(\sigma(t))-v(\sigma(t))\|} \\ &= \frac{\langle x(\sigma(t))-v(\sigma(t)), x^\Delta(t)-v^\Delta(t)\rangle}{\|x(\sigma(t))-v(\sigma(t))\|}. \end{aligned} \tag{3.5}$$

Let us denote $y(t) := x^\Delta(t) + x(\sigma(t)) - \overline{x}(\sigma(t)) \in F_0(t, x(\sigma(t)))$ Δ-a.e. on $\mathbb{T}_0$. Since (v, M) is a solution-tube of (1.1) and from (3.2), (3.4), (3.4), and Remark 3.2(b), we deduce that Δ-a.e. on $\{t \in A : M(\sigma(t)) > 0\}$,

$$\begin{aligned}(\|x(t) &- v(t)\| - M(t))^\Delta \\ &\leq \frac{\langle x(\sigma(t)) - v(\sigma(t)), y(t) + \overline{x}(\sigma(t)) - x(\sigma(t)) - v^\Delta(t)\rangle}{\|x(\sigma(t)) - v(\sigma(t))\|} - M^\Delta(t) \\ &= \frac{\langle \overline{x}(\sigma(t)) - v(\sigma(t)), y(t) - v^\Delta(t)\rangle}{M(\sigma(t))} \\ &\quad + M(\sigma(t)) - \|x(\sigma(t)) - v(\sigma(t))\| - M^\Delta(t) \\ &< \frac{M(\sigma(t))M^\Delta(t)}{M(\sigma(t))} - M^\Delta(t) = 0.\end{aligned}$$

On the other hand, if $M(\sigma(t)) = 0$, then $F_0(t, x(\sigma(t))) = \{v^\Delta(t)\}$ and Δ-a.e. on $\{t \in A : M(\sigma(t)) = 0\}$, we have

$$\begin{aligned}(\|x(t) &- v(t)\| - M(t))^\Delta \\ &\leq \frac{\langle x(\sigma(t)) - v(\sigma(t)), y(t) + \overline{x}(\sigma(t)) - x(\sigma(t)) - v^\Delta(t)\rangle}{\|x(\sigma(t)) - v(\sigma(t))\|} - M^\Delta(t) \\ &= \frac{\langle x(\sigma(t)) - v(\sigma(t)), v^\Delta(t) - v^\Delta(t)\rangle}{\|x(\sigma(t)) - v(\sigma(t))\|} \\ &\quad - \|x(\sigma(t)) - v(\sigma(t))\| - M^\Delta(t) < -M^\Delta(t) = 0.\end{aligned}$$

This last equality follows from Definition 3.1(c) and Proposition 2.13.

Therefore, $r(t) = \|x(t) - v(t)\| - M(t)$ satisfies $r^\Delta(t) < 0$ Δ-almost everywhere on $A = \{t \in \mathbb{T}_0 : r(\sigma(t)) > 0\}$. Moreover, since (v, M) is a solution-tube of (1.1), if (BC) denotes (1.2) (resp. (BC) denotes (1.3)), then $r(a) \leq 0$ (resp. $r(a) - r(b) \leq \|v(a) - v(b)\| - (M(a) - M(b)) \leq 0$). Lemma 2.14 implies that $A = \emptyset$. Therefore, $x \in T(v, M)$ and hence, x is a solution of (1.1). □

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