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WEIGHTED PSEUDO ALMOST AUTOMORPHIC SOLUTIONS TO NONAUTONOMOUS SEMILINEAR EVOLUTION EQUATIONS WITH DELAY AND S^p-WEIGHTED PSEUDO ALMOST AUTOMORPHIC COEFFICIENTS

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Abstract. By some new properties of Stepanov-like weighted pseudo almost automorphic functions established by Chang, Zhang and N'Guérékata recently, we shall deal with weighted pseudo almost automorphic solutions to the nonautonomous semilinear evolution equations with a constant delay: $u'(t) = A(t)u(t) + f(t, u(t-h))$, $t \in \mathbb{R}$ in a Banach space $\mathbb{X}$, where $A(t), t \in \mathbb{R}$ generates an exponentially stable evolution family $\{U(t,s)\}$ and $f: \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is a S^p-weighted pseudo almost automorphic function satisfying some suitable conditions. We obtain our main results by the Leray–Shauder Alternative theorem.

1. Introduction

In this paper, we deal with some existence results for weighted pseudo almost automorphic mild solutions to the following nonautonomous semilinear evolution equation with a constant delay:

$$u'(t) = A(t)u(t) + f(t, u(t-h)), \quad t \in \mathbb{R}, \tag{1.1}$$

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where $h \geq 0$ is a fixed constant, and $\{A(t)\}_{t\in\mathbb{R}}$ satisfies the "Acquistapace–Terreni" condition in [1], and $U(t,s)$ generated by $A(t)$, is exponentially stable, and $f \in \mathrm{WPAAS}^p(\mathbb{X}) \cap C(\mathbb{R},\mathbb{X})$ for $p > 1$.

In the earlier sixties, Bochner introduced the concept of almost automorphic functions in his papers [4]–[6] in relation to some aspects of differential geometry. G.M. N'Guérékata and Pankov introduced the concept of Stepanov-like almost automorphy and applied this concept to study the existence and uniqueness of an almost automorphic solution to the autonomous semilinear equation in [11]. Moreover, J. Blot introduce the notion of weighted pseudo almost automorphic functions with values in a Banach space in [3]. Very recently, motivated by the works [7], [11], [12], [15], [21], [16], Zhang, Chang and N'Guérékata established some new composition theorems of Stepanov-like weighted pseudo almost automorphic functions in [27].

In [19], Haewon Lee and Hadi Alkahby considered Stepanov-like almost automorphic solutions to equation (1.1). In this paper, we shall apply the concept of S^p-weighted pseudo almost automorphy and the composition theorems to obtain some existence results of weighted pseudo almost automorphic solution to the nonautonomous semilinear evolution equation (1.1) with S^p-weighted pseudo almost automorphic coefficients.

The rest of this paper is organized as follows. In Section 2, we recall some basic definitions, lemmas, and preliminary results which will be used throughout this paper. In Section 3, we prove the existence of weighted pseudo almost automorphic mild solutions to the nonautonomous semilinear evolution equation (1.1). In the last section, we give a suitable example.

2. Preliminaries

Throughout this paper, we consider $(\mathbb{X}, \|\cdot\|)$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ are two Banach spaces. The notation $C(\mathbb{R},\mathbb{X})$ stands for the space of all continuous functions from $\mathbb{R}$ into $\mathbb{X}$. We let $\mathrm{BC}(\mathbb{R},\mathbb{X})$ (respectively, $\mathrm{BC}(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) stand for the class of all bounded continuous functions from $\mathbb{R}$ into $\mathbb{X}$ (respectively, the class of all jointly bounded continuous functions from $\mathbb{R}\times\mathbb{Y}$ into $\mathbb{X}$). Note that $\mathrm{BC}(\mathbb{R},\mathbb{X})$ is a Banach space with the sup norm $\|\cdot\|_\infty$.

Definition 2.1 [23]. A continuous function $f\colon \mathbb{R} \to \mathbb{X}$ is said to be *almost automorphic* if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$ there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t+s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t-s_n) = f(t) \quad \text{for each } t \in \mathbb{R}.$$

The collection of all such functions is a Banach space and will be denoted by $\mathrm{AA}(\mathbb{X})$.

Definition 2.2 ([10]). A continuous function $f(t,s)\colon \mathbb{R}\times\mathbb{R}\to\mathbb{X}$ is called *bi-almost automorphic* if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$g(t,s) := \lim_{n\to\infty} f(t+s_n, s+s_n)$$

is well defined for each $t,s\in\mathbb{R}$, and

$$\lim_{n\to\infty} g(t-s_n, s-s_n) = f(t,s) \quad \text{for each } t,s\in\mathbb{R}.$$

The collection of all such functions will be denoted by $\mathrm{bAA}(\mathbb{R}\times\mathbb{R},\mathbb{X})$.

Now, like in [9], let $\mathbb{U}$ denote the set of all functions $\rho\colon\mathbb{R}\to(0,\infty)$, which are locally integrable over $\mathbb{R}$ such that $\rho>0$ almost everywhere. Set

$$m(T,\rho) := \int_{-T}^{T} \rho(t)\,dt.$$

Thus the space of weights $\mathbb{U}_\infty$ is defined by

$$\mathbb{U}_\infty := \Big\{\rho\in\mathbb{U} : \lim_{T\to\infty} m(T,\rho)=\infty\Big\}.$$

Now for $\rho\in\mathbb{U}_\infty$, we define

$$\mathrm{PAA}_0(\mathbb{X},\rho) := \Big\{ f\in \mathrm{BC}(\mathbb{R},\mathbb{X}) : \lim_{T\to\infty}\frac{1}{m(T,\rho)}\int_{-T}^{T}\|f(t)\|\rho(t)\,dt = 0\Big\};$$

$$\begin{aligned}\mathrm{PAA}_0(\mathbb{Y},\mathbb{X},\rho) := \Big\{ & f\in C(\mathbb{R}\times\mathbb{Y},\mathbb{X}) : f(\,\cdot\,,y) \text{ is bounded for each } y\in\mathbb{Y}\\ &\text{and}\quad \lim_{T\to\infty}\frac{1}{m(T,\rho)}\int_{-T}^{T}\|f(t,y)\|\rho(t)\,dt = 0 \text{ uniformly in } y\in\mathbb{Y}\Big\}.\end{aligned}$$

Definition 2.3 ([3]). Let $\rho\in\mathbb{U}_\infty$. A function $f\in\mathrm{BC}(\mathbb{R},\mathbb{X})$ (respectively, $f\in\mathrm{BC}(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) is called *weighted pseudo almost automorphic* if it can be expressed as $f=g+h$, where $g\in\mathrm{AA}(\mathbb{X})$ (respectively, $\mathrm{AA}(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) and $h\in\mathrm{PAA}_0(\mathbb{X},\rho)$ (respectively, $\mathrm{PAA}_0(\mathbb{Y},\mathbb{X},\rho)$). We denote by $\mathrm{WPAA}(\mathbb{X})$ (respectively, $\mathrm{WPAA}(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) the set of all such functions.

Lemma 2.4 ([22, Theorem 2.14]). *Let $\rho\in\mathbb{U}_\infty$. If $\mathrm{PAA}_0(\mathbb{X},\rho)$ is translation invariant, then $(\mathrm{WPAA}(\mathbb{X}),\|\cdot\|_\infty)$ is a Banach space.*

Definition 2.5 ([12], [24]). The Bochner transform $f^b(t,s)$, $t\in\mathbb{R}$, $s\in[0,1]$ of a function $f\colon\mathbb{R}\to\mathbb{X}$ is defined by

$$f^b(t,s) := f(t+s).$$

Definition 2.6 ([12]). The Bochner transform $f^b(t,s,u)$, $t\in\mathbb{R}$, $s\in[0,1]$, $u\in\mathbb{X}$ of a function $f\colon\mathbb{R}\times\mathbb{X}\to\mathbb{X}$ is defined by

$$f^b(t,s,u):=f(t+s,u)\quad\text{for each } u\in\mathbb{X}.$$

Definition 2.7 ([12], [24]). Let $p\in[1,\infty)$. The space $\mathrm{BS}^p(\mathbb{X})$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f\colon\mathbb{R}\to\mathbb{X}$ such that $f^b\in L^\infty(\mathbb{R},L^p(0,1;\mathbb{X}))$. This is a Banach space with the norm

$$\|f\|_{\mathrm{S}^p}=\|f^b\|_{L^\infty(\mathbb{R},L^p)}=\sup_{t\in\mathbb{R}}\bigg(\int_t^{t+1}\|f(\tau)\|^p\,d\tau\bigg)^{1/p}.$$

Definition 2.8 ([19], [24]). The space $\mathrm{AS}^p(\mathbb{X})$ of Stepanov-like almost automorphic (or S^p-almost automorphic) functions consists of all $f\in\mathrm{BS}^p(\mathbb{X})$ such that $f^b\in\mathrm{AA}(L^p(0,1;\mathbb{X}))$. In other words, a function $f\in L^p_{\mathrm{loc}}(\mathbb{R},\mathbb{X})$ is said to be S^p-almost automorphic if its Bochner transform $f^b\colon\mathbb{R}\to L^p(0,1;\mathbb{X})$ is almost automorphic in the sense that for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exist a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a function $g\in L^p_{\mathrm{loc}}(\mathbb{R},\mathbb{X})$ such that

$$\lim_{n\to\infty}\bigg(\int_t^{t+1}\|f(s+s_n)-g(s)\|^p\,ds\bigg)^{1/p}=0$$

and

$$\lim_{n\to\infty}\bigg(\int_t^{t+1}\|g(s-s_n)-f(s)\|^p\,ds\bigg)^{1/p}=0.$$

pointwise on $\mathbb{R}$.

Definition 2.9 ([19], [24]). A function $f\colon\mathbb{R}\times\mathbb{Y}\to\mathbb{X}$, $(t,u)\to f(t,u)$ with $f(\,\cdot\,,u)\in L^p_{\mathrm{loc}}(\mathbb{R},\mathbb{X})$ for each $u\in\mathbb{Y}$, is said to be S^p*-almost automorphic in* $t\in\mathbb{R}$ *uniformly in* $u\in\mathbb{Y}$ if $t\to f(t,u)$ is S^p-almost automorphic for each $u\in\mathbb{Y}$. That means, for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exist a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a function $g(\,\cdot\,,u)\in L^p_{\mathrm{loc}}(\mathbb{R},\mathbb{X})$ such that

$$\lim_{n\to\infty}\bigg(\int_t^{t+1}\|f(s+s_n,u)-g(s,u)\|^p\,ds\bigg)^{1/p}=0,$$

and

$$\lim_{n\to\infty}\bigg(\int_t^{t+1}\|g(s-s_n,u)-f(s,u)\|^p\,ds\bigg)^{1/p}=0,$$

pointwise on $\mathbb{R}$ and for each $u\in\mathbb{Y}$. We denote by $AS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ the set of all such functions.

Definition 2.10 ([27]). Let $\rho \in \mathbb{U}_\infty$. A function $f \in \mathrm{BS}^p(\mathbb{X})$ is said to be Stepanov-like weighted pseudo almost automorphic (or S^p-weighted pseudo almost automorphic) if it can be expressed as $f = g + h$, where $g \in \mathrm{AS}^p(\mathbb{X})$ and $h^b \in \mathrm{PAA}_0(L^p(0,1;\mathbb{X}),\rho)$. In other words, a function $f \in L^p_{\mathrm{loc}}(\mathbb{R},\mathbb{X})$ is said to be *Stepanov-like weighted pseudo almost automorphic relatively to the weight* $\rho \in \mathbb{U}_\infty$, if its Bochner transform $f^b: \mathbb{R} \to L^p(0,1;\mathbb{X})$ is weighted pseudo almost automorphic in the sense that there exist two functions $g, h: \mathbb{R} \to \mathbb{X}$ such that $f = g + h$, where $g \in \mathrm{AS}^p(\mathbb{X})$ and $h^b \in \mathrm{PAA}_0(L^p(0,1;\mathbb{X}),\rho)$. We denoted by $\mathrm{WPAAS}^p(\mathbb{X})$ the set of all such functions.

Definition 2.11 ([27]). Let $\rho \in \mathbb{U}_\infty$. A function $f: \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$, $(t,u) \to f(t,u)$ with $f(\,\cdot\,,u) \in L^p_{\mathrm{loc}}(\mathbb{R},\mathbb{X})$ for each $u \in \mathbb{Y}$, is said to be *Stepanov-like weighted pseudo almost automorphic* (or S^p*-weighted pseudo almost automorphic*) if it can be expressed as $f = g + h$, where $g \in \mathrm{AS}^p(\mathbb{R} \times \mathbb{Y},\mathbb{X})$ and $h^b \in \mathrm{PAA}_0(\mathbb{Y}, L^p(0,1;\mathbb{X}),\rho)$. We denoted by $\mathrm{WPAAS}^p(\mathbb{R} \times \mathbb{Y},\mathbb{X})$ the set of all such functions.

Lemma 2.12 ([27]). *Let $\rho \in \mathbb{U}_\infty$. Assume that $\mathrm{PAA}_0(L^p(0,1;\mathbb{X}),\rho)$ is translation invariant. Then the decomposition of a S^p-weighted pseudo almost automorphic function is unique.*

Lemma 2.13 ([27]). *If $f \in \mathrm{WPAA}(\mathbb{X})$, then $f \in \mathrm{WPAAS}^p(\mathbb{X})$ for each $1 \le p < \infty$. In other words, $\mathrm{WPAA}(\mathbb{X}) \subset \mathrm{WPAAS}^p(\mathbb{X})$.*

Lemma 2.14 ([27]). *Let $\rho \in \mathbb{U}_\infty$. The space $\mathrm{WPAAS}^p(\mathbb{X})$ equipped with the norm $\|\cdot\|_{\mathrm{S}^p}$ is a Banach space.*

Theorem 2.15 ([27]). *Let $\rho \in \mathbb{U}_\infty$ and let $f = g + h \in \mathrm{WPAAS}^p(\mathbb{R} \times \mathbb{X},\mathbb{X})$ with $g \in \mathrm{AS}^p(\mathbb{R} \times \mathbb{X},\mathbb{X})$, $h^b \in \mathrm{PAA}_0(\mathbb{X}, L^p(0,1;\mathbb{X}),\rho)$. Assume that the following conditions are satisfied:*

(a) *$f(t,x)$ is Lipschitzian in $x \in \mathbb{X}$ uniformly in $t \in \mathbb{R}$; that is, there exists a constant $L > 0$ such that*

$$\|f(t,x) - f(t,y)\| \le L\|x - y\| \quad \text{for all } x, y \in \mathbb{X} \text{ and } t \in \mathbb{R}.$$

(b) *$g(t,x)$ is uniformly continuous in any bounded subset $K' \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$.*

If $u = u_1 + u_2 \in \mathrm{WPAAS}^p(\mathbb{X})$, with $u_1 \in \mathrm{AS}^p(\mathbb{X})$, $u_2^b \in \mathrm{PAA}_0(L^p(0,1;\mathbb{X}),\rho)$ and $K = \overline{\{u_1(t) : t \in \mathbb{R}\}}$ is compact, then $\Lambda: \mathbb{R} \to \mathbb{X}$ defined by $\Lambda(\,\cdot\,) = f(\,\cdot\,, u(\,\cdot\,))$ belongs to $\mathrm{WPAAS}^p(\mathbb{X})$.

Theorem 2.16 ([27]). *Let* $\rho \in \mathbb{U}_\infty$ *and let* $f = g + h \in \mathrm{WPAAS}^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$ *with* $g \in \mathrm{AS}^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$, $h^b \in \mathrm{PAA}_0(\mathbb{X}, L^p(0,1;\mathbb{X}),\rho)$. *Assume that the following conditions are satisfied*:

(a) *there exists a nonnegative function* $L \in \mathrm{BS}^p(\mathbb{R})$ *with* $p > 1$ *such that for all* $u, v \in \mathbb{X}$ *and* $t \in \mathbb{R}$

$$\bigg(\int_t^{t+1} \|f(s,u) - f(s,v)\|^p\,ds\bigg)^{1/p} \le L(t)\|u - v\|.$$

(b) $\rho \in L^q_{\mathrm{loc}}(\mathbb{R})$ *satisfies*

$$\lim_{T\to\infty}\sup \frac{T^{1/p} m_q(T,\rho)}{m(T,\rho)} < \infty.$$

(c) $g(t,x)$ *is uniformly continuous in any bounded subset* $K \subset \mathbb{X}$ *uniformly for* $t \in \mathbb{R}$.

If $u = u_1 + u_2 \in \mathrm{WPAAS}^p(\mathbb{X})$, *with* $u_1 \in \mathrm{AS}^p(\mathbb{X})$, $u_2^b \in \mathrm{PAA}_0(L^p(0,1;\mathbb{X}),\rho)$ *and* $K = \overline{\{u_1(t) : t \in \mathbb{R}\}}$ *is compact, then* $\Lambda\colon \mathbb{R}\to\mathbb{X}$ *defined by* $\Lambda(\,\cdot\,) = f(\,\cdot\,, u(\,\cdot\,))$ *belongs to* $\mathrm{WPAAS}^p(\mathbb{X})$.

Theorem 2.17 ([27]). *Let* $\rho \in \mathbb{U}_\infty$ *and let* $f\colon \mathbb{R}\times\mathbb{X}\to\mathbb{X}$ *be a* S^p*-weighted pseudo almost automorphic function. Suppose that* f *satisfies the following conditions*:

(a) $f(t,x)$ *is uniformly continuous in any bounded subset* $K' \subset \mathbb{X}$ *uniformly for* $t \in \mathbb{R}$,

(b) $g(t,x)$ *is uniformly continuous in any bounded subset* $K' \subset \mathbb{X}$ *uniformly for* $t \in \mathbb{R}$,

(c) *For every bounded subset* $K' \subset \mathbb{X}$, $\{f(\,\cdot\,,x)\colon x \in K'\}$ *is bounded in* $\mathrm{WPAAS}^p(\mathbb{X})$.

If $x = \alpha + \beta \in \mathrm{WPAAS}^p(\mathbb{X})$, *with* $\alpha \in \mathrm{AS}^p(\mathbb{X})$, $\beta^b \in \mathrm{PAA}_0(L^p(0,1;\mathbb{X}),\rho)$ *and* $K = \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ *is compact, then the function* $f(\,\cdot\,, x(\,\cdot\,))$ *belongs to* $\mathrm{WPAAS}^p(\mathbb{X})$.

Now, we recall a useful compactness criterion.

Let $h'\colon \mathbb{R}\to\mathbb{R}$ be a continuous function such that $h'(t) \ge 1$ for all $t \in \mathbb{R}$ and $h'(t) \to \infty$ as $|t| \to \infty$. We consider the space

$$C_{h'}(\mathbb{X}) = \bigg\{u \in C(\mathbb{R},\mathbb{X}) : \lim_{|t|\to\infty} \frac{u(t)}{h'(t)} = 0\bigg\}.$$

Endowed with the norm $\|u\|_{h'} = \sup_{t\in\mathbb{R}} \|u(t)\|/h'(t)$, it is a Banach space (see [18] and [2]).

Lemma 2.18 ([2], [18]). *A subset $R \subseteq C_{h'}(\mathbb{X})$ is a relatively compact set if it verifies the following conditions*:

(a) *The set $R(t) = \{u(t) : u \in R\}$ is relatively compact in $\mathbb{X}$ for each $t \in \mathbb{R}$.*
(b) *The set R is equicontinuous.*
(c) *For each $\varepsilon > 0$ there exists $L > 0$ such that $\|u(t)\| \le \varepsilon h'(t)$ for all $u \in R$ and all $|t| > L$.*

Lemma 2.19 (Leray–Schauder Alternative Theorem, [17]). *Let D be a closed convex subset of a Banach space $\mathbb{X}$ such that $0 \in D$. Let $F\colon D \to D$ be a completely continuous map. Then the set $\{x \in D : x = \lambda F(x),\ 0 < \lambda < 1\}$ is unbounded or the map F has a fixed point in D.*

Theorem 2.20 ([25]). *Assume that $A(t)$, $t \in \mathbb{R}$ is a bounded linear operator on a Banach space $\mathbb{X}$ and $t \to A(t)$ is continuous in the uniform operator topology, then for $-\infty < s \le t < \infty$, $U(t,s)$ generated by $A(t)$, is a bounded linear operator satisfying the following*:

(a) $\|U(t,s)\| \le \exp(\int_s^t \|A(\tau)\|\, d\tau)$.
(b) $U(t,t) = I$, $U(t,s) = U(t,r)U(r,s)$, *for* $-\infty < s \le r \le t < \infty$.
(c) $(t,s) \to U(t,s)$ *is continuous in the uniform operator topology for* $-\infty < s \le t < \infty$.
(d) $\partial U(t,s)/\partial t = A(t)U(t,s)$ *for* $-\infty < s \le t < \infty$.
(e) $\partial U(t,s)/\partial s = -U(t,s)A(s)$ *for* $-\infty < s \le t < \infty$.

3. Main results

In this paper we assume that $\{A(t)\}_{t\in\mathbb{R}}$ satisfies the "Acquistapace–Terreni" conditions introduced in [1], [14], that is,

(H1) There exist constants $\lambda_0 \ge 0$, $\theta \in (\pi/2, \pi)$, $\mathcal{L}, \mathcal{K} \ge 0$, and $\alpha, \beta \in (0,1]$ with $\alpha + \beta > 1$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0), \qquad \|R(\lambda, A(t) - \lambda_0)\| \le \frac{\mathcal{K}}{1 + |\lambda|}$$

and

$$\|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\| \le \mathcal{L}|t - s|^\alpha |\lambda|^{-\beta}$$

for $t, s \in \mathbb{R}$, $\lambda \in \Sigma_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| \le \theta\}$.

Remark 3.1 (see [1], [13]). If the condition (H1) holds, then there exists a unique evolution family $\{U(t,s)\}_{-\infty<s\le t<\infty}$ on $\mathbb{X}$, which satisfies the homogeneous equation $u'(t) = A(t)u(t)$, $t \in \mathbb{R}$.

We further suppose that

(H2) The evolution family $U(t,s)$ generated by $A(t)$ is exponentially stable, that is, there are constants $K, \omega > 0$ such that $\|U(t,s)\| \le Ke^{-\omega(t-s)}$

for all $t \geq s$. And the function $\mathbb{R}\times\mathbb{R} \mapsto \mathbb{X}$, $(t,s) \mapsto U(t,s)x \in \mathrm{bAA}(\mathbb{R}\times\mathbb{R},\mathbb{X})$ uniformly for all x in any bounded subset of $\mathbb{X}$.

Consider in the Banach space $(\mathbb{X}, \|\cdot\|)$ the abstract differential equation:

$$u'(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R} \tag{3.1}$$

where $\{A(t)\}_{t\in\mathbb{R}}$ satisfies the condition (H1) and $f \in \mathrm{WPAAS}^p(\mathbb{X}) \cap C(\mathbb{R},\mathbb{X})$ for $p > 1$. Throughout this paper we set $1/q = 1 - 1/p$. Note that $q \neq 0$, as $p \neq 1$.

Lemma 3.2. *Assume that* (H1) *and* (H2) *hold. Then the equation* (3.1) *has a unique weighted pseudo almost automorphic mild solution given by*

$$u(t) = \int_{-\infty}^{t} U(t,\sigma)f(\sigma)\,d\sigma.$$

Proof. To prove Lemma 3.2, we refer Theorem 3.2 in [19]. First, we prove the uniqueness of the weighted pseudo almost automorphic solution. Assume that $u(\cdot)\colon \mathbb{R} \to \mathbb{X}$ is bounded and satisfies the homogeneous equation

$$u'(t) = A(t)u(t), \quad t \in \mathbb{R} \tag{3.2}$$

Then $u(t) = U(t,s)u(s)$, for any $t \geq s$. Thus $\|u(t)\| \leq MKe^{-\omega(t-s)}$, where $\|u(s)\| \leq M$. Take a sequence of real numbers $\{s_n\}$ such that $s_n \to -\infty$ as $n \to \infty$. For any $t \in \mathbb{R}$ fixed, one can find a subsequence $\{s_{nk}\} \subset \{s_n\}$ such that $s_{nk} < t$ for all $k = 1, 2, \ldots$ By letting $k \to \infty$, we get $u(t) = 0$. Now, if u_1, u_2 are bounded solution to the equation (3.1), then $v = u_1 - u_2$ is a bounded solution to the equation (3.2). In view of the above, $v = u_1 - u_2 = 0$, that is, $u_1 = u_2$.

Now let us investigate the existence of the weighted pseudo almost automorphic solution. Since $f \in \mathrm{WPAAS}^p(\mathbb{X})$, there exist $g \in \mathrm{AS}^p(\mathbb{X})$ and $h^b \in \mathrm{PAA}_0(L^p(0,1;\mathbb{X}),\rho)$ such that $f = g + h$. So

$$\begin{aligned} u(t) &= \int_{-\infty}^{t} U(t,\sigma)f(\sigma)\,d\sigma \\ &= \int_{-\infty}^{t} U(t,\sigma)g(\sigma)\,d\sigma + \int_{-\infty}^{t} U(t,\sigma)h(\sigma)\,d\sigma = \Phi(t) + \Psi(t), \end{aligned}$$

where $\Phi(t) = \int_{-\infty}^{t} U(t,\sigma)g(\sigma)\,d\sigma$, and $\Psi(t) = \int_{-\infty}^{t} U(t,\sigma)h(\sigma)\,d\sigma$. We just need to verify $\Phi(t) \in \mathrm{AA}(\mathbb{X})$ and $\Psi(t) \in \mathrm{PAA}_0(\mathbb{X},\rho)$. Above all, we prove that $\Phi(t) \in \mathrm{AA}(\mathbb{X})$. Consider for each $n = 1, 2, \ldots$ the integral

$$\Phi_n(t) = \int_{n-1}^{n} U(t,t-\sigma)g(t-\sigma)\,d\sigma.$$

We obtain

$$\Phi_n(t) = \int_{t-n}^{t-n+1} U(t,\sigma)g(\sigma)\,d\sigma.$$

Therefore

$$\|\Phi_n(t)\| \le K \int_{t-n}^{t-n+1} e^{-\omega(t-\sigma)} \|g(\sigma)\| \, d\sigma.$$

By using the Hölder's inequality and Weierstrass test (see [10, Lemma 11.2]), we obtain that $\sum_{n=1}^{\infty} \Phi_n(t)$ is uniformly convergent on $\mathbb{R}$. Now let

$$\Phi(t) = \sum_{n=1}^{\infty} \Phi_n(t), \quad t \in \mathbb{R}.$$

Observe that

$$\Phi(t) = \sum_{n=1}^{\infty} \Phi_n(t) = \int_{-\infty}^{t} U(t,\sigma) g(\sigma) \, d\sigma, \quad t \in \mathbb{R}.$$

Clearly, $\Phi(t) \in C(\mathbb{R}, \mathbb{X})$. Moreover, for any $t \in \mathbb{R}$, we have

$$\|\Phi(t)\| \le \sum_{n=1}^{\infty} \|\Phi_n(t)\| \le C_q(K, \omega) \|g\|_{\mathrm{S}^p},$$

where $C_q(K,\omega)$ depends only on the fixed constants q, K and ω, i.e. the parameters of the problem.

Now let us show that each $\Phi_n \in \mathrm{AA}(\mathbb{X})$, and then we conclude that $\Phi \in \mathrm{AA}(\mathbb{X})$. Indeed, let $\{s_m\}$ be a sequence of real numbers. Since $g \in \mathrm{AS}^p(\mathbb{X})$, there exists a subsequence $\{s_{mk}\}$ of $\{s_m\}$ and a function $\widetilde{g} \in L^p_{\mathrm{loc}}(\mathbb{X})$ such that

$$\left(\int_t^{t+1} \|g(\sigma + s_{mk}) - \widetilde{g}(\sigma)\|^p \, d\sigma \right)^{1/p} \to 0 \quad \text{as } k \to \infty.$$

Let

$$V_n(t) = \int_{n-1}^{n} U(t, t-\sigma) \widetilde{g}(t-\sigma) \, d\sigma.$$

Then using the Hölder's inequality we get

$$\begin{aligned}
&\|\Phi_n(t + s_{mk}) - V_n(t)\| \\
&\qquad = \left\| \int_{n-1}^{n} U(t, t-\sigma)[g(t + s_{mk} - \sigma) - \widetilde{g}(t-\sigma)] \, d\sigma \right\| \\
&\qquad \le K \int_{n-1}^{n} e^{-\omega\sigma} \|g(t + s_{mk} - \sigma) - \widetilde{g}(t-\sigma)\| \, d\sigma \\
&\qquad \le K \left(\int_{n-1}^{n} e^{-q\omega\sigma} \, d\sigma \right)^{1/q} \left(\int_{n-1}^{n} \|g(t + s_{mk} - \sigma) - \widetilde{g}(t-\sigma)\|^p \, d\sigma \right)^{1/p} \\
&\qquad = C' \left(\int_t^{t+1} \|g(s_{mk} + \sigma - n) - \widetilde{g}(\sigma - n)\|^p \, d\sigma \right)^{1/p},
\end{aligned}$$

where $C' = C'_q(K, \omega)$. Obviously the last integral goes to 0 as $k \to \infty$.

Similarly we can prove that $\|V_n(t-s_{mk})-\Phi_n(t)\|\to 0$ as $k\to\infty$. Thus we conclude that each $\Phi_n(t)\in \mathrm{AA}(\mathbb{X})$ and consequently their uniform limit $\Phi(t)\in \mathrm{AA}(\mathbb{X})$.

Next, we prove that $\Psi(t)\in \mathrm{PAA}_0(\mathbb{X},\rho)$. It is obvious that $\Psi(t)\in \mathrm{BC}(\mathbb{R},\mathbb{X})$, the left task is to show that

$$\lim_{T\to\infty}\frac{1}{m(T,\rho)}\int_{-T}^{T}\|\Psi(t)\|\rho(t)\,dt=0.$$

For this, we consider

$$\Psi_n(t)=\int_{t-n}^{t-n+1}U(t,\sigma)h(\sigma)\,d\sigma,$$

for each $t\in\mathbb{R}$ and $n=1,2,\ldots$ From triangle inequality, exponential dichotomy of $(U(t,s))_{t\geq s}$ and Hölder's inequality, it follows that

$$\begin{aligned}
\|\Psi_n(t)\| &\leq K\int_{t-n}^{t-n+1}e^{-\omega(t-\sigma)}\|h(\sigma)\|\,d\sigma\\
&\leq K\bigg(\int_{t-n}^{t-n+1}e^{-q\omega(t-\sigma)}\,d\sigma\bigg)^{1/q}\bigg(\int_{t-n}^{t-n+1}\|h(\sigma)\|^p\,d\sigma\bigg)^{1/p}\\
&\leq K\bigg(\int_{n-1}^{n}e^{-q\omega\sigma}\,d\sigma\bigg)^{1/q}\bigg(\int_{t-n}^{t-n+1}\|h(\sigma)\|^p\,d\sigma\bigg)^{1/p}\\
&\leq \frac{K}{\sqrt[q]{q\omega}}(e^{-q\omega(n-1)}-e^{-q\omega n})^{1/q}\bigg(\int_{t-n}^{t-n+1}\|h(\sigma)\|^p\,d\sigma\bigg)^{1/p}\\
&\leq \frac{Ke^{-\omega n}}{\sqrt[q]{q\omega}}(e^{q\omega}-1)^{1/q}\bigg(\int_{t-n}^{t-n+1}\|h(\sigma)\|^p\,d\sigma\bigg)^{1/p}\\
&\leq \frac{Ke^{-\omega n}}{\sqrt[q]{q\omega}}(e^{q\omega}+1)^{1/q}\bigg(\int_{t-n}^{t-n+1}\|h(\sigma)\|^p\,d\sigma\bigg)^{1/p}.
\end{aligned}$$

Then, for $T>0$, we see that

$$\begin{aligned}
&\frac{1}{m(T,\rho)}\int_{-T}^{T}\|\Psi_n(t)\|\rho(t)\,dt\\
&\qquad\leq \frac{Ke^{-\omega n}}{\sqrt[q]{q\omega}}(e^{q\omega}+1)^{1/q}\frac{1}{m(T,\rho)}\int_{-T}^{T}\bigg(\int_{t-n}^{t-n+1}\|h(\sigma)\|^p\,d\sigma\bigg)^{1/p}\rho(t)\,dt.
\end{aligned}$$

Since $h^b\in PAA_0(L^p(0,1;\mathbb{X}),\rho)$, the above inequality leads to $\Psi_n\in PAA_0(\mathbb{X},\rho)$. The above inequality leads also to

$$\|\Psi_n(t)\|\leq \frac{Ke^{-\omega n}}{\sqrt{[q]q\omega}}(e^{q\omega}+1)^{1/q}\|h\|_{\mathrm{S}^p}.$$

Since the series

$$\frac{K}{\sqrt[q]{q\omega}}(e^{q\omega}+1)^{1/q}\times\sum_{n=1}^{\infty}e^{-\omega n}$$

is convergent, then we deduce from the Weierstrass test that the series $\sum_{k=1}^{\infty} \Psi_n(t)$ is uniformly convergent on $\mathbb{X}$ and

$$\Psi(t) = \int_{-\infty}^{t} U(t,\sigma)h(\sigma)\,d\sigma = \sum_{n=1}^{\infty} \Psi_n(t).$$

Applying $\Psi_n \in \mathrm{PAA}_0(\mathbb{X},\rho)$ and the inequality

$$\begin{aligned}&\frac{1}{m(T,\rho)}\int_{-T}^{T} \|\Psi(t)\|\rho(t)\,dt \\ &\le \frac{1}{m(T,\rho)}\int_{-T}^{T} \|\Psi(t) - \sum_{n=1}^{m}\Psi_n(t)\|\rho(t)\,dt + \sum_{n=1}^{m}\frac{1}{m(T,\rho)}\int_{-T}^{T}\|\Psi_n(t)\|\rho(t)\,dt,\end{aligned}$$

we deduce that the uniformly limit $\Psi(t) = \sum_{n=1}^{\infty} \Psi_n(t) \in \mathrm{PAA}_0(\mathbb{X},\rho)$. Therefore $u(t) = \Phi(t) + \Psi(t)$ is weighted pseudo almost automorphic.

Finally, let us prove that $u(t)$ is a mild solution of the equation (3.1). Indeed, if we let

$$u(s) = \int_{-\infty}^{s} U(s,\sigma)f(\sigma)\,d\sigma, \tag{3.3}$$

and multiply both sides of (3.3) by $U(t,s)$, then

$$U(t,s)u(s) = \int_{-\infty}^{s} U(t,\sigma)f(\sigma)\,d\sigma.$$

If $t \ge s$, then

$$\begin{aligned}\int_{s}^{t} U(t,\sigma)f(\sigma)\,d\sigma &= \int_{-\infty}^{t} U(t,\sigma)f(\sigma)\,d\sigma - \int_{-\infty}^{s} U(t,\sigma)f(\sigma)\,d\sigma \\ &= u(t) - U(t,s)u(s).\end{aligned}$$

It follows that

$$u(t) = U(t,s)u(s) + \int_{s}^{t} U(t,\sigma)f(\sigma)\,d\sigma.$$

This completes the proof of the lemma. □

Lemma 3.3. *If $u \in \mathrm{WPAAS}^p(\mathbb{X})$ and $h \ge 0$, then $u(\cdot - h) \in \mathrm{WPAAS}^p(\mathbb{X})$.*

Proof. There exist $u_1 \in \mathrm{AS}^p(\mathbb{X})$ and $u_2^b \in \mathrm{PAA}_0(L^p(0,1;\mathbb{X}),\rho)$ such that $u = u_1 + u_2$, since $u \in \mathrm{WPAAS}^p(\mathbb{X})$. First, by [19, Lemma 3.3] we can obtain that $u_1(\cdot - h) \in \mathrm{AS}^p(\mathbb{X})$.

Now, we prove that $u_2^b(\cdot - h) \in \mathrm{PAA}_0(L^p(0,1;\mathbb{X}),\rho)$.

Since $u_2^b \in \mathrm{PAA}_0(L^p(0,1;\mathbb{X}),\rho)$ and the space $\mathrm{PAA}_0(\mathbb{X},\rho)$ is translation invariant, then for every $h \geq 0$ we have

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{m(T,\rho)}&\int_{-T}^{T}\bigg(\int_t^{t+1}\|u_2(s-h)\|^p\,ds\bigg)^{1/p}\rho(t)\,dt\\&=\lim_{T\to\infty}\frac{1}{m(T,\rho)}\int_{-T}^{T}\bigg(\int_{t-h}^{t-h+1}\|u_2(s)\|^p\,ds\bigg)^{1/p}\rho(t)\,dt\\&=\lim_{T\to\infty}\frac{1}{m(T,\rho)}\int_{-T-h}^{T-h}\bigg(\int_{t}^{t+1}\|u_2(s)\|^p\,ds\bigg)^{1/p}\rho(t+h)\,dt.\end{aligned}$$

Next, the rest of the proof is similar to that of [8, Lemma 3.1], i.e. $t \to u_2^b(t-h)$ belongs to $\mathrm{PAA}_0(L^p(0,1;\mathbb{X}),\rho)$. So, $u(\,\cdot\,-h) \in \mathrm{WPAAS}^p(\mathbb{X})$. $\square$

Let us list the following basic assumptions:

(H3) $f \in \mathrm{WPAAS}^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$, and there exists a constant $L_f > 0$, such that

$$\|f(t,x)-f(t,y)\| \leq L_f\|x-y\|$$

for all $t\in\mathbb{R}$ and each $x,y\in\mathbb{X}$.

(H4) The function $f \in \mathrm{WPAAS}^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$, and there exists a nonnegative function $L_f(\,\cdot\,) \in \mathrm{BS}^p(\mathbb{R})$, with $p>1$ such that

$$\|f(t,x)-f(t,y)\| \leq L_f(t)\|x-y\|$$

for all $t\in\mathbb{R}$ and each $x,y\in\mathbb{X}$.

(H5) Let $\rho \in L^q_{\mathrm{loc}}(\mathbb{R})$ satisfy

$$\lim_{T\to\infty}\frac{T^{1/p}m_q(T,\rho)}{m(T,\rho)} < \infty,$$

where $1/p+1/q=1$ and

$$m_q(T,\rho)=\bigg(\int_{-T}^{T}\rho^q(t)\,dt\bigg)^{1/q}.$$

(H6) The function $f=g+h \in \mathrm{WPAAS}^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$, where $g \in \mathrm{AS}^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$ is uniformly continuous in any bounded subset $M \subset \mathbb{X}$ uniformly on $t\in\mathbb{R}$ and $h^b \in \mathrm{PAA}_0(\mathbb{X},L^p(0,1;\mathbb{X}),\rho)$.

The following theorems are the main results of this section.

Theorem 3.4. *Let $\rho\in U_\infty$ and suppose that the conditions* (H1)–(H3) *and* (H6) *are satisfied. Then equation* (1.1) *has a unique weighted pseudo almost automorphic mild solution on $\mathbb{R}$ provided that $KL_f/\omega < 1$.*

Proof. Let $\Gamma\colon \mathrm{WPAA}(\mathbb{R},\mathbb{X}) \to \mathrm{WPAA}(\mathbb{R},\mathbb{X})$ be the nonlinear operator defined by

$$(\Gamma u)(t)=\int_{-\infty}^{t}U(t,s)f(s,u(s-h))\,ds,\quad t\in\mathbb{R}.$$

First, let us prove that $\Gamma(\mathrm{WPAA}(\mathbb{R},\mathbb{X}))\subset \mathrm{WPAA}(\mathbb{R},\mathbb{X})$. For each $u\in \mathrm{WPAA}(\mathbb{R},\mathbb{X})$, by using the fact that the range of an almost automorphic function is relatively compact combined with the above Lemma 2.13, Theorem 2.15 and Lemma 3.3, one can easily see that $f(\,\cdot\,,u(\,\cdot\,-h))\in \mathrm{WPAAS}^p(\mathbb{R},\mathbb{X})$. Hence, from the proof of Lemma 3.2, we know that $(\Gamma u)(\,\cdot\,)\in \mathrm{WPAA}(\mathbb{R},\mathbb{X})$. That is, Γ maps $\mathrm{WPAA}(\mathbb{R},\mathbb{X})$ into $\mathrm{WPAA}(\mathbb{R},\mathbb{X})$.

Now, let us prove that Γ has a unique fixed-point. To this end, for each $t\in\mathbb{R}$, $u,v\in \mathrm{WPAA}(\mathbb{R},\mathbb{X})$, we have

$$\begin{aligned}\|(\Gamma u)(t)-(\Gamma v)(t)\| &\le \int_{-\infty}^{t}\|U(t,s)[f(s,u(s-h))-f(s,v(s-h))]\|\,ds\\ &\le K\int_{-\infty}^{t} e^{-\omega(t-s)}\|f(s,u(s-h))-f(s,v(s-h))\|\,ds\\ &\le KL_f\int_{-\infty}^{t} e^{-\omega(t-s)}\|u(s-h)-v(s-h)\|\,ds\\ &\le KL_f\int_{-\infty}^{t} e^{-\omega(t-s)}\,ds\|u-v\|_\infty \le \frac{KL_f}{\omega}\|u-v\|_\infty.\end{aligned}$$

So $\|\Gamma u-\Gamma v\|_\infty\le (KL_f/\omega)\|u-v\|_\infty$. By the Banach contraction principle with $KL_f/\omega<1$, Γ has a unique fixed point u in $\mathrm{WPAA}(\mathbb{R},\mathbb{X})$, which is the weighted pseudo almost automorphic solution to equation (1.1). □

A different Lipschitz condition is considered in the following result.

Theorem 3.5. *Let $\rho\in U_\infty$ and assume that* (H1), (H2), (H4)–(H6) *hold. Then equation* (1.1) *has a unique weighted pseudo almost automorphic mild solution whenever*

$$\|L_f\|_{\mathrm{S}^p}<\frac{1-e^{-\omega}}{K}\bigg(\frac{\omega q}{1-e^{-\omega q}}\bigg)^{1/q}.$$

Proof. Consider the nonlinear operator Γ given by

$$(\Gamma u)(t)=\int_{-\infty}^{t}U(t,s)f(s,u(s-h))\,ds,\quad t\in\mathbb{R}.$$

Let $u\in \mathrm{WPAA}(\mathbb{R},\mathbb{X})$, with the range of an almost automorphic function being relatively compact combined with the above Lemma 2.13, Theorem 2.16 and Lemma 3.3, it follows that the function $s\to f(s,u(s-h))$ is in $\mathrm{WPAAS}^p(\mathbb{R},\mathbb{X})$. Moreover, from Lemma 3.2, we infer that $\Gamma u\in \mathrm{WPAA}(\mathbb{R},\mathbb{X})$, that is, Γ maps $\mathrm{WPAA}(\mathbb{R},\mathbb{X})$ into itself.

Next, we prove that the operator Γ has a unique fixed point in $\mathrm{WPAA}(\mathbb{R},\mathbb{X})$. Indeed, for each $t\in\mathbb{R}$, $u,v\in \mathrm{WPAA}(\mathbb{R},\mathbb{X})$, we have

$$\|\Gamma u(t)-\Gamma v(t)\|\le\bigg\|\int_{-\infty}^{t}U(t,s)[f(s,u(s-h))-f(s,v(s-h))]\,ds\bigg\|$$

$$
\begin{aligned}
&\le K \int_{-\infty}^{t} e^{-\omega(t-s)} \|f(s,u(s-h)) - f(s,v(s-h))\|\, ds \\
&\le K \int_{-\infty}^{t} e^{-\omega(t-s)} L_f(s)\, ds \|u-v\| \\
&= \sum_{n=1}^{\infty} \int_{t-n}^{t-n+1} K e^{-\omega(t-s)} L_f(s)\, ds \|u-v\| \\
&\le \sum_{n=1}^{\infty} \bigg(\int_{t-n}^{t-n+1} K^q e^{-\omega q(t-s)}\, ds \bigg)^{1/q} \|L_f\|_{\mathrm{S}^p} \|u-v\| \\
&\le \frac{K}{1-e^{-\omega}} \bigg(\frac{1-e^{-q\omega}}{\omega q} \bigg)^{1/q} \|L_f\|_{\mathrm{S}^p} \|u-v\|,
\end{aligned}
$$

which implies

$$
\|(\Gamma u)(t) - (\Gamma v)(t)\|_\infty \le \frac{K}{1-e^{-\omega}} \bigg(\frac{1-e^{-q\omega}}{\omega q} \bigg)^{1/q} \|L_f\|_{\mathrm{S}^p} \|u-v\|_\infty.
$$

Since $\|L_f\|_{\mathrm{S}^p} < (1-e^{-\omega}/K)(\omega q/(1-e^{-\omega q}))^{1/q}$, Γ has a unique fixed point $u \in \mathrm{WPAA}(\mathbb{R},\mathbb{X})$. □

We next investigate the existence of weighted pseudo almost automorphic mild solutions to equation (1.1) when the perturbation f is not necessarily Lipschitz continuous. For this, we require the following assumptions:

(H7) $f \in \mathrm{WPAAS}^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$ and $f(t,x)$ is uniformly continuous in any bounded subset $M \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$ and for every bounded subset $M \subset \mathbb{X}$, $\{f(\,\cdot\,,x) : x \in M\}$ is bounded in $\mathrm{WPAAS}^p(\mathbb{X})$.

(H8) There exists a continuous nondecreasing function $W\colon [0,\infty) \to (0,\infty)$ such that

$$
\|f(t,x)\| \le W(\|x\|) \quad \text{for all } t \in \mathbb{R} \text{ and } x \in \mathbb{X}.
$$

Remark 3.6. Let $f(t,x) = \sin(t) + \sin(\pi t) + \sin(x/\phi(t))$, where $\phi(t) = \max\{1,|t|\}$, $t \in \mathbb{R}$. According to [2, Remark 3.4], this defined function also satisfies the condition (H7) with $\rho(t) = 1+t^2$.

Remark 3.7. For condition (H8), an interesting result (see Corollary 3.1) is given for the perturbation f satisfying the Hölder type condition.

Theorem 3.8. *Let $\rho \in U_\infty$ and conditions* (H1) *and* (H2) *hold. Let $f\colon \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ be a function satisfying conditions* (H6)–(H8) *and the following additional assumptions*:

(a) *For each $z \ge 0$, the function*

$$
t \to \int_{-\infty}^{t} e^{-\omega(t-s)} W(zh'(s-h))\, ds
$$

belongs to BC($\mathbb{R}$). *We set*

$$\beta(z) = K\left\| \int_{-\infty}^{t} e^{-\omega(t-s)} W(zh'(s-h))\,ds \right\|_{h'},$$

where K *is the constant given in* (H2).

(b) *For each* $\varepsilon > 0$ *there is* $\delta > 0$ *such that for every* $u, v \in C_{h'}(\mathbb{X})$, $\|u - v\|_{h'} \le \delta$ *implies that, for all* $t \in \mathbb{R}$,

$$\int_{-\infty}^{t} e^{-\omega(t-s)} \|f(s, u(s-h)) - f(s, v(s-h))\|\,ds \le \varepsilon.$$

(c) $\liminf_{\xi\to\infty} \xi/\beta(\xi) > 1$.

(d) *For all* $a, b \in \mathbb{R}$, $a < b$ *and* $z > 0$, *the set* $\{f(s, h'(s-h)x) : a \le s-h \le b$, $x \in C_{h'}(\mathbb{X}), \|x\|_{h'} \le z\}$ *is relatively compact in* $\mathbb{X}$.

Then equation (1.1) *has at least one weighted pseudo almost automorphic mild solution.*

Proof. We define the nonlinear operator $\Gamma\colon C_{h'}(\mathbb{X}) \to C_{h'}(\mathbb{X})$ by

$$(\Gamma u)(t) := \int_{-\infty}^{t} U(t,s) f(s, u(s-h))\,ds, \quad t \in \mathbb{R}.$$

We will show that Γ has a fixed point in WPAA($\mathbb{R}, \mathbb{X}$). For the sake of convenience, we divide the proof into several steps.

(1) For $u \in C_{h'}(\mathbb{X})$, we have that

$$\begin{aligned}\|(\Gamma u)(t)\| &\le K \int_{-\infty}^{t} e^{-\omega(t-s)} W(\|u(s-h)\|)\,ds \\ &\le K \int_{-\infty}^{t} e^{-\omega(t-s)} W(\|u\|_{h'} h'(s-h))\,ds.\end{aligned}$$

It follows from condition (a) that Γ is well defined.

(2) The operator Γ is continuous. In fact, for any $\varepsilon > 0$, we take $\delta > 0$ involved in condition (b). If $u, v \in C_{h'}(\mathbb{X})$ and $\|u - v\|_{h'} \le \delta$, then

$$\|(\Gamma u)(t) - (\Gamma v)(t)\| \le K \int_{-\infty}^{t} e^{-\omega(t-s)} \|f(s, u(s-h)) - f(s, v(s-h))\|\,ds \le \varepsilon,$$

which shows the assertion.

(3) We will show that Γ is completely continuous. We set $B_z(\mathbb{X})$ for the closed ball with center at 0 and radius z in the space $\mathbb{X}$. Let $V = \Gamma(B_z(C_{h'}(\mathbb{X})))$ and $v = \Gamma(u)$ for $u \in B_z(C_{h'}(\mathbb{X}))$. Firstly, we will prove that $V(t)$ is a relatively compact subset of $\mathbb{X}$ for each $t \in \mathbb{R}$. It follows from condition (a) that the function $s \to Ke^{-\omega s} W(zh'(t-s-h))$ is integrable on $[0, \infty)$. Hence, for $\varepsilon > 0$, we can choose $a \ge 0$ such that $K\int_a^\infty e^{-\omega s} W(zh'(t-s-h))ds \le \varepsilon$. Since

$$v(t) = \int_0^a U(t, t-s) f(t-s, u(t-s-h))\,ds + \int_a^\infty U(t, t-s) f(t-s, u(t-s-h))\,ds$$

and

$$\left\| \int_a^\infty U(t,t-s) f(t-s, u(t-s-h))\, ds \right\| \leq K \int_a^\infty e^{-\omega s} W(zh'(t-s-h))\, ds \leq \varepsilon,$$

we get $v(t) \in a\overline{c_0(N)} + B_\varepsilon(\mathbb{X})$, where $c_0(N)$ denotes the convex hull of N and $N = \{U(t,t-s) f(\xi, h'(\xi - h)x) : 0 \leq s \leq a,\ t-a \leq \xi - h \leq t,\ \|x\|_{h'} \leq z\}$. Using the strong continuity of $U(t,s)$ and property (d) of f, we infer that N is a relatively compact set, and $V(t) \subseteq a\overline{c_0(N)} + B_\varepsilon(\mathbb{X})$, which establishes our assertion.

Secondly, we show that the set V is equicontinuous. In fact, we can decompose

$$\begin{aligned} v(t+s) - v(t) = {} & \int_0^s U(t,t-\sigma) f(t+s-\sigma, u(t+s-h-\sigma))\, d\sigma \\ & + \int_0^a [U(t,t-\sigma-s) - U(t,t-\sigma)] f(t-\sigma, u(t-h-\sigma))\, d\sigma \\ & + \int_a^\infty [U(t,t-\sigma-s) - U(t,t-\sigma)] f(t-\sigma, u(t-h-\sigma))\, d\sigma. \end{aligned}$$

For each $\varepsilon > 0$, we can choose $a > 0$ and $\delta_1 > 0$ such that

$$\begin{aligned} & \bigg\| \int_0^s U(t,t-\sigma) f(t+s-\sigma, u(t+s-h-\sigma))\, d\sigma \\ & \qquad + \int_a^\infty [U(t,t-\sigma-s) - U(t,t-\sigma)] f(t-\sigma, u(t-h-\sigma))\, d\sigma \bigg\| \\ & \leq K \int_0^s e^{-\omega\sigma} W(zh'(t+s-h-\sigma))\, d\sigma \\ & \qquad + K \int_a^\infty (e^{-\omega(\sigma+s)} + e^{-\omega\sigma}) W(zh'(t-h-\sigma))\, d\sigma \leq \frac{\varepsilon}{2} \end{aligned}$$

for $s \leq \delta_1$. Moreover, since $\{f(t-\sigma, u(t-h-\sigma)) : 0 \leq \sigma - h \leq a,\ u \in B_z(C_{h'}(\mathbb{X}))\}$ is a relatively compact set and $U(t,s)$ is strongly continuous, we can choose $\delta_2 > 0$ such that $\|[U(t,t-\sigma-s) - U(t,t-\sigma)] f(t-\sigma, u(t-h-\sigma))\| \leq \varepsilon/2a$ for $s \leq \delta_2$. Combining these estimates, we get $\|v(t+s) - v(t)\| \leq \varepsilon$ for s small enough and independent of $u \in B_z(C_{h'}(\mathbb{X}))$.

Finally, applying condition (a), we can see that

$$\frac{\|v(t)\|}{h'(t)} \leq \frac{K}{h'(t)} \int_{-\infty}^t e^{-\omega(t-s)} W(zh'(s-h))\, ds \to 0, \quad |t| \to \infty,$$

and this convergence is independent of $x \in B_z(C_{h'}(\mathbb{X}))$. Hence, by Lemma 2.18, V is a relatively compact set in $(C_{h'}(\mathbb{X}))$.

(4) Let us now assume that $u^\lambda(\,\cdot\,)$ is a solution of equation $u^\lambda = \lambda\Gamma(u^\lambda)$ for some $0 < \lambda < 1$. We can estimate

$$\begin{aligned}\|u^\lambda(t)\| &= \lambda\left\|\int_{-\infty}^{t} U(t,s)f(s,u^\lambda(s-h))\,ds\right\| \\ &\le K\int_{-\infty}^{t} e^{-\omega(t-s)}W(\|u^\lambda\|_{h'}h'(s-h))\,ds \le \beta(\|u^\lambda\|_{h'})h'(t-h).\end{aligned}$$

Hence, we get

$$\frac{\|u^\lambda\|_{h'}}{\beta(\|u^\lambda\|_{h'})} \le 1$$

and combining with condition (c), we conclude that the set $\{u^\lambda : u^\lambda = \lambda\Gamma(u^\lambda),\ \lambda \in (0,1)\}$ is bounded.

(5) It follows from Lemma 2.13, (H6)–(H7), Theorem 2.17 and Lemma 3.3 that the function $t \to f(t,u(t-h))$ belongs to $\mathrm{WPAAS}^p(\mathbb{R},\mathbb{X})$, whenever $u \in \mathrm{WPAA}(\mathbb{R},\mathbb{X})$. Moreover, from Lemma 3.2 we infer that $\Gamma(\mathrm{WPAA}(\mathbb{R},\mathbb{X})) \subset \mathrm{WPAA}(\mathbb{R},\mathbb{X})$ and noting that $\mathrm{WPAA}(\mathbb{R},\mathbb{X})$ is a closed subspace of $C_{h'}(\mathbb{X})$, consequently, we can consider $\Gamma\colon \mathrm{WPAA}(\mathbb{R},\mathbb{X}) \to \mathrm{WPAA}(\mathbb{R},\mathbb{X})$.

Using properties (1)–(3), we deduce that this map is completely continuous. Applying Lemma 2.19, we infer that Γ has a fixed point $u \in \mathrm{WPAA}(\mathbb{R},\mathbb{X})$, which completes the proof. $\square$

Remark 3.9. The main technique in the above Theorem 3.8 can be seen as a generation of that in [2, Theorem 3.4] to a weighted pseudo almost automorphic function.

As a direct consequence of Theorem 3.8, we can obtain the following interesting corollary for the the Hölder type condition (see also [18, Corollary 4.10]).

Corollary 3.10. *Let $\rho \in U_\infty$. Assume that conditions* (H1)–(H2) *hold. Let $f\colon \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ be a function satisfying assumptions* (H6)–(H7) *and the Hölder type condition:*

$$\|f(t,u) - f(t,v)\| \le \gamma\|u-v\|^\alpha, \quad 0 < \alpha < 1,$$

for all $t \in \mathbb{R}$ and $u, v \in \mathbb{X}$, where $\gamma > 0$ is a constant. Moreover, assume the following conditions are satisfied:

(a) $f(t,0) = q$.
(b) $\sup\limits_{t\in\mathbb{R}} K\int_{-\infty}^{t} e^{-\omega(t-s)}h'(s-h)^\alpha\,ds = \gamma_2 < \infty$.
(c) *For all $a, b \in \mathbb{R}$, $a < b$ and $z > 0$, the set $\{f(s,h'(s-h)x) : a \le s-h \le b,\ x \in C_{h'}(\mathbb{X}),\ \|x\|_{h'} \le z\}$ is relatively compact in $\mathbb{X}$.*

Then equation (1.1) *admits at least one weighted pseudo almost automorphic mild solution.*

Proof. Let $\gamma_0 = \|q\|$, $\gamma_1 = \gamma$, and we take $W(\xi) = \gamma_0 + \gamma_1\xi^\alpha$. Then condition (H8) is satisfied. It follows from (b), we can see that function f satisfies (a) in Theorem 3.8. Note that for each $\varepsilon > 0$ there is $0 < \delta^\alpha < (\varepsilon/\gamma_1)\gamma_2$ such that for every $u, v \in C_{h'}(\mathbb{X})$, $\|u - v\|_{h'} \le \delta$ implies that

$$K\int_{-\infty}^{t} e^{-\omega(t-s)}\|f(s,u(s-h)-f(s,v(s-h))\|\,ds \le \varepsilon \quad \text{for all } t \in \mathbb{R}.$$

The assumption (c) in Theorem 3.8 can be easily verified by the definition of W. So, from Theorem 3.8, we can conclude that equation (1.1) has a weighted pseudo almost automorphic mild solution. $\square$

4. An application

As an application, we consider the following equations with Dirichlet boundary conditions:

$$\tag{4.1} \begin{cases} \dfrac{\partial u}{\partial t}u(t,x) = \dfrac{\partial^2 u}{\partial^2 x}u(t,x) \\ \qquad\qquad + u(t,x)\sin\dfrac{1}{2+\cos t+\cos\sqrt{2}t} + f(t,u(t-h,x)), \\ u(t,0) = u(t,1) = 0, \quad t \in \mathbb{R}, \end{cases}$$

where $h > 0$, $\mathbb{X} = L^2(0,1)$ and $D(B) := \{x \in C^1[0,1] : x'$ is absolutely continuous on $[0,1]$, $x'' \in \mathbb{X}$, $x(0) = x(1) = 0\}$, $Bx(r) = x''(r)$, $r \in (0,1)$, $x \in D(B)$. Then B generates a C_0-semigroup $T(t)$ on $\mathbb{X}$ given by

$$(T(t)x)(r) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\langle x, e_n\rangle_{L^2} e_n(r),$$

where $e_n(r) = \sqrt{2}\sin n\pi r$, $n = 1, 2, \ldots$ Moreover, $\|T(t)\| \le e^{-\pi^2 t}$, $t \ge 0$.

Define a family of linear operators $A_1(t)$ by

$$\begin{cases} D(A_1(t)) = D(B), & t \in \mathbb{R}, \\ A_1(t)x = \left(B + \sin\dfrac{1}{2+\cos t+\cos\sqrt{2}t}\right)x, & x \in D(A_1(t)). \end{cases}$$

Then, $\{A_1(t),\ t \in \mathbb{R}\}$ generates an evolution family $\{U_1(t,s)\}_{t\ge s}$ such that

$$U_1(t,s)x = T(t-s)e^{\int_s^t \sin(1/(2+\cos\tau+\cos\sqrt{2}\tau))\,d\tau}x.$$

Hence

$$\|U_1(t,s)\| \le e^{-(\pi^2-1)(t-s)}, \quad t \ge s.$$

It is easy to see that $U_1(t,s)$ satisfies (H1)–(H2) with $K = 1$, $\omega = \pi^2 - 1$. Set

$$f(t,u) = u\sin\frac{1}{\cos^2 t + \cos^2 \pi t} + \max_{k\in\mathbb{Z}}\{e^{-(t\pm k^2)^2}\}\sin u, \quad t \in \mathbb{R}.$$

According to [20], [26], f clearly satisfies conditions (H3) and (H6). From Theorem 3.4, the problem (4.1) has a unique weighted pseudo almost automorphic mild solution.

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