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EQUIVALENT FORMS OF THE BROUWER FIXED POINT THEOREM I

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To Professor Lech Górniewicz on the occasion of Honoris Causa Doctorate granted to Him by University of Zielona Góra

Abstract. In this paper we survey a set of Brouwer fixed point theorem equivalents. These equivalents are divided into four loops related to (1) the Borsuk retraction theorem, (2) the Himmelberg fixed point theorem, (3) the Gale lemma and (4) the Nash equilibrium theorem.

1. Introduction

In this paper we show the equivalence of many forms (some of them are classic) of the following Brouwer fixed point theorem

Theorem 1.1 [4]. *For the unit ball $B^n := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ and a continuous function $f\colon B^n \to B^n$ there exists $x \in B^n$ such that $x = f(x)$.*

The motivation for writing the paper is as follows: the first reason is that – as we believe – some of these equivalents and/or connections between these equivalents are new and the second reason is gathering in one paper different approaches of applications of the Brouwer theorem.

In the next section we present notation, necessary definitions and auxiliary results to be used later. The third section contains the before mentioned equivalents. The first set of equivalent forms covers some classic results connected to

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surjectivity property of continuous functions under proper assumptions on their boundary behavior. Then we show some results for multifunctions which are related to the Himmelberg fixed point theorem. The third loop involves equivalence of the existence of economic equilibrium and the Brouwer theorem. The last loop studies relations among simplex coverings, Maynard Smith equilibrium and Nash equilibrium.

2. Preliminaries

Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}^n$ denote the n-dimensional Euclidean space, and $[n] := \{1, \ldots, n\}$, $n \in \mathbb{N}$.

For vectors $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ we write $x \geq y$, if $x_i \geq y_i$, $i \in [n]$; $x > y$ is for strict component-wise inequalities $x_i > y_i$, $i \in [n]$. We write $x \in \mathbb{R}^n_+$ for $x \geq 0$ and $x \in \mathbb{R}^n_{++}$ for $x > 0$. By e^i we denote the ith unit vector of the standard basis for $\mathbb{R}^n$, $n \in \mathbb{N}$.

In what follows, for $n \in \mathbb{N}_0$ the set

$$\Delta^n = \left\{ x \in \mathbb{R}^{n+1}_{++} : \sum_{i=1}^{n+1} x_i = 1 \right\}$$

is the standard n-dimensional open unit simplex.

If $A \subset \mathbb{R}^n$, then the closure of A and the interior of A are denoted by $\overline{A}$ and $\operatorname{int} A$, respectively. For vectors $x, y \in \mathbb{R}^n$ their scalar product is $xy = \sum_{i=1}^{n} x_i y_i$. If $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, then $xA := \{xa \in \mathbb{R} : a \in A\}$. If $A, B \subset \mathbb{R}^n$, then $A + B := \{a + b \in \mathbb{R}^n : a \in A,\ b \in B\}$. $|a|$ is the Euclidean norm of $a \in \mathbb{R}^n$. For any number $n \in \mathbb{N}$, $B^n \subset \mathbb{R}^n$ denotes the closed n-dimensional unit ball and $\partial(B^n)$ denotes its boundary. For a set $X \subset \mathbb{R}^n$, $\operatorname{conv}(X)$ is its convex hull.

We now introduce some definitions:

- [9] Let $X \subset \mathbb{R}^n$, $n \in \mathbb{N}$. X is said to be *almost convex* if for any neighbourhood V of $0 \in \mathbb{R}^n$ and any finite set $x_1, \ldots, x_k \in X$ there exist points $v_1, \ldots, v_k \in V$ such that $\operatorname{conv}(\{x_1 + v_1, \ldots, x_k + v_k\}) \subset X$.
- A function $f\colon X \to \mathbb{R}^n$, $X \subset \mathbb{R}^n$, $n \in \mathbb{N}$, satisfies the *Walras law* if, for $x \in X$, $xf(x) = 0$.
- A function $f\colon X \to \mathbb{R}^n$, $X \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is *bounded from below* if $\inf_{x \in X} f_i(x) > -\infty$ for all coordinates f_i of f, $i \in [n]$.
- A function $f\colon X \to \mathbb{R}$, where X is a convex subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, is *concave* if, for $x, x' \in X$, $\alpha \in [0, 1]$,

$$f(\alpha x + (1 - \alpha)x') \geq \alpha f(x) + (1 - \alpha) f(x').$$

- A function $f\colon X \to \mathbb{R}$, where X is a convex subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, is *quasi concave* (q.c.) if, for $x, x' \in X$, $\alpha \in [0, 1]$,

$$f(\alpha x + (1 - \alpha)x') \geq \min\{f(x), f(x')\}.$$

- A function $f\colon X \to \mathbb{R}$, where X is a convex subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, is *strictly quasi concave* (s.q.c.) if for $x, x' \in X$, $x \neq x'$, $\alpha \in (0,1)$,
$$f(\alpha x + (1-\alpha)x') > \min\{f(x), f(x')\}.$$
- [10] For a nonempty convex and closed set $X \subset \mathbb{R}^n$, $n \in \mathbb{N}$, a nonempty proper subset D of X is called a *face of* X if there exists a vector $0 \neq p \in \mathbb{R}^n$ and a number $\alpha \in \mathbb{R}$ such that $D = X \cap \{x \in \mathbb{R}^n : px = \alpha\}$ and $px > \alpha$ for $x \in X \setminus D$.
- A function $f\colon X \to X$, where $\emptyset \neq X \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is a convex and closed set, *preserves faces* if, for any face D of X, $f(D) \subset D$.
- Let $f = (f_1, \dots, f_n)\colon X_1 \times \ldots \times X_n \to \mathbb{R}^n$, $X_i \subset \mathbb{R}^{n_i}$, $n_i \in \mathbb{N}$, $i \in [n]$, $n \in \mathbb{N}$, be given. A point $x = (x_1, \dots, x_n) \in X_1 \times \ldots \times X_n$ is *Nash equilibrium for* f if $f_i(x) \geq f_i(x_1, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_n)$ for all $x'_i \in X_i$, $i \in [n]$.
- [1, p. 109] A multifunction $f\colon X \multimap Y$, where $X, Y \subset \mathbb{R}^n$ are nonempty sets, is upper semicontinuous at $x \in X$ if for each open set $W \subset Y$ such that $f(x) \subset W$ there exists an open neighbourhood $U \subset X$ of x with $f(u) \subset W$ for all $u \in U$. The multifunction f is upper semicontinuous (u.s.c.) if it is upper semicontinuous at each $x \in X$.
- [1, p. 109] A multifunction $f\colon X \multimap Y$, where $X, Y \subset \mathbb{R}^n$ are nonempty sets, is lower semicontinuous at $x \in X$ if for each open set $W \subset Y$ such that $f(x) \cap W \neq \emptyset$ there exists an open neighbourhood $U \subset X$ of x with $f(u) \cap W \neq \emptyset$ for all $u \in U$. The multifunction f is lower semicontinuous (l.s.c.) if it is lower semicontinuous at each $x \in X$.

We need the following lemmata:

Lemma 2.1 [1, Corollary, p. 112]. *Suppose that f is a multifunction from $X \subset \mathbb{R}^n$ to $Y \subset \mathbb{R}^m$ $(m, n \in \mathbb{N})$, with nonempty closed values, and Y is compact. Multifunction $f\colon X \multimap Y$ is u.s.c. if and only if the graph of f, $\{(x,y) \in X \times Y : y \in f(x)\}$, is closed in $\mathbb{R}^n \times \mathbb{R}^m$.*

Lemma 2.2. *Suppose that A is an almost convex dense subset of $B \subset \mathbb{R}^n$, $n \in \mathbb{N}$. Then $\overline{A}$ is convex and $\overline{A} = \overline{B}$.*

Proof. The equality $\overline{A} = \overline{B}$ is obvious. For convexity of $\overline{A}$, see [11, Corollary 2.6]. $\square$

Lemma 2.3 [1, Maximum theorem, p. 115–117]. *Let $g\colon X \multimap Y$, $\emptyset \neq X \subset \mathbb{R}^n$, $\emptyset \neq Y \subset \mathbb{R}^m$ $(m, n \in \mathbb{N})$ be a continuous multifunction, i.e. u.s.c. and l.s.c., with nonempty compact values, and let f be a continuous real valued function defined on the graph of g. Then the multifunction $X \ni x \multimap \operatorname{argmax}_{y \in g(x)} f(x,y) \subset Y$ is u.s.c. with nonempty compact values. If for each $x \in X$ the set $\operatorname{argmax}_{y \in g(x)} f(x,y)$*

is a one-element set, then the just mentioned mapping is the continuous function from X to Y.

Lemma 2.4. *Let $\emptyset \neq X \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a compact and convex set with $\operatorname{int} X \neq \emptyset$. There exists a homeomorphism $h\colon X \to B^n$ satisfying $h(\partial(X)) = \partial(B^n)$ and $h(x) \neq -h(x')$, for $x, x' \in D$, where D is a face of X.*

Proof. Without loss of generality assume that $0 \in \operatorname{int} X$. For $x \in X \setminus \{0\}$ we define $t(x) := \max\{t \geq 0 : tx \in X\}$ and let $h\colon X \to B^n$ be defined for $x \in X$ by

$$h(x) := \begin{cases} \dfrac{1}{t(x)|x|}x, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

It is obvious that h is a homeomorphism with $h(\partial(X)) = \partial(B^n)$.

Now, let D be a face of X and let $x, x' \in D$. Suppose that p and α are the vector and the number defining D, respectively. Thus $px = px' = \alpha$ and since 0 is in the interior of X, $\alpha \neq 0$. Assuming $h(x) = -h(x')$ we get a contradiction.□

3. The equivalents

3.1. The first loop. The following theorems are equivalent:

(1) If $f\colon \partial(B^{n+1}) \to \partial(B^{n+1})$, $n \in \mathbb{N}_0$, is a continuous function such that for $x \in \partial(B^{n+1}) : x \neq f(x)$, then f is surjective.

(2) If $f\colon \partial(B^{n+1}) \to \partial(B^n) \times \{0\}$, $n \in \mathbb{N}$, is continuous, then, for some $x \in \partial(B^{n+1})$, $x = f(x)$.

(3) There is no continuous function $f\colon B^n \to \partial(B^n)$, $n \in \mathbb{N}$, such that $x \neq f(x)$ for $x \in \partial(B^n)$.

(4) [18] If $f\colon B^n \to B^n$, $n \in \mathbb{N}$, is a continuous function with $f(\partial(B^n)) \subset \partial(B^n)$ and for $x \in \partial(B^n)$ we have $x \neq f(x)$, then $f(B^n) = B^n$.

(5) If $f\colon X \to X$ is a continuous function, where X is a compact and convex subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, $\operatorname{int}(X) \neq \emptyset$, and f preserves faces, then f is surjective.

(6) [3] If $f\colon \overline{\Delta^n} \to \overline{\Delta^n}$, $n \in \mathbb{N}_0$, is continuous and preserves faces, then f is surjective.

(7) [5, p. 95, Theorem 7.2 (Borsuk)] There exists no retraction $r\colon B^n \to \partial(B^n)$, $n \in \mathbb{N}$.

Proof. (1)⇒(2) Since $\partial(B^n) \times \{0\} \subsetneq \partial(B^{n+1})$ and f is continuous, the claim follows.

(2)⇒(3) Let $g\colon \partial(B^{n+1}) \to B^n$, $g(x_1, \ldots, x_n, x_{n+1}) := (x_1, \ldots, x_n)$ and define $h\colon \partial(B^n) \to \partial(B^n) \times \{0\}$ by $h(x) := (x, 0)$. For a continuous function $f\colon B^n \to \partial(B^n)$, consider the composition

$$h \circ f \circ g\colon \partial(B^{n+1}) \to \partial(B^n) \times \{0\}.$$

The function $h \circ f \circ g$ satisfies assumptions of (2), so there exists $x' \in \partial(B^{n+1})$ such that $h(f(g(x'))) = x'$. But $x' \in \partial(B^n) \times \{0\}$ so $x' = (x, 0) \in \partial(B^n) \times \{0\}$. Therefore $f(g(x, 0)) = x$ and $f(x) = x \in \partial(B^n)$. This proves (3).

(3)⇒(4) If $f(B^n) \subsetneq B^n$ then there exists $\overline{y} \in B^n \setminus f(B^n)$. Define $g\colon B^n \to \partial(B^n)$ by $g(x) = \overline{y} + t(x)(f(x) - \overline{y})$, where $t(x) = \max\{t \geq 0 : \overline{y} + t(f(x) - \overline{y}) \in B^n\}$. The function $g(x)$ is continuous and (3) implies that there exists $x \in \partial(B^n)$ such that $x = g(x)$ (1). Since $x \in \partial(B^n)$ then by assumptions of (4) $f(x) \in \partial(B^n)$ and $g(x) = f(x)$, so $f(x) = x \in \partial(B^n)$, but this is impossible.

(4)⇒(5) Suppose that $y \in X \setminus f(X)$. Since f is continuous and X is compact and convex we can assume that $y \in \operatorname{int}(X)$, and $y = 0 \in \operatorname{int}(X)$ (after a translation if needed). Let h be the homeomorphism existing by Lemma 2.4. Define $g\colon B^n \to B^n$ by $g(x) := (h \circ f \circ h^{-1})(-x)$ for $x \in B^n$. Since faces of X are contained in the boundary of X, f preserves faces and h maps $\partial(X)$ onto $\partial(B^n)$, then it follows that $g(x) \in \partial(B^n)$ for $x \in \partial(B^n)$. Observe that $g(x) = x$, or equivalently $f(h^{-1}(-x)) = h^{-1}(x)$, implies $h^{-1}(-x)$ and $h^{-1}(x)$ belong to the same face of X and consequently, by Lemma 2.4, $-x = -h(h^{-1}(x)) \neq h(h^{-1}(-x)) = -x$, which is not possible. Thus for $x \in \partial(B^n)$ $g(x) \neq x$, g satisfies assumptions of Theorem (4) and g is surjective. By the definition of g, the function f is also surjective.

(5)⇒(6) Obvious.

(6)⇒(7) Suppose that $r\colon B^n \to \partial(B^n)$ is a retraction. For $x = (x_1, \ldots, x_{n+1})$ in $\overline{\Delta^n}$ let $p(x) := (x_1, \ldots, x_n)$. The function p is a homeomorphism between $\overline{\Delta^n}$ and $X = \Big\{x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i \leq 1\Big\}$. Let $h\colon X \to B^n$ be the homeomorphism in Lemma 2.4. Define $g\colon \overline{\Delta^n} \to \overline{\Delta^n}$ by $g(x) := ((p^{-1} \circ h^{-1}) \circ r \circ (h \circ p))(x)$, $x \in \overline{\Delta^n}$. The function g is the identity on $\partial(\overline{\Delta^n})$ and satisfies assumptions of (6). Thus g is surjective and by definition of g, $g(\overline{\Delta^n}) \subset \partial(\overline{\Delta^n})$. But this is impossible.

(7)⇒(1) Suppose f satisfies assumptions in (1) but is not surjective. Let $\overline{y} \in \partial(B^{n+1}) \setminus f(\partial(B^{n+1}))$. Since f is continuous, then there exists $\varepsilon > 0$, such that $B_\varepsilon(\overline{y}) := \{x \in \partial(B^{n+1}) : |x - \overline{y}| < \varepsilon\}$ is disjoint from the image of f. Obviously, $\partial(B^{n+1}) \setminus B_\varepsilon(\overline{y})$ is homeomorphic to B^n.

Let $h\colon B^n \to \partial(B^{n+1}) \setminus B_\varepsilon(\overline{y})$ be a homeomorphism. Define $g\colon B^n \to B^n$ by $g(x) := (h^{-1} \circ f \circ h)(x)$ for $x \in B^n$. Observe that $g(x) = x$ for some $x \in B^n$ implies $x = (h^{-1} \circ f \circ h)(x)$ and $h(x) = (f \circ h)(x)$. The last fact contradicts assumptions on f. Thus $g(x) \neq x$ for all $x \in B^n$.

Let $r\colon B^n \to \partial(B^n)$ be defined by $r(x) := g(x) + t(x)(x - g(x))$, where $t(x) := \max\{t \geq 0 : g(x) + t(x - g(x)) \in B^n\}$, $x \in B^n$. One can check that r is a retraction from B^n to $\partial(B^n)$. □

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(1) One can find a proof of continuity of the function $t(\cdot)$ in [15, pp. 413, 416].

3.2. The second loop. The following theorems are equivalent:

(1) [13, Kakutani] If $f\colon B^n \multimap B^n$, $n \in \mathbb{N}$, is an u.s.c. multifunction with nonempty convex and closed values, then f has a fixed point: $x \in f(x)$ for some $x \in B^n$.
(2) [9, Theorem 1] If D is a nonempty convex and compact subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, and $f\colon D \multimap D$ is an u.s.c. multifunction such that $f(x)$ is closed for $x \in D$ and convex for $x \in C$, where C is an almost convex dense subset of D, then f has a fixed point.
(3) [12, Theorem 3.2] Suppose that C, D are nonempty and almost convex subsets of $\mathbb{R}^n$, $n \in \mathbb{N}$, and C is a dense subset of D. Let $f\colon D \multimap D$ be an u.s.c. multifunction with nonempty closed values and such that $f(x)$ is convex for $x \in C$. If $\overline{f(D)}$ is a bounded subset of D, then f has a fixed point.
(4) [9, Theorem 2] If D is a nonempty and convex subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, $f\colon D \multimap D$ is an u.s.c. multifunction such that $f(x)$ is a nonempty convex and closed set for $x \in D$, $\overline{f(D)} \subset D$ and $f(D)$ is bounded, then f has a fixed point.
(5) If $f\colon \Delta^n \multimap \Delta^n$, $n \in \mathbb{N}_0$, is an u.s.c. multifunction with nonempty convex and closed values, then there exist sequences $x^q, y^q \in \Delta^n$ with $y^q \in f(x^q)$, $q \in \mathbb{N}$, satisfying $\lim_{q\to\infty}(y^q - x^q) = 0$.
(6) [25, Nash-2], [27] Let $f = (f_1, \ldots, f_n) : \underbrace{X_1 \times \ldots \times X_n}_{X} \to \mathbb{R}^n$, where $X_i \subset \mathbb{R}^{n_i}$, $n_i \in \mathbb{N}$, is a nonempty convex and compact set, $i \in [n]$, $n \in \mathbb{N}$, be a continuous function. Suppose that each coordinate function f_i is quasiconcave in x_i, for fixed $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$, $i \in [n]$. Then there exists Nash equilibrium for f.

Proof. (1)$\Rightarrow$(2) For a proof, see [9, Theorem 1].

(2)$\Rightarrow$(3) Let $E := \operatorname{conv}(\overline{f(D)})$. E is a compact convex subset of $\overline{D}$ (Lemma 2.2) and since $\overline{C} = \overline{D}$, then for a set $C' := C \cap E'$, where $E' := (E + \operatorname{int} B^n) \cap D$ and $\operatorname{int} B^n$ denotes the interior of B^n, it holds $\overline{C'} = \overline{E'}$ and C', E' are nonempty almost convex and bounded sets with $C' \subset E'$.

Now, let $G := \{(x,y) \in E' \times E' : y \in f(x)\}$ and define a multifunction $g\colon \overline{E'} \multimap \overline{E'}$ by $g(x) := \{y \in \overline{E'} : (x,y) \in \overline{G}\}$. By Lemma 2.1, the multifunction g is u.s.c. and since $\overline{E'}$ is compact and convex, then g satisfies assumptions of (2) (with C' being the dense subset). So there exists $x \in \overline{E'}$ such that $x \in g(x)$. Observe that $f(E') \subset f(D) \subset D$ and $f(E') \subset \underline{f(D)} \subset E \subset E + B^n$, hence $f(E') \subset E'$ and $\overline{f(E')} \subset \overline{E'}$. Furthermore, $G \subset \overline{E'} \times \overline{f(E')}$. By definition of g, $g(\overline{E'}) \subset \overline{f(E')} \subset \overline{f(D)}$. Thus, for the fixed point $x \in \overline{E'}$, $x \in g(x) \subset \overline{f(D)} \subset D$. Therefore $(x,x) \in \overline{G}$ and there exists a sequence $(x^q, y^q) \in G$, $y^q \in f(x^q)$, $q \in \mathbb{N}$

with $\lim_{q\to\infty}(x^q, y^q) = (x, x)$. By u.s.c. of the multifunction f and because $x \in D$, $x \in f(x)$.

(3)⇒(4) Obvious.

(4)⇒(5) Let us fix $q \in \mathbb{N}$ and define

$$e_q^i := \left(\frac{1}{nq}, \dots, \frac{1}{nq}, 1 - \frac{1}{q}, \frac{1}{nq}, \dots, \frac{1}{nq}\right) \in \mathbb{R}^{n+1},$$

where $1 - 1/q$ occurs on the ith coordinate, $i \in [n+1]$. Observe that $e_q^i \in \Delta^n$ for $q > 1$ and $\lim_{q\to\infty} e_q^i = e^i$, $i \in [n+1]$. Let us also define $\alpha^q(x) = \sum_{i=1}^{n+1} x_i e_q^i$ for $x \in \Delta^n$ and $D^q = \overline{\alpha^q(\Delta^n)}$. It is clear that D^q is a nonempty convex and compact subset of Δ^n and α^q is a continuous function on Δ^n, mapping convex compact subsets of Δ^n onto convex compact subsets of D^q. Furthermore, let $f^q\colon \Delta^n \multimap \Delta^n$ be defined by $f^q(x) := \alpha^q(f(x))$, $x \in \Delta^n$. As the composition of a continuous function and an u.s.c. mapping the mapping f^q is u.s.c. Moreover, $f^q(x)$ is a convex and compact subset of a compact set $D^q \subset \Delta^n$ for $x \in \Delta^n$ and $\overline{f^q(\Delta^n)} \subset D^q$. Theorem (4) now guarantees that there exists $x^q \in D^q$ such that $x^q \in f^q(x^q)$. Assume that $x^q \in f^q(x^q)$ for each $q \in \mathbb{N}$. It holds $x^q = \alpha^q(y^q)$ for some $y^q \in f(x^q)$ and therefore (for convergent subsequences if necessary)

$$\begin{aligned} 0 = \lim_{q\to\infty}(\alpha^q(y^q) - x^q) &= \lim_{q\to\infty}\left(\left(\sum_{i=1}^{n+1} y_i^q e_q^i\right) - x^q\right) \\ &= \lim_{q\to\infty}\left(\left(\sum_{i=1}^{n+1} y_i^q e^i\right) - x^q\right) = \lim_{q\to\infty}(y^q - x^q), \end{aligned}$$

which completes the proof.

(5)⇒(6) Suppose that $f = (f_1, \dots, f_n)$ satisfies the assumptions of Theorem (6). Since function f_i is continuous on X and q.c. with respect to the ith variable, then Lemma 2.3 implies that

$$g_i(x) := \operatorname*{argmax}_{x_i' \in X_i} f_i(x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_n)$$

is an u.s.c. multifunction from X to X_i with nonempty, convex and compact values, $i \in [n]$. Let $g(x) := g_1(x) \times \ldots \times g_n(x)$, $x \in X$. Notice that $g(x) \subset X$ for $x \in X$. Without loss of generality we assume that X is a retract of $\Delta^{n'}$ for some $n' \in \mathbb{N}$. Theorem (5) now implies the existence of points $x^q \in X$, $y^q \in g(x^q)$, $q \in \mathbb{N}$, such that $\lim_{q\to\infty}(x^q - y^q) = 0$. The compactness of X and u.s.c. of g imply that we may assume $\lim_{q\to\infty} x^q = x \in X$ and $\lim_{q\to\infty} y^q = x \in g(x)$. The point x is Nash equilibrium for f.

(6)⇒(1) For the implication [Nash-2]⇒[Kakutani], see [25]. □

3.3. The third loop. The following theorems are equivalent:

(1) [15, p. 137] Let $f\colon \overline{\Delta^n} \to \mathbb{R}^{n+1}$, $n \in \mathbb{N}_0$, be a continuous function. Then there exists $x \in \overline{\Delta^n}$ such that either $f(x) \leq 0$ or for $i \in [n+1]$: $x_i > 0$ if and only if $f_i(x) > 0$.

(2) Let $f\colon \Delta^n \to \mathbb{R}^{n+1}$, $n \in \mathbb{N}_0$, be a bounded from below and continuous function satisfying the Walras law and the following boundary condition: $x = \lim\limits_{q\to\infty} x^q$, $x^q \in \Delta^n$, $q \in \mathbb{N}$ and $x_i = 0$, imply $\liminf\limits_{q\to\infty} f_i(x^q) > 0$, $i \in [n+1]$. Then there exists $x \in \Delta^n$ such that $f(x) = 0$.

(3) Let $f\colon \Delta^n \to \mathbb{R}^{n+1}$, $n \in \mathbb{N}_0$, be a bounded from below and continuous function satisfying the Walras law and the following boundary condition: $x = \lim\limits_{q\to\infty} x^q$, $x^q \in \Delta^n$, $q \in \mathbb{N}$ and $x_i = 0$, imply $\lim\limits_{q\to\infty} f_i(x^q) = +\infty$, $i \in [n+1]$. Then there exists $x \in \Delta^n$ such that $f(x) = 0$.

(4) [19, Theorem 3] Let $f : \Delta^n \to \mathbb{R}^{n+1}$, $n \in \mathbb{N}_0$, be a bounded from below continuous function satisfying the Walras law. There exists a sequence $x^q \in \Delta^n$, $q \in \mathbb{N}$, satisfying $\lim\limits_{q\to\infty} f_i(x^q) \leq 0$, $i \in [n+1]$.

(5) Let $f\colon \overline{\Delta^n} \to \mathbb{R}^{n+1}$, $n \in \mathbb{N}_0$, be a continuous function and suppose that $g\colon \overline{\Delta^n} \to \overline{\Delta^n}$ is a homeomorphism satisfying $g(x)f(x) = 0$ for $x \in \overline{\Delta^n}$. There exists $x \in \overline{\Delta^n}$ satisfying $f(x) \leq 0$.

(6) [8, Principal Lemma, p. 159] Let $f\colon \overline{\Delta^n} \to \mathbb{R}^{n+1}$ for $n \in \mathbb{N}_0$, be a continuous function satisfying the Walras law. There exists $x \in \overline{\Delta^n}$ satisfying $f(x) \leq 0$.

Proof. (1)⇒(2) Suppose that assumptions of (2) are satisfied. For $q \in \mathbb{N}$, let $e^i_q \in \mathbb{R}^{n+1}$ be defined as in the proof of the implication (4)⇒(5) in the second loop, Section 3.2, $i \in [n+1]$, and let $\alpha^q(x) := \sum\limits_{i=1}^{n+1} x_i e^i_q$ for $x \in \overline{\Delta^n}$. Notice that $\alpha^q(x) \in \Delta^n$, $x \in \overline{\Delta^n}$, for $q > 1$. The composition $f \circ \alpha^q$ is a continuous function from $\overline{\Delta^n}$ to $\mathbb{R}^{n+1}$, $q \in \mathbb{N}$, $q > 1$. By Theorem 1 for each q there exists $x^q \in \overline{\Delta^n}$ such that either $f \circ \alpha^q(x^q) \leq 0$ or for $i \in [n+1]$: $f_i(\alpha^q(x^q)) > 0$ is equivalent to $x^q_i > 0$.

Suppose that for infinitely many q: $f_i(\alpha^q(x^q)) > 0$ if and only if $x^q_i > 0$, $i \in [n+1]$. We may assume that $\lim\limits_{q\to\infty} \alpha^q(x^q) = x \in \overline{\Delta^n}$. If $x_i = 0$ for some $i \in [n+1]$, then by the boundary condition it holds $f_i(\alpha^q(x^q)) > 0$ for large q and we conclude that our assumption implies $f(\alpha^q(x^q)) > 0$ for large q. But this is impossible in view of the fact that $\alpha^q(x^q) > 0$ and $\alpha^q(x^q)f(\alpha^q(x^q)) = 0$ (the Walras law) for all $q \in \mathbb{N}$. Therefore there exists $q \in \mathbb{N}$ such that $f(\alpha^q(x^q)) \leq 0$. Thus, the Walras law and the inequality $\alpha^q(x^q) > 0$ imply $f(\alpha^q(x^q)) = 0$.

(2)⇒(3) Obvious.

(3)⇒(4) Suppose that the function f satisfies assumptions of (4). For every $q \in \mathbb{N}$ define $f^q\colon \Delta^n \to \mathbb{R}^{n+1}$ by $f^q(x) := f(x) + q^{-1}g(x)$, where

$$g(x) := \left(\frac{1}{(n+1)x_1} - 1, \dots, \frac{1}{(n+1)x_{n+1}} - 1\right).$$

It can be easily checked that f^q satisfies assumptions of Theorem (3). So that for $q \in \mathbb{N}$ there is $x^q \in \Delta^n$ such that $f^q(x^q) = 0$. Suppose we have chosen a sequence $x^q \in \overline{\Delta^n}$ with $\lim_{q\to\infty} x^q = x \in \overline{\Delta^n}$ and $f^q(x^q) = 0$. For $i \in [n+1]$ it holds:

$$(*) \qquad f_i(x^q) = -q^{-1}\left(\frac{1}{(n+1)x_i^q} - 1\right).$$

If $x_i = 0$, then for large values of q the right-hand side term of the equation $(*)$ is negative so that in the limit the left-hand side term must be non positive (and can not be divergent to $-\infty$ since F is bounded from below). If $x_i > 0$, then the limit of the left-hand side of the equation $(*)$ is 0.

(4)⇒(5) Suppose that f and g meet the assumptions of Theorem (5). Let g^{-1} denote the inverse of the function g. Notice that $g(x)f(x) = g(x)(f \circ g^{-1})(g(x)) = 0$, hence $x(f \circ g^{-1})(x) = 0$ for $x \in \overline{\Delta^n}$, because g is a homeomorphism. Theorem (4) applied to $f \circ g^{-1}$ and the continuity of $f \circ g^{-1}$ defined on the compact set $\overline{\Delta^n}$, imply that there exists $x \in \overline{\Delta^n}$ satisfying $f(g^{-1}(x)) \le 0$. Obviously, $g^{-1}(x) \in \overline{\Delta^n}$.

(5)⇒(6) Obvious.

(6)⇒(1) Suppose that there is no x satisfying $f(x) \le 0$. Define $h : \overline{\Delta^n} \to \mathbb{R}^{n+1}$ by $h(x) := f(x) - xf(x)/(xx)x$. Observe h satisfies assumptions of (6) and there exists $x \in \overline{\Delta^n}$ such that $h(x) \le 0$. We have $f(x) \le xf(x)/(xx)x$, or equivalently $f_i(x) \le xf(x)/(xx)x_i$, $i \in [n+1]$. So $xf(x) > 0$, since $x \ge 0$. Obviously, if $f_i(x) > 0$ then $x_i > 0$. If for some j, $x_j > 0$ but $f_j(x) \le 0$, then $f_j(x)x_j < xf(x)/(xx)(x_j)^2$. But for $i \in [n+1]$ $f_i(x)x_i \le xf(x)/(xx)(x_i)^2$. Adding the inequalities and using the strict inequality for j we get

$$f(x)x = \sum_{i=1}^{n+1} f_i(x)x_i < \frac{xf(x)}{xx}\sum_{i=1}^{n+1}(x_i)^2 = \frac{xf(x)}{xx}xx = xf(x),$$

which is impossible. □

3.4. The fourth loop. The following theorems are equivalent:

(1) [7, Theorem 1] Let $X \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a nonempty convex compact set and suppose that $\{A_i\}_{i\in I}$ is a locally finite family of closed subsets of X such that $X = \bigcup_{i\in I} A_i$. Then for any family $\{x^i\}_{i\in I}$ of points of X, there

exists a non-empty finite subset J of I such that

$$\operatorname{conv}(\{x^j : j \in J\}) \cap \Big(\bigcap_{j\in J} A_j\Big) \neq \emptyset.$$

(2) [15, p. 185] Let $A_1,\dots,A_{n+1} \subset \mathbb{R}^{n+1}$, $n \in \mathbb{N}_0$, be an open (a closed) covering of $\overline{\Delta^n}$. Then there exists a subset of indices $\{i_1,\dots,i_k\} \subset [n+1]$, $i_1 < \dots < i_k$, such that

$$\operatorname{conv}(\{e^{i_1},\dots,e^{i_k}\}) \cap A_{i_1} \cap \dots \cap A_{i_k} \neq \emptyset.$$

(3) [14, KKM Lemma] If a family of open (closed) sets $A_1,\dots,A_{n+1} \subset \mathbb{R}^{n+1}$, $n \in \mathbb{N}_0$, admits $\operatorname{conv}(\{e^{i_1},\dots,e^{i_k}\}) \subset A_{i_1} \cup \dots \cup A_{i_k}$ for any subset of indices $\{i_1,\dots,i_k\} \subset [n+1]$, then $A_1 \cap \dots \cap A_{n+1} \neq \emptyset$.

(4) [16, Theorem on Indexed Families] Let $f\colon \overline{\Delta^n} \to X$, $n \in \mathbb{N}_0$, where $X \neq \emptyset$ is a topological Hausdorff space, be a continuous function. Then for any open cover $A_1,\dots,A_{n+1}$ of X there exists a subset of indices $\{i_1,\dots,i_k\} \subset [n+1]$ satisfying

$$f(\operatorname{conv}(\{e^{i_1},\dots,e^{i_k}\})) \cap A_{i_1} \cap \dots \cap A_{i_k} \neq \emptyset.$$

(5) [6, Corollary 1], [17, Maynard Smith Theorem] Let $X \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a nonempty convex compact set and suppose that $f\colon X \times X \to \mathbb{R}$ is a continuous function such that $f(\,\cdot\,, y)$ is q.c. for any fixed $y \in X$. Then there exists $\overline{y} \in X$ such that $f(x,\overline{y}) \le f(\overline{y},\overline{y})$, $x \in X$.

(6) Let $f = (f_1,\dots,f_n)\colon \underbrace{X_1 \times \dots \times X_n}_{X} \to \mathbb{R}^n$, where $\emptyset \neq X_i \subset \mathbb{R}^{n_i}$ and $n_i \in \mathbb{N}$, is a nonempty convex and compact set, $i \in [n]$, $n \in \mathbb{N}$, be a continuous function. Suppose further that each coordinate function f_i is concave in x_i given any $x_1,\dots,x_{i-1},x_{i+1},\dots,x_n$ fixed. Then there exists $\overline{x} \in X$ such that

$$\sum_{i\in I} f_i(\overline{x}_1,\dots,\overline{x}_{i-1},x_i,\overline{x}_{i+1},\dots,\overline{x}_n) \le \sum_{i\in I} f_i(\overline{x})$$

for any subset $I \subset [n]$, $x_i \in X_i$, $i \in I$ (2).

(7) [15, p. 141] Suppose that f is as in (6). Then there exists Nash equilibrium for f.

(8) [25, Nash-1] Suppose that f is as in (6), with 'concave' changed to 'strictly quasi concave'. Then there exists Nash equilibrium for f.

Proof. (1)⇒(2) If the sets $A_1,\dots,A_{n+1}$ in the formulation of (2) are closed, then (2) is obviously implied by (1). If the sets $A_1,\dots,A_{n+1}$ are open it suffices to consider the case where $\bigcap_{i\in[n+1]} A_i = \emptyset$. Let $A_i^\varepsilon := \{x \in A_i : \operatorname{dist}(x,\partial A_i) \ge \varepsilon\}$, where $\operatorname{dist}(x,\partial A_i)$ denotes the distance of $x \in \overline{\Delta^n}$ from the boundary of A_i,

(2) This statement is motivated by the results contained in [21].

$i \in [n+1]$ (3). It is easy to see that there exists $\varepsilon > 0$ for which the closed sets A_i^ε, $i \in [n+1]$, cover $\overline{\Delta^n}$ and since for $i \in [n+1]$ $A_i^\varepsilon \subset A_i$, claim (2) is true for open sets, too.

(2)⇒(3) Suppose that Theorem (3) is false. For a family of sets A_i satisfying assumptions of Theorem (3) with $\bigcap_{i=1}^{n+1} A_i = \emptyset$ define $B_i := \overline{\Delta^n} \setminus A_i$, $i \in [n+1]$, and apply Theorem (2) with B_i in place of A_i to get a contradiction.

(3)⇒(4) Let $B_i := f^{-1}(A_i)$, $i \in [n+1]$. By continuity of f, the family B_i, $i \in [n+1]$, is an open cover for $\overline{\Delta^n}$. Suppose that there is no nonempty subset $I \subset [n+1]$ for which $conv(\{e^i : i \in I\}) \cap \bigcap_{i\in I} B_i \neq \emptyset$. This implies that for any subset $I \subset [n+1]$ it holds $\mathrm{conv}(\{e^i : i \in I\}) \subset \bigcup_{i\in I} \overline{\Delta^n} \setminus B_i$. Therefore the family of closed sets $\overline{\Delta^n} \setminus B_i$, $i \in [n+1]$, satisfies assumptions of (3) and we get

$$\bigcap_{i=1}^{n+1} \overline{\Delta^n} \setminus B_i \neq \emptyset.$$

Moreover,

$$\bigcap_{i=1}^{n+1} \overline{\Delta^n} \setminus B_i = \overline{\Delta^n} \setminus \left(\bigcup_{i=1}^{n+1} B_i \right) = \overline{\Delta^n} \setminus \left(\bigcup_{i=1}^{n+1} f^{-1}(A_i) \right) = \overline{\Delta^n} \setminus \overline{\Delta^n} = \emptyset$$

which is not possible.

(4)⇒(5) Just apply the proof of the Maynard Smith theorem in [15, p. 140] (4).

(5)⇒(6) Suppose that $f = (f_1, \ldots, f_n)$ satisfies the assumptions of Theorem (6). Define $g\colon X \times X \to \mathbb{R}$ by

$$g(x, x') := \sum_{i\in[n]} f_i(x'_1, \ldots, x'_{i-1}, x_i, x'_{i+1}, \ldots, x'_n).$$

One can easily check that g is continuous in its domain and concave in its first variable for any fixed $x' \in X$. Theorem (5) implies the existence of $\overline{x} \in X$ such that $g(x, \overline{x}) \leq g(\overline{x}, \overline{x})$ for $x \in X$. Let $\emptyset \neq I \subset [n]$ and take any vector $x \in X$ such that $x_i = \overline{x}_i$, $i \in I$. It holds

$$\begin{aligned} g(x, \overline{x}) = \sum_{i\in[n]\setminus I} f_i(\overline{x}) + \sum_{i\in I} f_i(\overline{x}_1, \ldots, \overline{x}_{i-1}, x_i, \overline{x}_{i+1}, \ldots, \overline{x}_n) \\ \leq g(\overline{x}, \overline{x}) = \sum_{i\in[n]\setminus I} f_i(\overline{x}) + \sum_{i\in I} f_i(\overline{x}) \end{aligned}$$

(3) The standard simplex $\overline{\Delta^n}$ is treated here as a topological subspace of $\mathbb{R}^n$ endowed with the natural topology.

(4) The proof in [15] is carried out under the assumption of concavity, but it is also valid in the case of quasiconcavity.

and canceling the same terms on both sides we get

$$\sum_{i\in I} f_i(\overline{x}_1,\dots,\overline{x}_{i-1},x_i,\overline{x}_{i+1},\dots,\overline{x}_n) \le \sum_{i\in I} f_i(\overline{x}).$$

(6)⇒(7) It suffices to consider all one element subsets of $[n]$.

(7)⇒(8) Suppose that $f = (f_1,\dots,f_n)$ satisfies the assumptions of Theorem (8). Functions f_i are continuous and s.q.c. in variable x_i for any fixed $x_1,\dots,x_{i-1},x_{i+1},\dots,x_n$, which in view of convexity and compactness of the domain of f and Lemma 2.3 implies that multifunctions

$$g_i(x) := \operatorname*{argmax}_{x_i'\in X_i} f_i(x_1,\dots,x_{i-1},x_i',x_{i+1},\dots,x_n), \quad i\in[n],$$

are continuous functions from X to X_i. Let $\overline{f}_i\colon X\to\mathbb{R}^n$ be defined by $\overline{f}_i(x) := f_i(x_1,\dots,x_{i-1},g_i(x),x_{i+1},\dots,x_n)$ for $x\in X$ and $i\in[n]$. The functions $\overline{f}_i$ are continuous and each $\overline{f}_i$ is constant with respect to the ith variable, while all other variables are fixed, so in particular it is concave with respect to the ith variable. Thus, the function $\overline{f} = (\overline{f}_1,\dots,\overline{f}_n) : X\to\mathbb{R}^n$ satisfies assumptions of Theorem (7) and there exists Nash equilibrium $\overline{x}\in X$ for $\overline{f}$. Therefore $\overline{f}_i(\overline{x}_1,\dots,\overline{x}_{i-1},x_i,\overline{x}_{i+1},\dots,\overline{x}_n)\le\overline{f}_i(\overline{x})$ and

$$\begin{aligned} &f_i(\overline{x}_1,\dots,\overline{x}_{i-1},x_i,\overline{x}_{i+1},\dots,\overline{x}_n)\\ &\qquad\qquad \le f_i(\overline{x}_1,\dots,\overline{x}_{i-1},g_i(\overline{x}),\overline{x}_{i+1},\dots,\overline{x}_n) = \overline{f}_i(\overline{x})\end{aligned}$$

for $x_i\in X_i$, $i\in[n]$. We obtain:

$$\overline{x}_i = \operatorname*{argmax}_{x_i\in X_i} f_i(\overline{x}_1,\dots,\overline{x}_{i-1},x_i,\overline{x}_{i+1},\dots,\overline{x}_n),$$

$i\in[n]$, so $\overline{x}$ is Nash equilibrium for f.

(8)⇒(1) If (8) is true, then the Brouwer fixed point theorem follows from [25]. From the second loop we know that the Brouwer fixed point theorem implies the Kakutani fixed point theorem. And from the Kakutani fixed point theorem we can derive (1) – see [7] for a simple proof. □

4. Final comments

Equivalence of the first loop and the Brouwer fixed point theorem comes easily from equivalence of the Borsuk theorem and the Brouwer theorem – see [5, p. 95]. It is clear that the Kakutani fixed point theorem (Theorem (1) in the second loop) implies the Brouwer theorem; the implication in reverse direction is also well-known (e.g. see [22, p. 67]). For equivalence of Theorem (6) in the third loop and the Brouwer theorem see [2, p. 47], and [26]. It is also known that the KKM lemma (Theorem (3) in the last loop) is equivalent to the Brouwer theorem [2, p. 28, p. 44]. A recent non-constructive proof of the Brouwer theorem is presented in [24]. In [20] a new constructive proof of Theorem (3) from the third loop is given.

Finally, in the paper [23] many equivalent formulations and a historical background of the Brouwer fixed point theorem can be found.

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