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EXISTENCE OF SOLUTIONS FOR SINGULARLY PERTURBED HAMILTONIAN ELLIPTIC SYSTEMS WITH NONLOCAL NONLINEARITIES

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Abstract. In the present paper we study singularly perturbed Hamiltonian elliptic systems with nonlocal nonlinearities

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u = \left(\displaystyle\int_{\mathbb{R}^N} \frac{|z|^p}{|x-y|^\mu}\,dy\right)|z|^{p-2}u, \\ -\varepsilon^2\Delta v + V(x)v = -\left(\displaystyle\int_{\mathbb{R}^N} \frac{|z|^p}{|x-y|^\mu}\,dy\right)|z|^{p-2}v, \end{cases}$$

where $z=(u,v)\in H^1(\mathbb{R}^N,\mathbb{R}^2)$, $V(x)$ is a continuous real function on $\mathbb{R}^N$, $0<\mu<N$ and $2-\mu/N<p<(2N-\mu)/(N-2)$. Under suitable assumptions on the potential $V(x)$, we can prove the existence of solutions for small parameter ε by variational methods. Moreover, if $N>2$ and $2+(2-\mu)/(N-2)<p<(2N-\mu)/(N-2)$ then the solutions $z_\varepsilon\to 0$ as the parameter $\varepsilon\to 0$.

1. Introduction and main results

As we all know the nonlinear Schrödinger equation

$$i\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\phi + W(x)\phi - f(x,|\phi|)\phi \tag{1.1}$$

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has been given huge attention during the last twenty years. Here m is the mass of the bosons, $\hbar$ is the Planck constant and $W(x)$ is the external potential. The nonlinearity $f(x,|\phi|)\phi$ is used to describe the interactions between the particles. In many physical applications, nonlocal Hartree type nonlinearities appear naturally, i.e.

$$f(x,|\phi|)\phi = \bigg(\int_{\mathbb{R}^N} K(x-y)|\phi|^p\,dy\bigg)|\phi|^{p-2}\phi,$$

$K(x)$ is usually called the response function which appears naturally in the propagation of electromagnetic waves in plasmas [11] and accounts for the finite-range many-body interaction in the theory of Bose–Einstein condensation [14].

In recent years many people are interested in the existence of standing waves of (1.1), i.e. solutions of the form

$$\phi(x,t) = u(x)e^{-iEt/\hbar}.$$

With this ansatz, the Schrödinger equation can be reduced to a semilinear elliptic equation. There are a large number of literature considering the existence and multiplicity of nontrivial solutions for the semilinear elliptic equation

$$-\varepsilon^2\Delta u + V(x)u = \bigg(\int_{\mathbb{R}^N} K(x-y)|u(y)|^p\,dy\bigg)|u|^{p-2}u. \tag{1.2}$$

In general, if the response function $K(x)$ is the delta function, then the equation (1.2) becomes a local one

$$-\varepsilon^2\Delta u + V(x)u = |u|^{2p-2}u, \quad x\in\mathbb{R}^N. \tag{1.3}$$

As we all know the property of the potential affects the existence of solutions for problem (1.3) greatly. Assuming that $V(x)$ satisfying $\inf V(x) > 0$ is a globally bounded potential with a nondegenerate critical point, Floer and Weinstein [23] studied firstly the existence of single and multiple spike solutions based on a Lyapunov–Schmidt reduction. Since then, many mathematicians are interested in the existence and the concentration of equation (1.3) under various assumptions on the potential $V(x)$. In [6] Ambrosetti et. al. studied the problem with polynomial degenerate potential $V(x)$. In [28], still assuming that $\inf V(x) > 0$, Rabinowitz proved the existence of a positive ground state for any $\varepsilon > 0$ by further assuming that

$$0 < a \le V(x) \le \liminf V(x), \quad \text{for all } x\in\mathbb{R}^N \text{ and some } a > 0, \tag{1.4}$$

with strict inequality on a set of positive measure. Using a local variational approach, del Pino and Felmer [17]–[19] constructed positive solutions by assuming $\Lambda\subset\mathbb{R}^3$ is a bounded open set such that

$$V_0 := \inf_{x\in\Lambda} V(x) < \inf_{x\in\partial\Lambda} V(x).$$

Then if $\inf V(x) > 0$ there also exists a positive semiclassical solution. We also refer authors to [13] for the case $\inf V(x) = 0$ and [22] for the case where the potential $V(x)$ was allowed to change sign.

If the response function $K(x)$ is a function of Coulomb type, for example $1/|x|$, then we arrive at the nonlocal Schrödinger equation

$$-\varepsilon^2 \Delta u + V(x)u = \left(\int_{\mathbb{R}^N} \frac{u^2(y)}{|x-y|}\, dy \right) u. \tag{1.5}$$

If $\varepsilon = 1$, assuming 0 is not in the spectrum of $-\Delta + V(x)$ where the potential $V(x)$ is periodic in x, Buffoni et. al. firstly obtained the existence of one nontrivial solution in [12]. Ackermann [1] proposed an approach to prove the existence of infinitely many geometrically distinct weak solutions. However there is few works about the existence of semiclassical solutions for equation (1.5), since little is known about the ground states of the corresponding autonomous limit problem

$$-\Delta u + u = \left(\int_{\mathbb{R}^N} \frac{u^2(y)}{|x-y|}\, dy \right) u. \tag{1.6}$$

Recently Wei and Winter [33] proved the non-degeneracy property of the ground state solution of (1.6), and then they studied the existence of multi-bump solutions for equation (1.5) under the assumptions that $\inf V(x) > 0$ and $V(x) \in \mathcal{C}^2(\mathbb{R}^3)$.

On the other hand, Schrödinger systems of Hamiltonian type have been also widely considered, see [24], [15], [16], [29], [30], [32] for example. Many works also consider the singularly perturbed Hamiltonian systems

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = |v|^{p-1}v & \text{in } \mathbb{R}^N, \\ -\varepsilon^2 \Delta v + W(x)v = |u|^{q-1}u & \text{in } \mathbb{R}^N. \end{cases} \tag{1.7}$$

In [8], Ávila and Yang established existence results for strongly indefinite elliptic systems with Neumann boundary condition, and they studied the limiting behavior of the positive solutions of the singularly perturbed Hamiltonian problem. In [5], Alves et. al. established the existence and concentration behavior for the singularly perturbed Hamiltonian systems

$$\begin{cases} -\varepsilon^2 \Delta u + u = W_1(x)|v|^{p-1}v & \text{in } \mathbb{R}^N, \\ -\varepsilon^2 \Delta v + v = W_2(x)|u|^{q-1}u & \text{in } \mathbb{R}^N. \end{cases} \tag{1.8}$$

In [4], Alves and Soares studied the existence and concentration behavior for a class of the singularly perturbed gradient systems. Existence problems for radially invariant Hamiltonian systems were also studied by de Figueiredo and Yang [16], and Sirakov [30]. For systems with general nonlinearities, Alves and Soares [3] obtained solutions of (1.7) by using the Legendre–Fenchel transformations and the Mountain Pass Theorem. Existence results for (1.7) with general nonlinearities were also obtained in [29] by assuming that $V(x) = W(x)$ and (1.4) while the growth of the nonlinearities satisfying

$$\frac{1}{p} + \frac{1}{q} > \frac{N-2}{N}.$$

In [32] Sirakov and Soares considered trapping (or "well"-type) potentials, there the authors obtained the existence of solution for small ε by using the Legendre–Fenchel transformation and Fourier analysis methods.

Inspired by the works mentioned above, the aim of this paper is to study the existence of solutions of the perturbed Hamiltonian elliptic systems under the effects of the nonlocal nonlinearities and trapping type potentials

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = \bigg(\displaystyle\int_{\mathbb{R}^N} \frac{|z|^p}{|x-y|^\mu}\,dy\bigg)|z|^{p-2}u, \\ -\varepsilon^2 \Delta v + V(x)v = -\bigg(\displaystyle\int_{\mathbb{R}^N} \frac{|z|^p}{|x-y|^\mu}\,dy\bigg)|z|^{p-2}v. \end{cases} \tag{1.9}$$

Here $z = (u, v) \in H^1(\mathbb{R}^N, \mathbb{R}^2)$, the potential $V(x)$ is a continuous real function on $\mathbb{R}^N$, $0 < \mu < N$ and $2 - \mu/N < p < (2N - \mu)/(N - 2)$. Under suitable assumptions on the potential $V(x)$ we prove that, for small ε, there is at least one nontrivial solution z_ε for (1.9). Moreover, $z_\varepsilon \to 0$ as $\varepsilon \to 0$ if the exponent p is in a suitable range. Set

$$\mathcal{S}z = \begin{pmatrix} -\varepsilon^2 \Delta u + V(x)u \\ \varepsilon^2 \Delta v - V(x)v \end{pmatrix}$$

and

$$\Psi(z) = \frac{1}{2p}\bigg(\int_{\mathbb{R}^N} \frac{|z|^p}{|x-y|^\mu}\,dy\bigg)|z|^p,$$

the systems (1.9) can be restated in the form

$$\mathcal{S}z = \Psi_z(z), \quad z \in H^1(\mathbb{R}^N, \mathbb{R}^2). \tag{1.10}$$

To establish the existence results, we assume that the potential $V(x)$ satisfies

- (P_1) $V(x) \in \mathcal{C}(\mathbb{R}^3)$ and there is $b > 0$ such that the set $\mathcal{V}^b := \{x \in \mathbb{R}^N : V(x) < b\}$ has finite Lebesgue measure.
- (P_2) $0 = V(0) = \min V \leq V(x)$.
- (P_3) There exists $0 < \tau < 1/2$ such that

$$\lim_{|x|\to 0} \frac{V(x)}{|x|^{(1-2\tau)/\tau}} = 0.$$

The assumptions (P_1) and (P_2) are introduced in [31] by Sirakov, the standard example is $b(x) \sim |x - x_0|^2$ in a neighbourhood of some $x_0 \in \mathbb{R}^N$. A particular example satisfying (P_1) and (P_2) is

$$0 = \inf_{x\in\mathbb{R}^N} V(x) < \liminf_{|x|\to\infty} V(x),$$

which means the potential is a "well" type one. The main results of this paper are

Theorem 1.1. *Let* (P_1)–(P_3) *satisfied. Then for any* $\sigma > 0$ *there is* $\mathcal{E}_\sigma > 0$ *such that if* $\varepsilon \le \mathcal{E}_\sigma$, *equation* (1.9) *has at least one solution* z_ε *satisfying*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|z_\varepsilon(x)|^p |z_\varepsilon(y)|^p}{|x-y|^\mu}\,dy\,dx \le \sigma \varepsilon^{((N-2)p-2N+\mu+2)/(p-1)}$$

and

$$\|z_\varepsilon\|^2 \le \varepsilon^{((4-N)p+2N-\mu-4)/(p-1)}.$$

Moreover, if $N > 2$ *and* $2 + (2-\mu)/(N-2) < p < (2N-\mu)/(N-2)$ *we have* $z_\varepsilon \to 0$ *in* $H^1(\mathbb{R}^N, \mathbb{R}^2)$ *as* $\varepsilon \to 0$.

Remark 1.2. We need remark that the behavior of the potential plays an important role in proving the main results and we believe it can not be applied to the case $V(0) > 0$.

We first note that there is no compactness for the Sobolev imbedding, since the problem is set in $\mathbb{R}^N$. Second, the systems of Hamiltonian type is quite different from the Lagrangian type systems in the sense that the energy function associated to a Lagrangian system possesses the "mountain-pass" geometry, while the energy function associated to the Hamiltonian type systems is strongly indefinite without such a geometry, that is, the leading part the energy functional is respectively coercive and anti-coercive on infinitely dimensional subspaces of the energy space, thus the classical critical point theory cannot be applied directly.

This paper is organized as follows. In Section 2, we introduce the variational framework and restate the problems in equivalent forms. In Section 3, we will analysis the behaviors of the bounded $(\mathrm{PS})_c$ sequences. In Section 4, we will prove the existence of semiclassical solutions for the Hamiltonian system (1.10).

2. Notations and variational framework

In this paper we use C, C_i to denote different positive constants and B_R the open ball centered at the origin with radius $R > 0$. $C_0^\infty(\mathbb{R}^N)$ denotes functions infinitely differentiable with compact support in $\mathbb{R}^N$. $H^1(\mathbb{R}^N)$ is the usual Sobolev spaces with norm

$$\|u\|_{H^1} := \bigg(\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)\,dx\bigg)^{1/2}$$

and $L^s(\mathbb{R}^N)$, $1 \le s < \infty$, denotes the Lebesgue space with the norms

$$|u|_s := \bigg(\int_{\mathbb{R}^N} |u|^s\,dx\bigg)^{1/s}.$$

Let $2^* = 2N/(N-2)$, the best Sobolev constant S is defined by:

$$S|u|_{2^*}^2 \leq \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \quad \text{for all } u \in H^1(\mathbb{R}^N).$$

The following inequality will be frequently used to study the nonlocal problems.

Proposition 2.1 (Hardy–Littlewood–Sobolev inequality). *Let $p, r > 1$ and $0 < \mu < N$ with $1/p + \mu/N + 1/r = 2$. Let $f \in L^p(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$, then there exists a sharp constant $C(p, \mu, r)$, independent of f, h, such that*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\mu} \, dy \, dx \leq C(p, \mu, r)|f|_p|h|_r.$$

To prove the existence of semiclassical solutions of (1.10) for small ε, we may rewrite (1.10) in an equivalent form, let $\lambda = \varepsilon^{-2}$, (1.9) reads then as

$$\begin{cases} -\Delta u + \lambda V(x)u = \lambda \bigg(\displaystyle\int_{\mathbb{R}^N} \frac{|z|^p}{|x-y|^\mu} \, dy \bigg) |z|^{p-2}u, \\ -\Delta v + \lambda V(x)v = -\lambda \bigg(\int_{\mathbb{R}^N} \frac{|z|^p}{|x-y|^\mu} \, dy \bigg) |z|^{p-2}v \end{cases} \tag{2.1}$$

for $\lambda \to \infty$. And then the existence results can be restated as

Theorem 2.2. *Let* (P_1)–(P_3) *satisfied. Then for any $\sigma > 0$ there is $\Lambda_\sigma > 0$ such that if $\lambda \geq \Lambda_\delta$, equation* (2.1) *has at least one solution z_λ satisfying*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|z_\lambda(x)|^p |z_\lambda(y)|^p}{|x-y|^\mu} \, dy \, dx \leq \frac{2p}{p-1} \sigma \lambda^{-((N-2)p-2N+\mu+2)/(2(p-1))}$$

and

$$\|z_\lambda\|_\lambda^2 \leq \frac{2p}{p-1} \sigma \lambda^{-((4-N)p+2N-\mu-4)/(2(p-1))}.$$

Moreover, if $N > 2$ and $2 + (2-\mu)/(N-2) < p < (2N-\mu)/(N-2)$ then we have $z_\lambda \to 0$ as $\lambda \to \infty$.

To solve the problem we will apply variational methods. To this end, we introduce the Hilbert spaces

$$E := \bigg\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 \, dx < \infty \bigg\}$$

with inner products

$$(u, v) := \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv) \, dx$$

and the associated norms $\|u\|^2 = (u, u)$.

Obviously, it follows from (P_1) that E embeds continuously in $H^1(\mathbb{R}^N)$ (see [21], [31]). Note that the norm $\|\cdot\|$ is equivalent to $\|\cdot\|_\lambda$ deduced by the inner product

$$(u, v)_\lambda := \int_{\mathbb{R}^N} (\nabla u \nabla v + \lambda V(x)uv) \, dx$$

for each $\lambda > 0$. It is thus clear that, for each $s \in [2, 2^*]$, there is $\nu_s > 0$ (independent of λ) such that if $\lambda \geq 1$ then

$$|u|_s \leq \nu_s \|u\| \leq \nu_s \|u\|_\lambda \quad \text{for all } u \in E. \tag{2.2}$$

In order to investigate the problems in suitable variational framework, we use $A_\lambda := -\Delta + \lambda V$ in $L^2(\mathbb{R}^N)$ to denote the selfadjoint operator related to the Schrödinger equation. By $\sigma(A_\lambda)$, $\sigma_e(A_\lambda)$ and $\sigma_d(A_\lambda)$ we denote the spectrum, the essential spectrum and the eigenvalues of $A_{i,\lambda}$ below $\lambda_e := \inf \sigma_e(A_\lambda)$, respectively. Note that each $\mu \in \sigma_d(A_\lambda)$ is of finite multiplicity. The following two lemmas are proved in [22], we sketch the proofs here for the completeness of the paper.

Lemma 2.3 ([22]). *Suppose that the condition* (P_1) *is satisfied, then there holds* $\lambda_e \geq \lambda b$.

Proof. Set $W_\lambda(x) = \lambda(V(x) - b), W_\lambda^\pm = \max\{\pm W_\lambda, 0\}$ and $D_\lambda = -\Delta + \lambda b + W_\lambda^+$. By (P_1), the multiplicity operator W_λ^- is compact relative to D_λ, hence $\sigma_e(A_\lambda) \subset \sigma_e(D_\lambda) \subset [\lambda b, \infty)$. □

Fix in the following a number b' that is close to b with $0 < b' < b$ and k_λ be the numbers of the eigenvalues of A_λ which is smaller than $\lambda b'$. We write $\eta_{\lambda,j}$ and $h_{\lambda,j}$ $(1 \leq j \leq k_\lambda)$ for the eigenvalues and eigenfunctions and define

$$L_\lambda^d = \operatorname{span}\{h_{\lambda,1}, \ldots, h_{\lambda,k_\lambda}\}.$$

We will also use the following orthogonal decomposition

$$L^2(\mathbb{R}^N) = L_\lambda^d \oplus L_\lambda^e, \quad u = u^d + u^e.$$

Correspondingly, one has

$$E = E_\lambda^d \oplus E_\lambda^e \quad \text{with } E_\lambda^d = L_\lambda^d \cap E \quad \text{and} \quad E_\lambda^e = L_\lambda^e \cap E \tag{2.3}$$

orthogonal with respect to $(\,\cdot\,,\,\cdot\,)_{L^2}$ and $(\,\cdot\,,\,\cdot\,)_\lambda$. From Lemma 2.3, we have

$$\lambda b' |u|_2^2 \leq \|u\|_\lambda^2 \quad \text{for all } u \in E_\lambda^e. \tag{2.4}$$

Lemma 2.4 ([22]). *For each* $s \in [2, 2^*]$, *there is* $c_s > 0$ *independent of* λ *such that*

$$c_s \lambda^{(2^*-s)/(2^*-2)} |u|_s^s \leq \|u\|_\lambda^s \quad \textit{for all } u \in E_\lambda^e.$$

Proof. For $s \in (2, 2^*)$, by Sobolev inequality and (2.4), there holds

$$\begin{aligned} |u|_s^s &\leq \bigg(\int_{\mathbb{R}^N} |u|^2\,dx\bigg)^{(2*-s)/(2*-2)} \bigg(\int_{\mathbb{R}^N} |u|^{2^*}\,dx\bigg)^{(s-2)/(2*-2)} \\ &\leq ((b'\lambda)^{-1}\|u\|_\lambda^2)^{(2*-s)/(2*-2)} (S^{-1}\|u\|_\lambda^{2*})^{(s-2)/(2*-2)} \end{aligned}$$

for all $u \in E_\lambda^e$, thus the conclusion follows. □

Now let $H = E \times E$ and

$$I_\lambda(z) = \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{2}\|v\|_\lambda^2 - \frac{\lambda}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|z(x)|^p|z(y)|^p}{|x-y|^\mu}\,dy\,dx$$

for $z = (u, v) \in H$. Since $0 < \mu < N$ and $2 - \mu/N < p < (2N - \mu)/(N - 2)$, from Proposition 2.1, we know the energy functional $I_\lambda(z)$ is well defined and belongs to $\mathcal{C}^1(H, \mathbb{R})$. Consequently, in order to obtain solutions of system (2.1), we only need to look for critical points of the energy functional $I_\lambda(z)$. However, the functional I_λ is respectively coercive and anti-coercive on H^+ and H^- where

$$H^+ = \{z = (u, 0) \in H : u \in E\}, \qquad H^- = \{z = (0, v) \in H : v \in E\}$$

and $H = H^+ \oplus H^-$. So $\dim H^\pm = \infty$ and each $z \in H$ may be represented as

$$z = z^+ + z^- = (u, 0) + (0, v) \quad \text{where } z^\pm \in H^\pm.$$

Hence we can write

$$I_\lambda(z) = \frac{1}{2}\|z^+\|_\lambda^2 - \frac{1}{2}\|z^-\|_\lambda^2 - \frac{\lambda}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|z(x)|^p|z(y)|^p}{|x-y|^\mu}\,dy\,dx$$

with the norm on H defined by $\|z\|_\lambda^2 = \|u\|_\lambda^2 + \|v\|_\lambda^2$ for $z = (u, v) \in H$. On H there is the orthogonal decomposition

$$H = H_\lambda^d \oplus H_\lambda^e \quad \text{where } H_\lambda^d = E_\lambda^d \times E_\lambda^d \text{ and } H_\lambda^e = E_\lambda^e \times E_\lambda^e. \tag{2.5}$$

Note that $\dim H_\lambda^d < \infty$. Accordingly, we write $z = z^d + z^e$ for $z = (u, v) \in H$ with $z^d = (u^d, v^d)$ and $z^e = (u^e, v^e)$. It follows from Lemma 2.4 that for each $s \in [2, 2^*]$,

$$c_s\lambda^{(2^*-s)/(2^*-2)}|z|_s^s \le \|z\|_\lambda^s \quad \text{for all } z = (u, v) \in H_\lambda^e,$$

where c_s is a constant independent of λ.

To prove the existence of solutions for system (2.1), we introduce the following generalized linking theorem for strongly indefinite problems developed by Bartsch and Ding [9].

Let E be a Banach space with direct sum decomposition $E = Y \oplus X$ and corresponding projections P_Y, P_X onto Y, X, respectively. For a functional $I \in \mathcal{C}^1(E, \mathbb{R})$ we write $I_a = \{z \in E : I(z) \ge a\}$, $I^b = \{z \in E : I(z) \le b\}$ and $I_a^b = I_a \cap I^b$. Recall that a sequence $(z_n) \subset E$ is said to be a $(\mathrm{PS})_c$ sequence if $I(z_n) \to c$ and $I'(z_n) \to 0$. I is said to satisfy the (PS) condition at c if any $(\mathrm{PS})_c$ sequence has a convergent subsequence.

From now on we assume that X is separable and reflexive, and we fix a dense subset $\mathcal{S} \subset X^*$. For each $s \in \mathcal{S}$ there is a semi-norm on E defined by

$$p_s \colon E \to \mathbb{R}, \qquad p_s(z) = \|y\| + |s(x)| \quad \text{for } z = y + x \in Y \oplus X.$$

We denote by $\mathcal{T}_\mathcal{S}$ the induced topology. Let w^* denote the weak*-topology on E^*.

Suppose:

(I_1) For any $c \in \mathbb{R}$, I_c is $\mathcal{T}_{\mathcal{S}}$-closed, and $I' : (I_c, \mathcal{T}_{\mathcal{S}}) \to (E^*, w^*)$ is continuous.

(I_2) There exists $\rho > 0$ with $\kappa := \inf I(S_\rho Y) > 0$ where $S_\rho Y := \{z \in Y : \|z\| = \rho\}$.

The following theorem is a special case of the Theorem 4.2 of [20] (see also [9]).

Theorem 2.5. *Let* (I_1)–(I_2) *satisfied and suppose there are* $R > \rho > 0$ *and* $e \in Y$ *with* $\|e\| = 1$ *such that* $\sup I(\partial Q) \le \kappa$ *where* $Q = \{z = te + x : t \ge 0,\ x \in X,\ \|z\| < R\}$. *Then* I *has a* $(\mathrm{PS})_c$ *sequence with* $\kappa \le c \le \sup I(Q)$.

In our applications we take $\mathcal{S} = X^*$ so that $\mathcal{T}_{\mathcal{S}}$ is the product topology on $E = Y \oplus X$ given by the strong topology on Y and the weak topology on X. The hypothesis (I_0) follows from the following

Proposition 2.6. *Suppose* $I \in \mathcal{C}^1(E, \mathbb{R})$ *is of the form*

$$I(z) = \frac{1}{2}(\|y\|^2 - \|x\|^2) - \Psi(z) \quad \text{for } z = y + x \in E = Y \oplus X$$

such that

(a) $\Psi \in \mathcal{C}^1(E, \mathbb{R})$ *is bounded from below*;

(b) $\Psi : (E, \mathcal{T}_w) \to \mathbb{R}$ *is sequentially lower semicontinuous, that is,* $z_n \rightharpoonup z$ *in* E *implies* $\Psi(z) \le \liminf \Psi(z_n)$;

(c) $\Psi' : (E, \mathcal{T}_w) \to (E^*, \mathcal{T}_{w^*})$ *is sequentially continuous*;

(d) $\nu : E \to \mathbb{R}$, $\nu(z) = \|z\|^2$, *is* $\mathcal{C}^1$ *and* $\nu' : (E, \mathcal{T}_w) \to (E^*, \mathcal{T}_{w^*})$ *is sequentially continuous.*

Then I *satisfies* (I_1).

A proof can be found in [9], Proposition 4.1.

3. Behaviors of the (PS) sequences

In this section we will analysis the behaviors of the (PS) sequences of the functional I_λ.

Lemma 3.1. *Suppose that the condition* (P_1) *is satisfied. For fixed* $\lambda \ge 1$, *let* (z_n) *be a* $(\mathrm{PS})_c$ *sequence for* I_λ. *Then* $c \ge 0$ *and* (z_n) *is bounded in* H.

Proof. Let $z_n = (u_n, v_n)$ be a $(\mathrm{PS})_c$ sequence:

$$I_\lambda(z_n) \to c \quad \text{and} \quad I_\lambda'(z_n) \to 0,$$

then

$$I_\lambda(z_n) - \frac{1}{2}(I_\lambda'(z_n), z_n) = \left(\frac{1}{2} - \frac{1}{2p}\right)\lambda \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|z_n(x)|^p |z_n(y)|^p}{|x-y|^\mu}\, dy\, dx.$$

Noting that $z_n = (u_n, v_n)$, we have $z_n^+ - z_n^- = (u_n, -v_n)$ and then $|z_n^+ - z_n^-| = |z_n|$. Consequently, from the definition of I_λ, we have

$$\begin{aligned}\|z_n\|_\lambda^2 &= (I'_\lambda(z_n), z_n^+ - z_n^-) \\ &\quad + \lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|z_n(x)|^{p-2} z_n(x)(z_n^+ - z_n^-)(x)|z_n(y)|^p}{|x-y|^\mu}\, dy\, dx \\ &\le \lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|z_n(x)|^p |z_n(y)|^p}{|x-y|^\mu}\, dy\, dx + o(1)\|z_n\|_\lambda \\ &\le C_0\bigg(I_\lambda(z_n) - \frac{1}{2} I'_\lambda(z_n) z_n\bigg) + o(1)\|z_n\|_\lambda,\end{aligned}$$

therefore we must have that (z_n) is bounded in H and $c \ge 0$. □

Hence, without loss of generality, we may assume $z_n \rightharpoonup z$ in H and $L^2(\mathbb{R}^N, \mathbb{R}^2)$, $z_n \to z$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^2)$ for $1 \le s < 2^*$, and $z_n(x) \to z(x)$ almost everywhere for $x \in \mathbb{R}^N$. Clearly z is a critical point of I_λ.

Lemma 3.2. *One has along a subsequence*:

(a) $I_\lambda(z_n - z) \to c - I_\lambda(z)$;
(b) $I'_\lambda(z_n - z) \to 0$.

Proof. (a) Direct computation shows that

$$\begin{aligned}I_\lambda(z_n - z) &= \frac{1}{2}\|z_n^+\|_\lambda^2 - \frac{1}{2}\|z_n^-\|_\lambda^2 - \frac{1}{2}\|z^+\|_\lambda^2 + \frac{1}{2}\|z^-\|_\lambda^2 \\ &\quad - \frac{\lambda}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|z_n(x) - z(x)|^p |z_n(y) - z(y)|^p}{|x-y|^\mu}\, dy\, dx + o(1) \\ &= I_\lambda(z_n) - I_\lambda(z) + \Gamma_n + o(1)\end{aligned}$$

where

$$\begin{aligned}\Gamma_n = \frac{\lambda}{2p}\bigg\{ &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|z_n(x)|^p |z_n(y)|^p}{|x-y|^\mu}\, dy\, dx \\ &- \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|z(x)|^p |z(y)|^p}{|x-y|^\mu}\, dy\, dx \\ &- \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|z_n(x) - z(x)|^p |z_n(y) - z(y)|^p}{|x-y|^\mu}\, dy\, dx\bigg\}.\end{aligned}$$

From the nonlocal Brezis–Lieb type Lemma 3.4 in [1], we know $\Gamma_n \to 0$.

(b) For any $w = (\phi, \varphi) \in H$ with $\|w\|_\lambda \le 1$, we have

$$\begin{aligned}(I'_\lambda(z_n - z), w) &= \int_{\mathbb{R}^N} (\nabla(u_n - u)\nabla\phi + \lambda V(x)(u_n - u)\phi)\, dx \\ &\quad - \int_{\mathbb{R}^N} (\nabla(v_n - v)\nabla\varphi + \lambda V(x)(v_n - v)\varphi)\, dx\end{aligned}$$

$$
\begin{aligned}
&- \lambda \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|z_n(y) - z(y)|^p |z_n(x) - z(x)|^{p-2} [(u_n - u)\phi + (v_n - v)\varphi](x)}{|x-y|^\mu} \, dy \, dx \\
&= (I'_\lambda(z_n), w) - (I'_\lambda(z), w) + \widetilde{\Gamma}_n + o(1)
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{\Gamma}_n = \lambda \bigg\{ &\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|z_n(y)|^p |z_n(x)|^{p-2} [u_n \phi + v_n \varphi](x)}{|x-y|^\mu} \, dy \, dx \\
&- \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|z(y)|^p |z(x)|^{p-2} [u\phi + v\varphi](x)}{|x-y|^\mu} \, dy \, dx \\
&- \int_{\mathbb{R}^N} \frac{|z_n(y) - z(y)|^p |z_n(x) - z(x)|^{p-2} [(u_n - u)\phi + (v_n - v)\varphi](x)}{|x-y|^\mu} \, dy \, dx \bigg\}.
\end{aligned}
$$

From Lemma 3.4 in [1], we also know $\widetilde{\Gamma}_n \to 0$ uniformly in w with $\|w\|_\lambda \le 1$. □

In the following we will utilize the decomposition (2.5): $H = H^d_\lambda \oplus H^e_\lambda$. Write $w_n := z_n - z$ and decompose w_n by

$$
w_n = w^d_n + w^e_n, \quad \text{with } w^d_n \in H^d_\lambda \text{ and } w^e_n \in H^e_\lambda.
$$

From $w_n \rightharpoonup 0$ it is easy to see $w^d_n \to 0$ since $\dim(H^d_\lambda) < \infty$. And therefore $z_n \to z$ if and only if $w_n \to 0$.

Lemma 3.3. *Suppose that the assumption* (P$_1$) *holds. There is a constant $\alpha_0 > 0$ independent of λ such that, for any* (PS)$_c$ *sequence (z_n) for I_λ with $z_n \rightharpoonup z$, either $z_n \to z$ along a subsequence or*

$$
c - I_\lambda(z) \ge \alpha_0 \lambda^{-((N-2)p - 2N + \mu + 2)/(2(p-1))}.
$$

Proof. Assume that (z_n) has no convergent subsequence, then

$$
\liminf_{n\to\infty} \|w_n\|_\lambda > 0.
$$

Lemma 3.2 implies that along a subsequence, one has

$$
I_\lambda(w_n) \to c - I_\lambda(z) \quad \text{and} \quad I'_\lambda(w_n) \to 0.
$$

It follows that

$$
I_\lambda(w_n) - \frac{1}{2}(I'_\lambda(w_n), w_n) = \frac{\lambda(p-1)}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w_n(x)|^p |w_n(y)|^p}{|x-y|^\mu} \, dy \, dx,
$$

i.e.

$$
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w_n(x)|^p |w_n(y)|^p}{|x-y|^\mu} \, dy \, dx \le \frac{2p}{p-1} \cdot \frac{c - I_\lambda(z) + o(1)}{\lambda}.
$$

Noting that $w_n = (u_n - u, v_n - v)$, we have $w^+_n - w^-_n = (u_n - u, v - v_n)$ and then $|w^+_n - w^-_n| = |w_n|$. Since $I'_\lambda(w_n) \to 0$, we must have

$$
\begin{aligned}
o(1) = (I'_\lambda(w_n), w^+_n - w^-_n) &= \|w_n\|^2_\lambda \\
&- \lambda \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w_n(x)|^{p-2} w_n(x)(w^+_n - w^-_n)(x) |w_n(y)|^p}{|x-y|^\mu} \, dy \, dx.
\end{aligned}
$$

Therefore, using the Hardy–Littlewood–Sobolev inequality again, we know

$$
\begin{aligned}
\|w_n\|_\lambda^2 &\le \lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_n(x)|^p|w_n(y)|^p}{|x-y|^\mu}\,dy\,dx + o(1) \\
&\le C_0\lambda\bigg(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_n(x)|^p|w_n(y)|^p}{|x-y|^\mu}\,dy\,dx\bigg)^{1/p'} \\
&\quad\cdot\bigg(\int_{\mathbb{R}^N}|w_n|^{2Np/(2N-\mu)}\,dx\bigg)^{(2N-\mu)/(Np)} + o(1) \\
&\le C_1\lambda^{1/p}(c - I_\lambda(z) + o(1))^{1/p'} \\
&\quad\cdot\bigg(\int_{\mathbb{R}^N}|w_n|^{2Np/(2N-\mu)}\,dx\bigg)^{(2N-\mu)/(Np)} + o(1).
\end{aligned} \tag{3.1}
$$

Since $w_n^d \to 0$, it follows from Lemma 2.4 that

$$
\begin{aligned}
\|w_n^e\|_\lambda^2 + o(1) &\le C_1\lambda^{1/p}(c - I_\lambda(z) + o(1))^{1/p'} \\
&\quad\cdot\bigg(\int_{\mathbb{R}^N}|w_n^e|^{2Np/(2N-\mu)}\,dx\bigg)^{(2N-\mu)/(Np)} + o(1) \\
&\le C_2\lambda^{((N-2)p-2N+\mu+2)/(2p)}(c - I_\lambda(z) + o(1))^{1/p'}\|w_n^e\|_\lambda^2,
\end{aligned}
$$

consequently

$$
1 + o(1) \le C_2\lambda^{((N-2)p-2N+\mu+22p)}(c - I_\lambda(z) + o(1))^{1/p'}.
$$

We thus get

$$
\alpha_0\lambda^{-((N-2)p-2N+\mu+2)/(2(p-1))} \le c - I_\lambda(z)
$$

with $\alpha_0 > 0$ independent of λ, proving the lemma. □

From Lemma 3.3, we have the following convergence criterion for the (PS) sequences.

Corollary 3.4. *Let the potential $V(x)$ satisfies the assumption* (P_1). *Then I_λ satisfies the* $(\mathrm{PS})_c$ *condition for all $c < \alpha_0\lambda^{-((N-2)p-2N+\mu+2)/(2(p-1))}$.*

4. Proof of the main results

In the following we will prove that I_λ satisfies the geometry conditions of the generalized linking theorem. And then we can construct small minimax values for I_λ at levels where the $(\mathrm{PS})_c$ condition holds if the parameter λ is large enough.

Proposition 4.1.

$$
\inf\bigg\{\int_{\mathbb{R}^N}|\nabla\varphi|^2\,dx : \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N),\ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p|u(y)|^p}{|x-y|^\mu}\,dy\,dx = 1\bigg\} = 0. \tag{4.1}
$$

Lemma 4.2. *Assume that* (P_1) *and* (P_2) *hold. Then*

$$\int_{\mathbb{R}^N} \Psi(z)dx = \frac{\lambda}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|z(x)|^p|z(y)|^p}{|x-y|^\mu}\,dy\,dx \in \mathcal{C}^1(H,\mathbb{R})$$

is weakly sequentially lower semicontinuous and is weakly sequentially continuous.

Proof. The conclusion follows easily because $H \hookrightarrow H^1(\mathbb{R}^N,\mathbb{R}^2)$, so H embeds continuously into L^q for $q\in[2,2^*]$ and compactly into L^q_{loc} for $q\in[1,2^*)$.□

Lemma 4.3. *Assume that the potential* $V(x)$ *satisfies condition* (P_1)–(P_3), *then the functional* I_λ *satisfies the geometry generalized linking theorem*:

(a) *for each* $\lambda\geq 1$, $I_\lambda(0)=0$ *there exists* $\rho_\lambda>0$ *such that*

$$\kappa_\lambda := \inf I_\lambda(S_{\rho_\lambda}H^+)>0$$

where $S_{\rho_\lambda}=\{z\in H^+ : \|z\|_\lambda=\rho_\lambda\}$;

(b) *for any* $e\in H^+\setminus\{0\}$ *there is* $R>0$ *such that* $I_\lambda|_{\partial Q}\leq 0$ *where* $Q:=\{z=se+w : w\in H^-,\ s\geq 0,\ \|z\|_\lambda\leq R\}$.

Proof. (a) First note that, for each fixed λ, $I_\lambda(0)=0$. By the the Hardy–Littlewood–Sobolev inequality, for each $z\in H^+$, we know

$$\begin{aligned} I_\lambda(z) &= \frac{1}{2}\|z\|_\lambda^2 - \frac{\lambda}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|z(x)|^p|z(y)|^p}{|x-y|^\mu}\,dy\,dx \\ &\geq \frac{1}{2}\|z\|_\lambda^2 - \frac{\lambda}{2p}\bigg(\int_{\mathbb{R}^N}|z|^{2Np/(2N-\mu)}\,dx\bigg)^{(2N-\mu)/N} \\ &\geq \frac{1}{2}\|z\|_\lambda^2 - C_0\|z\|_\lambda^{2p}, \end{aligned} \tag{4.2}$$

since $p>1$, the conclusion follows if $\|z\|_\lambda$ is small enough.

(b) For $z=se+w$ such that $e\in H^+$, $w\in H^-$, $s\geq 0$, from the definition of $H^\pm$, we know that $e=(e_1,0)$, $w=(0,w_1)$, for some $e_1,w_1\in E$. Therefore,

$$\begin{aligned} I_\lambda(z) &= \frac{1}{2}\|z\|_\lambda^2 - \frac{\lambda}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|z(x)|^p|z(y)|^p}{|x-y|^\mu}\,dy\,dx \\ &\leq \frac{1}{2}(s^2\|e_1\|_\lambda^2-\|w_1\|_\lambda^2) - \frac{2^{p-1}\lambda s^{2p}}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|e_1(x)|^p|e_1(y)|^p}{|x-y|^\mu}\,dy\,dx, \end{aligned}$$

consequently, $I_\lambda(z)\to-\infty$ as $\|z\|_\lambda\to\infty$. □

In order to prove the existence results, we need to construct mini-max values below the levels where the (PS) condition holds. In the following we will construct the small Minimax values for large λ below the level in the Corollary 3.4. First let us define $J_\lambda(u)\colon E\to\mathbb{R}$ by

$$J_\lambda(u) = \frac{1}{2}\|u\|_\lambda^2 - \frac{\lambda}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p|u(y)|^p}{|x-y|^\mu}\,dy\,dx.$$

Then we have the following lemma.

Lemma 4.4. *Assume that $V(x)$ satisfies condition* (P$_1$)–(P$_3$)*, then the functional J_λ satisfies: For any $\sigma > 0$ there exists $\Lambda_\delta > 0$, such that, for each $\lambda \geq \Lambda_\delta$, there is $e_\lambda \in E_\lambda$ such that $J_\lambda(e_\lambda) < 0$ and*

$$\max_{t\in\mathbb{R}} J_\lambda(te_\lambda) \leq (1-\frac{1}{p})2^{(p+1)/(p-1)}\delta\lambda^{-((N-2)p-2N+\mu+2)/(2(p-1))}.$$

Proof. From Proposition 4.1, we know

$$\inf\bigg\{\int_{\mathbb{R}^N}|\nabla\varphi|^2dx : \varphi\in\mathcal{C}_0^\infty(\mathbb{R}^N),\ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi(x)|^p|\varphi(y)|^p}{|x-y|^\mu}\,dy\,dx = 1\bigg\} = 0.$$

Thus for any $\delta > 0$ one can choose $\varphi_\delta \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ with $\operatorname{supp}\varphi_\delta \subset B_{r_\delta}(0)$ such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi(x)|^p|\varphi(y)|^p}{|x-y|^\mu}\,dy\,dx = 1$$

and $|\nabla\varphi_\delta|_2^2 < 2\delta$. For any $\alpha \geq (N-\mu)/(4(p-1))$ and τ in assumption (P$_3$), set

$$e_\lambda(x) := \lambda^\alpha\varphi_\delta(\lambda^\tau x), \tag{4.3}$$

then $\operatorname{supp} e_\lambda \subset B_{\lambda^{-\tau}r_\delta}(0)$. It is easy to see that

$$\begin{aligned}
\int_{\mathbb{R}^N}|\nabla e_\lambda(x)|^2\,dx &= \lambda^{2\alpha-(N-2)\tau}\int_{\mathbb{R}^N}|\nabla\varphi_\delta|^2\,dx,\\
\int_{\mathbb{R}^N}V(x)e_\lambda^2(x)\,dx &= \lambda^{2\alpha-N\tau}\int_{\mathbb{R}^N}V(\lambda^{-\tau}x)\varphi_\delta^2(x)\,dx,\\
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|e_\lambda(x)|^p|e_\lambda(y)|^p}{|x-y|^\mu}\,dy\,dx &= \lambda^{2p\alpha+(\mu-2N)\tau}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi_\delta(x)|^p|\varphi_\delta(y)|^p}{|x-y|^\mu}\,dy\,dx.
\end{aligned}$$

From $\operatorname{supp}\varphi_\delta \subset B_{r_\delta}(0)$ and

$$\lim_{|x|\to 0}\frac{V(x)}{|x|^{(1-2\tau)/\tau}} = 0,$$

we know that there is $\Lambda_{\delta,0} > 0$ such that $V(\lambda^{-\tau}x) \leq 2\delta/(\lambda^{1-2\tau}|\varphi_\delta|_2^2)$ uniformly for $x \in B_{r_\delta}(0)$. Then from the above equalities, we get

$$\begin{aligned}
J_\lambda(e_\lambda) &= \frac{1}{2}\|e_\lambda\|_\lambda^2 - \frac{\lambda}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|e_\lambda(x)|^p|e_\lambda(y)|^p}{|x-y|^\mu}\,dy\,dx\\
&= \frac{\lambda^{2\alpha-(N-2)\tau}}{2}\int_{\mathbb{R}^N}|\nabla\varphi_\delta|^2\,dx + \frac{\lambda^{2\alpha-N\tau+1}}{2}\int_{\mathbb{R}^N}V(\lambda^{-\tau}x)\varphi_\delta^2\,dx\\
&\quad - \frac{\lambda^{2p\alpha+(\mu-2N)\tau+1}}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi_\delta(x)|^p|\varphi_\delta(y)|^p}{|x-y|^\mu}\,dy\,dx\\
&\leq 2\delta\lambda^{2\alpha-(N-2)\tau} - \frac{\lambda^{2p\alpha+(\mu-2N)\tau+1}}{2p}.
\end{aligned}$$

Since $\alpha \geq (N-\mu)/(4(p-1))$ and $0 < \tau < 1/2$, then

$$2\alpha-(N-2)\tau < 2p\alpha+(\mu-2N)\tau+1,$$

thus we know there exists $\Lambda_\delta > \Lambda_{\delta,0}$ such that, for any $\lambda > \Lambda_\delta > 1$ there is a e_λ such that $J_\lambda(e_\lambda) < 0$.

Since $J_\lambda(te_\lambda) > 0$ for t is small enough and $J_\lambda(te_\lambda) < 0$ for $t \geq 1$, we know

$$\max_{t\in\mathbb{R}} J_\lambda(te_\lambda) = \max_{t\in[0,1]} J_\lambda(te_\lambda).$$

Moreover, for such fixed $\lambda > \Lambda_\delta$, since $0 < \delta < 1$,

$$\begin{aligned}
\max_{t\in[0,1]} J_\lambda(te_\lambda) = \max_{t\in[0,1]} \bigg\{ & t^2\lambda^{2\alpha-(N-2)\tau} \int_{\mathbb{R}^N} |\nabla\varphi_\delta(x)|^2\,dx \\
&+ t^2\lambda^{2\alpha-N\tau+1} \int_{\mathbb{R}^N} V(\lambda^{-\tau}x)\varphi_\delta^2(x)\,dx \\
&- \frac{t^{2p}\lambda^{2p\alpha+(\mu-2N)\tau+1}}{2p} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varphi_\delta(x)|^p|\varphi_\delta(y)|^p}{|x-y|^\mu}\,dy\,dx \bigg\} \\
\leq{}& \delta\lambda^{-((N-2)p-2N+\mu+2)/(2(p-1))} \\
&\cdot \max_{t\in[0,1]} \bigg\{ 2t^2\lambda^{2\alpha-(N-2)\tau+((N-2)p-2N+\mu+2)/(2(p-1))} \\
&- \frac{t^{2p}\lambda^{2p\alpha+(\mu-2N)\tau+1+((N-2)p-2N+\mu+2)/(2(p-1))}}{2p} \bigg\}.
\end{aligned}$$

Direct computation shows $t^2 = 2^{2/(p-1)}\lambda^{(-2(p-1)\alpha-(\mu-N-2)\tau-1)/(p-1)}$, and then

$$\begin{aligned}
\max_{t\in[0,1]} J_\lambda(te_\lambda) &\leq \left(1-\frac{1}{p}\right) 2^{(p+1)/(p-1)}\delta\lambda^{-((N-2)p-2N+\mu+2)/(2(p-1))} \\
&\quad \cdot \lambda^{((1-2\tau)(p(N-2)+\mu-2N))/(2(p-1))} \\
&\leq \left(1-\frac{1}{p}\right) 2^{(p+1)/(p-1)}\delta\lambda^{-((N-2)p-2N+\mu+2)/(2(p-1))},
\end{aligned}$$

since $(2N-\mu)/N < p < (2N-\mu)/(N-2)$, $\tau \leq 1/2$ and $\lambda > 1$. □

Using this estimate we can prove easily the following lemma.

Lemma 4.5. *Under the assumptions of Lemma 4.4, for any $\sigma > 0$ there exist $\Lambda_\sigma > 0$, such that, for each $\lambda \geq \Lambda_\delta$, there exists $\widetilde{e}_\lambda \in H^+$ such that*

$$\sup_{z\in\mathbb{R}\widetilde{e}_\lambda\times H^-} I_\lambda(z) \leq \sigma\lambda^{-((N-2)p-2N+\mu+2)/(2(p-1))}.$$

Proof. For any $\delta > 0$, set $\widetilde{e}_\lambda = (e_\lambda, 0) \in H^+$. Let $z = t\widetilde{e}_\lambda + w$, $w = (0, w_1) \in H^-$, $t \geq 0$, we know

$$\begin{aligned}
I_\lambda(z) &= \frac{t^2}{2}\|z^+\|_\lambda^2 - \frac{1}{2}\|z^-\|_\lambda^2 - \frac{\lambda}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(t\widetilde{e}_\lambda+w)(x)|^p|(t\widetilde{e}_\lambda+w)(y)|^p}{|x-y|^\mu}\,dy\,dx \\
&\leq \frac{t^2}{2}\|e_\lambda\|_\lambda^2 - \frac{1}{2}\|w_1\|_\lambda^2 - \frac{2^{p-1}\lambda t^{2p}}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|e_\lambda(x)|^p|e_\lambda(y)|^p}{|x-y|^\mu}\,dy\,dx \\
&= J_\lambda(te_\lambda) - \frac{1}{2}\|w_1\|_\lambda^2.
\end{aligned}$$

Thus, for any $\sigma > 0$ there exist $\Lambda_\sigma > 0$, such that, for each $\lambda \geq \Lambda_\sigma$, there holds

$$\sup_{z\in\mathbb{R}\widetilde{e}_\lambda\times H^-} I_\lambda(z) \leq \max_{t\in\mathbb{R}} J_\lambda(te_\lambda) \leq \sigma\lambda^{-((N-2)p-2N+\mu+2)/(2(p-1))}. \qquad \square$$

Proof of Theorem 2.2. With $Y = H^+$ and $X = H^-$ the conditions (I_1) and (I_2) hold by Lemmas 4.2 and 4.3. Lemma 4.3 also shows that I_λ possesses the linking structure of Theorem 2.4. In particular, by Lemma 4.5 for any $\sigma > 0$ there is $\Lambda_\sigma > 0$ so that if $\lambda \geq \Lambda_\sigma$ we may choose $\widetilde{e}_\lambda \in H^+$ with the associated Q_λ satisfying

$$\sup I_\lambda(Q_\lambda) \leq \sigma\lambda^{-((N-2)p-2N+\mu+2)/(2(p-1))}.$$

Hence Corollary 3.4 implies that I_λ verifies the (PS) condition for all $c \leq \sup I_\lambda(Q_\lambda)$. Therefore, there exists a critical point z_λ satisfying

$$\kappa_\lambda \leq I_\lambda(z_\lambda) \leq \sigma\lambda^{-((N-2)p-2N+\mu+2)/(2(p-1))}.$$

Since z_λ is a critical point of I_λ, we know

$$\begin{aligned}\sigma\lambda^{-((N-2)p-2N+\mu+2)/(2(p-1))} &\geq I_\lambda(z_\lambda) - \frac{1}{2}(I'_\lambda(z_\lambda), z_\lambda)\\ &\geq \lambda\bigg(\frac{1}{2} - \frac{1}{2p}\bigg)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|z_\lambda(x)|^p|z_\lambda(y)|^p}{|x-y|^\mu}\,dy\,dx\end{aligned}$$

and therefore

$$\frac{p-1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|z_\lambda(x)|^p|z_\lambda(y)|^p}{|x-y|^\mu}\,dy\,dx \leq \sigma\lambda^{-((N-2)p-2N+\mu+2)/(2(p-1))-1}.$$

From $(I'_\lambda(z_\lambda), z^+_\lambda - z^-_\lambda) = 0$ we have

$$\begin{aligned}\|z_\lambda\|_\lambda &= \lambda\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|z_\lambda(x)|^{p-2}z_\lambda(x)(z^+_\lambda - z^-_\lambda)(x)|z_\lambda(y)|^p}{|x-y|^\mu}\,dy\,dx\\ &\leq \lambda\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|z_\lambda(x)|^p|z_\lambda(y)|^p}{|x-y|^\mu}\,dy\,dx \leq \frac{2p}{p-1}\sigma\lambda^{-((N-2)p-2N+\mu+2)/(2(p-1))},\end{aligned}$$

then if $N > 2$ and $2 + (2-\mu)/(N-2) < p < (2N-\mu)/(N-2)$ we must have $((N-2)p-2N+\mu+2)/(2(p-1)) > 0$ and consequently $z_\lambda \to 0$ as $\lambda \to \infty$, the conclusions are thus proved. $\square$

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