ON SECOND ORDER ELLIPTIC EQUATIONS
AND VARIATIONAL INEQUALITIES WITH ANISOTROPIC
PRINCIPAL OPERATORS

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ABSTRACT. This paper is about boundary value problems of the form

$$\begin{cases} -\operatorname{div}[\nabla\Phi(\nabla u)] = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Φ is a convex function of $\xi \in \mathbb{R}^N$, rather than a function of the norm $|\xi|$. The problem is formulated appropriately in an anisotropic Orlicz–Sobolev space associated with Φ. We study the existence of solutions and some other properties of the above problem and its corresponding variational inequality in such space.

1. Introduction

We are concerned in this paper with the following boundary value problem:

$$(1.1) \qquad \begin{cases} -\operatorname{div}[\nabla\Phi(\nabla u)] = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded region in $\mathbb{R}^N$ ($N \geq 1$) with sufficiently smooth boundary $\partial\Omega$, the function $\Phi\colon \mathbb{R}^N \to \mathbb{R}$ defines the principal term, and $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is the lower order term.

If $\Phi(\xi) = |\xi|^2/2$ ($|\cdot|$ is the Euclidean norm in $\mathbb{R}^N$) then $\mathrm{div}(\nabla\Phi(\nabla u)) = \mathrm{div}(\nabla u) = \Delta u$ is the classical Laplacian. The case $\Phi(\xi) = |\xi|^p/p$ ($p > 1$) corresponds to equations with the p-Laplacian. In the more general case where $\Phi(\nabla u) = \phi(|\nabla u|)$ where ϕ is a convex function from $[0,\infty)$ into itself, we have problems with ϕ-Laplacian. In the above cases, $\Phi(\xi)$ does not depend on ξ directly but instead on its norm $|\xi|$ and thus the growth of Φ is the same in all directions (isotropic). In this paper we study equation (1.1) and a more general variational inequality in the anisotropic case, that is, when Φ has different growths in different directions in $\mathbb{R}^N$. This means that Φ is a function in all components $\xi_1, \ldots, \xi_N$ of ξ, not only in $|\xi|$. A simple example of such function is

$$\Phi(\xi) = \sum_{j=1}^{N} |\xi_j|^{p_j}, \quad \text{for all } \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N \text{ with } p_1, \ldots, p_n \in (1,\infty),$$

or more generally

$$(1.2) \qquad\qquad \Phi(\xi) = \sum_{j=1}^{N} A_j(|\xi_j|), \quad \text{for all } \xi \in \mathbb{R}^N,$$

where $A_1, \ldots, A_N$ are possibly different N-functions.

Several interesting problems in Physics and Mechanics have anisotropic natures and have been formulated recently in anisotropic Orlicz spaces or cones, for example in [9] and [18]–[21] (problems in Fluid Mechanics) and in [44], [5] (problems in Elasto–Plasticity). Differential equations with anisotropic operators have been also studied recently in various situations, cf. e.g. [41], [4], [12] and the references therein. After finishing this work, we have learnt of the very recent works [17] and [22] in which similar equations with anisotropic growths were studied in general settings. It was studied in [17] the existence of solutions to the boundary value problem

$$\begin{cases} -\mathrm{div}(A(x, \nabla u)) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in generalized Orlicz–Musielak spaces generated by inhomogeneous and anisotropic N-functions without Δ_2 or ∇_2 conditions. In [22], the authors proved the existence of renormalized and weak solutions of the inclusion

$$\begin{cases} \beta(x, u) - \mathrm{div}(a(x, \nabla u) + F(u)) \ni f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $f \in L^1(\Omega)$ and β being maximal monotone with respect to u in the framework of generalized Orlicz–Musielak spaces. Compared to these interesting works, our approaches, goals, and results here are quite different. More detailed discussions are presented in Remark 3.12.

In this paper, we propose some steps for a functional analytic study of second order partial differential equations and inclusions with general principal parts having anisotropic growths, that is, different growths in different directions of the gradient and not necessarily given in the form (1.2). We are, in particular, interested in what extend the standard tools in classical cases such as Sobolev spaces or regular Orlicz–Sobolev spaces can be applied in this more general and complicated case and what adaptation and generalization are needed for such extension. We present here basic definitions and properties of anisotropic Orlicz and Orlicz–Sobolev spaces associated with G-functions and consider weak formulations of equation (1.1) also in the more general case where Φ is nonsmooth. As will be seen later, in this case an appropriate formulation of (1.1) is an inclusion or a variational inequality. We study the existence of solutions and properties of the resolvent mapping of (1.1) and its associated variational inequality by extending the classical monotonicity and topological/fixed point methods to our present problems.

The paper is organized as follows. In Section 2, we present the definitions of G-functions following [47] and of anisotropic Orlicz and Orlicz–Sobolev spaces associated with G-functions, together with some of their basic properties. Section 3 is about weak formulations of (1.1) in anisotropic Orlicz–Sobolev spaces. Abstract existence theorems and properties of resolvent mappings of (1.1) and a more general variational inequality are next established in this general setting. Section 4 is devoted to fixed point formulations of our problems. We show that under certain appropriate growth condition on the lower order term, extending the classical sub-critical growth condition, the problem is equivalent to a fixed point inclusion which has suitable compactness property such that a multivalued Leray–Schauder topological degree for the involved operators is well defined. An illustrative example of the abstract existence theorem in Section 3 when the lower order term has a more specific growth is presented in Section 5.

2. Preliminaries

Assume Φ is a G-function in the sense of [47], that is $\Phi\colon \mathbb{R}^N \to [0,\infty]$ such that:

(G1) $\Phi(0) = 0$,

(G2) $\lim\limits_{|x|\to\infty} \Phi(x) = \infty$,

(G3) Φ is convex on $\mathbb{R}^N$,

(G4) Φ is symmetric, i.e. $\Phi(-x) = \Phi(x)$, for all $x \in \mathbb{R}^N$,

(G5) The set $\Phi^{-1} = \{x \in \mathbb{R}^N : \Phi(x) = \infty\}$ is separated from 0,

(G6) Φ is lower semicontinuous.

In what follows, we need stronger conditions than (G2) and (G5). In fact, instead of (G2) and (G5), we assume that

(G2') $\displaystyle\lim_{|x|\to\infty} \frac{\Phi(x)}{|x|} = \infty$, and

(G5') $\Phi^{-1}(\infty) = \emptyset$, i.e. $\Phi(x) \neq \infty$ for all $x \in \mathbb{R}^N$.

In short, we assume that Φ is a symmetric, convex, lower semicontinuous function from $\mathbb{R}^N$ to $[0,\infty)$ that satisfies (G1) and (G2') (cf. Hypothesis 2.1, [9], Definitions 2.1–2.2, [46], and [47]). As a consequence of (G5') and (G6), Φ is locally Lipschitz and therefore continuous on $\mathbb{R}^N$. Let Φ^* be the (Fenchel) conjugate of Φ,

$$(2.1) \qquad \Phi^*(y) = \sup_{x \in \mathbb{R}^N} [x \cdot y - \Phi(x)], \quad y \in \mathbb{R}^N.$$

Our assumptions on Φ imply that Φ^* also satisfies (G1), (G2'), (G3), (G4), (G5'), (G6). It follows from the definition of Φ^* that $x \cdot y \leq \Phi(x) + \Phi^*(y)$, for all $x, y \in \mathbb{R}^N$ and for each $x \in \mathbb{R}^N$, there exists $y \in \mathbb{R}^N$ such that

$$(2.2) \qquad x \cdot y = \Phi(x) + \Phi^*(y).$$

Furthermore (cf. e.g. [3]),

$$(2.3) \qquad \Phi^{**} = \Phi.$$

We have definitions about some behaviors of G-functions.

DEFINITION 2.1. (a) A G-function G is said to satisfy a Δ_2 condition (at ∞) if there exist $L, M \geq 0$ such that

$$(2.4) \qquad \Phi(2x) \leq L\Phi(x), \quad \text{for all } x \in \mathbb{R}^N \text{ with } |x| \geq M.$$

If $M = 0$ in (2.4) then we say that Φ satisfies a Δ_2 condition globally.

(b) Given G-functions $G_1, G_2 \colon \mathbb{R}^N \to [0,\infty]$. We say that $G_1 \prec G_2$ if there exist $L, M \geq 0$ such that

$$(2.5) \qquad G_1(x) \leq G_2(Lx), \quad \text{for all } x \in \mathbb{R}^N \text{ with } |x| \geq M.$$

If $G_1 \prec G_2$ and $G_2 \prec G_1$ then we say that G_1 and G_2 are equivalent and denote $G_1 \sim G_2$.

We define now the anisotropic Orlicz space $L_G(\Omega)$ and first order Sobolev–Orlicz space $W^1 L_G(\Omega)$. Let $L^0(\Omega)$ be the set of all (equivalent classes of) measurable functions from Ω to $\mathbb{R}$.

DEFINITION 2.2. (a) Let $G \colon \mathbb{R}^N \to [0,\infty]$ be a G-function. The anisotropic Orlicz space associated with G is defined as:

$$L_G(\Omega) = \left\{ u \in [L^0(\Omega)]^N : \text{exists } \lambda > 0 \text{ such that } \int_\Omega G(\lambda u)\, dx < \infty \right\}.$$

(b) Let $G \colon \mathbb{R}^{N+1} \to [0,\infty]$ be a G-function. The first order anisotropic Orlicz–Sobolev space $W^1 L_G(\Omega)$ associated with G is defined as the set of all weakly differentiable functions $u \colon \Omega \to \mathbb{R}$ such that $(u, \partial_1 u, \ldots, \partial_N u) \in L_G(\Omega)$.

$L_G(\Omega)$, equipped with the Luxemburg norm

$$(2.6) \qquad \|u\|_{L_G(\Omega)} = \|u\|_G = \inf\left\{\lambda > 0 : \int_\Omega G\left(\frac{u}{\lambda}\right) dx \leq 1\right\} \quad (u \in L_G(\Omega)),$$

is a Banach space. In Definition 2.2(b), $W^1 L_G$ with the norm

$$(2.7) \qquad \|u\|_{W^1 L_G(\Omega)} = \|u\|_{1,G} = \|(u, \partial_1 u, \ldots, \partial_N u)\|_G \quad (u \in W^1 L_G),$$

can be identified with a closed subspace of $L_G(\Omega)$ and is therefore a Banach space itself. It is clear that $C_0^1(\Omega) \subset W^1 L_G(\Omega)$. We have next the following definitions.

DEFINITION 2.3. (a) The space $W_0^1 L_G(\Omega)$ is the closure of $C_0^1(\Omega)$ in $W^1 L_G$ with respect to the norm $\|\cdot\|_{1,G}$.

(b) ([47]) Let $G\colon \mathbb{R}^N \to [0,\infty]$ be a G-function. The space $H^0(G,\Omega)$ is defined as the completion of the space $C_0^1(\Omega)$ with respect to the norm

$$\|\cdot\|_{H^0(G,\Omega)} = \|\nabla(\,\cdot\,)\|_G\colon u \mapsto \|\nabla u\|_G, \quad \text{for all } u \in C_0^1(\Omega).$$

First, we have the following properties of anisotropic Orlicz spaces and norms, whose proofs are straightforward and thus omitted.

PROPOSITION 2.4. *Let G_1, G_2 be G-functions.*

(a) *If $G_1 \prec G_2$ then $L_{G_2}(\Omega) \subset L_{G_1}(\Omega)$ with continuous embedding.*

(b) *If $G_1 \sim G_2$ then $L_{G_2}(\Omega) = L_{G_1}(\Omega)$ and the norms $\|\cdot\|_{G_1}$ and $\|\cdot\|_{G_2}$ are equivalent.*

(c) *If $G_1 \prec G_2$ and G_1 satisfies condition (G5') then $G_2^* \prec G_1^*$.*

PROPOSITION 2.5. *Assume $G_i\colon \mathbb{R}^{N_i} \to [0,\infty]$ is a G-function ($i = 1, 2$). Then $G\colon \mathbb{R}^N \equiv \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \to [0,\infty]$, $G(x) = G(x_1, x_2) = G_1(x_1) + G_2(x_2)$, for all $x = (x_1, x_2) \in \mathbb{R}^N$ ($N = N_1 + N_2$, $x_i \in \mathbb{R}^{N_i}$, $i = 1, 2$), is a G-function on $\mathbb{R}^N$ and*

(a) $L_G(\Omega) = L_{G_1}(\Omega) \times L_{G_2}(\Omega).$

(b) *For $u = (u_1, u_2) \in L_G(\Omega)$ with $u_i \in L_{G_i}(\Omega)$, $i = 1, 2$, the norms $u \mapsto \|u\|_G$ and $u \mapsto \|u_1\|_{G_1} + \|u_2\|_{G_2}$ are equivalent.*

Next, let us consider some relations between the spaces introduced in Definition 2.3. Assume $G\colon \mathbb{R}^N \to [0,\infty]$ is a G-function. Define $G_1\colon \mathbb{R}^{N+1} \to [0,\infty]$ by

$$(2.8) \qquad G_1(x_0, x) = |x_0| + G(x), \quad \text{for all } x_0 \in \mathbb{R}, \ x \in \mathbb{R}^N.$$

Then G_1 is a G-function with

$$(2.9) \qquad H^0(G,\Omega) = W_0^1 L_{G_1}(\Omega),$$

and moreover, the norm $u \mapsto \|u\|_{W^1 L_{G_1}(\Omega)}$ is equivalent to the norm

$$(2.10) \qquad u \mapsto \|\nabla u\|_{L_G(\Omega)},$$

in this space. In fact, straightforward calculations show that G_1 is a G-function. To verify (2.9), let us assume $u \in H^0(G, \Omega)$. Then there exists a sequence $\{u_n\} \subset C_0^1(\Omega)$ such that $\|u_n - u\|_{H^0(G,\Omega)} = \|\nabla u_n - \nabla u\|_{L_G(\Omega)} \to 0$. We know ([47]) that $H^0(G, \Omega) \hookrightarrow W_0^{1,1}(\Omega)$ (with continuous embedding) and $\|v\|_{L^1(\Omega)} \leq C\|v\|_{H^0(G,\Omega)}$, for all $v \in H^0(G, \Omega)$ for some constant $C > 0$. Hence for some $C_1, C_2 > 0$ independent of u, u_n,

$$\|u_n - u\|_{W^1 L_{G_1}(\Omega)} \leq C_1 (\|u_n - u\|_{L^1(\Omega)} + \|\nabla u_n - \nabla u\|_{L_G(\Omega)}) \leq C_2 \|\nabla u_n - \nabla u\|_{L_G(\Omega)}.$$

This shows that $u_n \to u$ with respect to the norm $\|\cdot\|_{W^1 L_{G_1}(\Omega)}$, i.e. $u \in W_0^1 L_{G_1}(\Omega)$. Conversely, assume $u \in W_0^1 L_{G_1}(\Omega)$, i.e. there exists $\{u_n\} \subset C_0^1(\Omega)$ such that $\|u_n - u\|_{W^1 L_{G_1}(\Omega)} \to 0$.

Because of the equivalence between $\|\cdot\|_{W^1 L_{G_1}(\Omega)}$ and the norm in (2.10) (see Proposition 2.5), we see that $\|u_n - u\|_{H^0(G,\Omega)} = \|\nabla u_n - \nabla u\|_{L_G(\Omega)} \to 0$, i.e. $u \in H^0(G, \Omega)$, which completes the proof of (2.9). These arguments also prove the equivalence between the norms $\|\cdot\|_{W^1 L_{G_1}(\Omega)}$ and $\|\nabla(\cdot)\|_{L_G(\Omega)}$.

In what follows, we concentrate on G-functions $G : \mathbb{R}^{N+1} \to [0, \infty]$ with

$$G(u_0, u) = \Phi_0(u_0) + \Phi(u), \quad u_0 \in \mathbb{R}, \ u \in \mathbb{R}^N,$$

where Φ_0 is a Young function from $\mathbb{R}$ to $[0, \infty]$ and Φ a G-function from $\mathbb{R}^N$ to $[0, \infty]$. As in [47], we denote $\Phi_+^*(u) = \sup\{u \cdot v - \Phi(v) : v \in \mathbb{R}_+^N\}$ where $\mathbb{R}_+^N = \{v \in \mathbb{R}^N : v_i \geq 0, \text{ for all } i \in \{1, \ldots, N\}\}$. Let A be a Young function that satisfies Theorem 1 in [47] for $H^0(G, \Omega)$, i.e. there are continuous, nondecreasing functions $f_1, \ldots, f_N$ from $[0, \infty)$ into itself such that

$$(2.11) \qquad G_+^*(0, f_1(s), \ldots, f_N(s))(= \Phi_+^*(f_1(s), \ldots, f_N(s))) \leq s,$$

for all $s \in [0, \infty)$, and for some $k > 0$,

$$(2.12) \qquad \frac{1}{k} \int_0^t \frac{ds}{s \left(\prod\limits_{i=1}^N f_i(s) \right)^{1/n}} \leq A^{-1}(t), \quad \text{for all } t \geq 0.$$

According to Theorem 1 and the remark in pages 29–30 of [47], the embedding

$$(2.13) \qquad W_0^1 L_G(\Omega) \hookrightarrow L_A(\Omega)$$

is continuous with

$$(2.14) \qquad \|u\|_{L_A(\Omega)} \leq C \| |\nabla u| \|_{L_\Phi(\Omega)}, \quad \text{for all } u \in W_0^1 L_G(\Omega).$$

From (2.13) we see that if

$$(2.15) \qquad \Phi_0 \prec A,$$

then the norms $\|\cdot\|_{W^1 L_G}$ and $\|\nabla(\cdot)\|_{L_\Phi}$ are equivalent on $W_0^1 L_G$. In fact, we have from Theorem 1 in [47],

$$\|u\|_{L_{\Phi_0}} \leq C_1 \|u\|_{L_A} \leq C_2 \|\nabla u\|_{L_\Phi},$$

for all $u \in H^0(\Phi, G) \supset W_0^1 L_\Phi$. Therefore, the norms $\| \cdot \|_{L_{\Phi_0}} + \|\nabla(\cdot)\|_{L_\Phi}$ and $\|\nabla(\cdot)\|_{L_\Phi}$ are equivalent on $W_0^1 L_\Phi$. From Proposition 2.5, the norms $\| \cdot \|_{L_{\Phi_0}} + \|\nabla(\cdot)\|_{L_\Phi}$ and $\| \cdot \|_{W^1 L_G}$ are equivalent on $W^1 L_G$. Hence the norms $\| \cdot \|_{W^1 L_G}$ and $\|\nabla(\cdot)\|_{L_\Phi}$ are equivalent. These arguments also imply that if (2.15) holds then $H^0(\Phi, \Omega) = W_0^1 L_G$.

In the particular case where Φ is given by (1.2), it was proved in [47] that the function A such that (2.12) and thus (2.13) hold can be chosen as

$$A^{-1}(t) = \int_0^t \frac{ds}{\overline{m}(s)} \quad (t \geq 0),$$

where

$$\overline{m}(s) = s^{1+1/N} \left[\prod_{j=1}^N A_j^{-1}\left(\frac{s}{N}\right) \right]^{-1/N}.$$

REMARK 2.6. Instead of using the classical embedding theorem in [47], we can use alternatively a sharp embedding theorem due to Cianchi ([7], see also [6], [8]). If Φ is as above then A can be constructed as follows. Let $\Phi_\star \colon [0, \infty] \to [0, \infty]$ be defined by

$$\Phi_\star(s) = \sup\{t : |\{\xi \in \mathbb{R}^N : \Phi(\xi) \leq t\}| < C_n s^n\} \quad \text{for } s \geq 0,$$

where $C_n = \pi^{n/2} \Gamma(1+n/2)$ is the measure of the n-dimensional unit ball. Assume

$$\int_0^a \left(\frac{t}{\Phi_\star(t)}\right)^{1/(N-1)} dt < \infty$$

for some $a > 0$ and define $H \colon [0, \infty) \to [0, \infty)$ by

$$H(r) = \left(\int_0^r \left(\frac{t}{\Phi_\star(t)}\right)^{1/(N-1)} dt \right)^{(N-1)/N} \quad \text{for } r \geq 0.$$

Then for the Young function A given by

(2.16) $$A = \Phi_\star \circ H^{-1},$$

where H^{-1} is the left-continuous inverse of H, the embedding (2.13) holds (when $\partial\Omega$ is sufficiently smooth). More importantly, the embedding (2.13) is sharp in the class of Young functions with A given by (2.16).

Note that our results in the sequel still hold if A is given by (2.16) instead by (2.11)–(2.12).

Let us conclude this section with a coercivity result for integrals in anisotropic Orlicz spaces that will be used later. This result is obvious for Lebesgue spaces and was proved in [16] for Orlicz spaces.

PROPOSITION 2.7. *Assume G is a G-function that satisfies* (G2') *and* (G5').
If G^* satisfies a Δ_2 condition (*at infinity*) *then $\int_\Omega G(u)\,dx$ is coercive on $L_G(\Omega)$
in the following sense*:

$$(2.17) \qquad \lim_{\|u\|_G \to \infty} \frac{1}{\|u\|_G} \int_\Omega G(u)\,dx = \infty.$$

To prove this proposition, we need a simple result on G-functions with
Δ_2 conditions, a one-dimensional version of which (for N-functions) was given
in [26].

PROPOSITION 2.8. *Assume G is a G-function satisfying condition* (G5') *and
a Δ_2 condition at ∞. Then there exists a G-function H satisfying* (G5') *and
a Δ_2 condition globally such that $G \sim H$.*

PROOF. Assume (2.4) holds with $M > 0$. By the continuity of G, $K :=
\sup\{G(x) : |x| \le M\}+1 \in [1, \infty)$. Let $B = G^{-1}([0, K])$. Since G is a continuous
G-function, B is a closed, bounded, convex, symmetric (about 0) subset of $\mathbb{R}^N$.
Since $K > 0$, the continuity of G implies that $0 \in B^\circ$. It also follows from the
convexity of G that

$$(2.18) \qquad G(x) = K \Leftrightarrow x \in \partial B.$$

Next, we prove that

(2.19)   For each $x \in B \setminus \{0\}$, there exists a unique $t = t_x \ge 1$

$$\text{such that } t_x x \in \partial B.$$

In fact, consider the function $G^x \colon \mathbb{R} \to [0, \infty)$, $G^x(t) = G(tx)$. Since $G^x(0) = 0$
and $G^x(tx) \to \infty$ as $t \to \infty$, there must exists $t_0 \in (0, \infty)$ such that $G^x(t_0) = K$,
i.e. $t_0 x \in \partial B$ by (2.18). Moreover, from the convexity of G implies that G^x is
nondecreasing on $[0, \infty)$, i.e. if $0 \le t_1 < t_2$ then $G^x(t_1) \le G^x(t_2)$. Moreover, if
$G^x(t_2) > 0$ then $G^x(t_1) < G^x(t_2)$.

This property implies that $t_0 \ge 1$. In fact, if $t_0 < 1$ then $0 < K = G^x(t_0) \le
G^x(1)$ and thus $K = K^x(t_0) < G^x(1) = G(x) = K$.

To see the uniqueness of t_0, assume there are t_0, t_1 such that $1 \le t_0 < t_1$
and $G^x(t_0) = G^x(t_1) = K$. This is impossible since $G^x(t_1) > 0$ and thus
$K = G^x(t_0) < G^x(t_1) = K$. This contradiction proves the uniqueness of t_0 and
thus (2.19).

Let us define $G_1 \colon B \to \mathbb{R}$ as follows. $G_1(0) = 0$ and for $x \in B \setminus \{0\}$,
$G_1(x) = K/t_x$ where t_x is the unique number in $[1, \infty)$ such that $t_x x \in \partial B$. It
is clear that G_1 is symmetric and $G_1(x) = K$ for all $x \in \partial B$. Moreover,

$$(2.20) \qquad\qquad G_1 \text{ is convex on } B.$$

In fact, let $x, y \in B$, $x = tx_0$, $y = sy_0$ with $x_0, y_0 \in \partial B$ and $s, t \in [0, 1]$. We have
by definition, $G_1(x) = tK$, $G_1(y) = sK$. Let $z = \lambda x + (1 - \lambda)y$ with $\lambda \in (0, 1)$.

We need to prove that

(2.21) $$G_1(z) \leq \lambda G_1(x) + (1 - \lambda)G_1(y).$$

Without loss of generality, we can assume that either $t > 0$ or $s > 0$. Consider $\sigma = [\lambda t + (1 - \lambda)s]^{-1}$. We have $\sigma \geq 1$ and $\sigma z = \sigma \lambda t x_0 + \sigma(1 - \lambda)s y_0$ with $\sigma \lambda t + \sigma(1 - \lambda)s = 1$ and $\sigma \lambda t, \sigma(1 - \lambda)s \geq 0$. Since B is convex, we have $\sigma z \in B$, i.e. $G_1(\sigma z) \leq K$. Note that $G_1(\sigma z) = \sigma G_1(z)$ and thus

$$G_1(z) \leq \frac{K}{\sigma} = \lambda t K + (1 - \lambda)s K = \lambda G_1(x) + (1 - \lambda)G_1(y),$$

proving (2.21) and therefore (2.20). It follows that

(2.22) $$\mathrm{epi}(G_1) = \{X = (x, x^*) \in B \times \mathbb{R} : G_1(x) \leq x^*\} \text{ is convex}.$$

Let us define $H \colon \mathbb{R}^N \to [0, \infty)$ by

(2.23) $$H(x) = \begin{cases} G_1(x) & \text{if } x \in B, \\ G(x) & \text{if } x \notin B. \end{cases}$$

Straightforward arguments show that H is continuous in $\mathbb{R}^N$. Let us prove that H is convex in $\mathbb{R}^N$, i.e. to prove that $\mathrm{epi}\, H = \{(x, x^*) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N \text{ and } x \geq H(x)\}$ is a convex set. First, note that

(2.24) $$\mathrm{epi}\, H \subset \mathrm{epi}\, G.$$

In fact, let $(x, x^*) \in \mathrm{epi}\, H$. If $x \notin B$ then $G(x) = H(x)$ and $(x, x^*) \in \mathrm{epi}\, G$. If $x \in B$ with $x = tx_0$, $x_0 \in \partial B$ and $t \in [0, 1]$, we have $(x, G_1(x)) = (x, tK) = t(x_0, G(x_0))$. Since $\mathrm{epi}\, G$ is convex and contains $(0, 0)$, we have $(x, G_1(x)) \in \mathrm{epi}\, G$, i.e. $G_1(x) \geq G(x)$. It follows that $x^* \geq H(x) = G_1(x) \geq G(x)$ and thus $(x, x^*) \in \mathrm{epi}\, G$, proving (2.24). Let $X = (x, x^*), Y = (y, y^*) \in \mathrm{epi}\, H$. We prove that

(2.25) $$[X, Y] := \{\lambda X + (1 - \lambda)Y : \lambda \in [0, 1]\} \subset \mathrm{epi}\, H.$$

We first note the following equivalences, whose proofs are straightforward and thus omitted. For $Z = (z, z^*) \in \mathbb{R}^{N+1}$ ($z \in \mathbb{R}^N$ and $z^* \in \mathbb{R}$), we have

(2.26) $$[Z \in \mathrm{epi}\, H \text{ and } z^* \geq K] \Leftrightarrow [Z \in \mathrm{epi}\, G \text{ and } z^* \geq K],$$

and

(2.27) $$[Z \in \mathrm{epi}\, H \text{ and } z^* \leq K] \Leftrightarrow [Z \in \mathrm{epi}\, G_1 \text{ and } z^* \leq K].$$

To prove (2.25), we consider 3 cases.

(i) Both $x^*, y^* \geq K$. It follows from (2.26) that $X, Y \in \mathrm{epi}\, G$. Since $\mathrm{epi}\, G$ is convex, $[X, Y] \subset \mathrm{epi}\, G$. For any $Z = (z, z^*) \in [X, Y]$, since z^* is a convex combination of x^* and y^*, we have $z^* \geq K$. Thus from (2.26), $Z = (z, z^*) \in \mathrm{epi}\, H$ and (2.25) is proved in this case.

(ii) Both $x^*, y^* \leq K$. For this case, the equivalence in (2.27) implies that $X, Y \in \operatorname{epi} G_1$. Using again the convexity of $\operatorname{epi} G_1$ (cf. (2.22)), we have $[X, Y] \subset \operatorname{epi} G_1$, and thus for any $Z = (z, z^*) \in [X, Y]$, we have $z^* \leq K$. Since $Z \in \operatorname{epi} G_1$, (2.27) implies that $Z \in \operatorname{epi} H$.

(iii) $\min\{x^*, y^*\} \leq K < \max\{x^*, y^*\}$. Assume for example $x^* \leq K < y^*$. Then there exists a unique $Z = (z, z^*) \in [X, Y]$ such that $z^* = K$. It is clear that

$$(2.28) \qquad\qquad [X, Y] = [X, Z] \cup [Z, Y].$$

From (2.24), we have $[X, Y] \subset \operatorname{epi} G$ and thus $Z \in \operatorname{epi} G$. As a consequence, $K = z^* \geq G(z)$ and thus $z \in B$. It follows from the definition of H and G_1 that $H(z) = G_1(z) = tK$ where $t \in [0, 1]$ is such that $z = tz_0$, $z_0 \in \partial B$. We thus have $H(z) = tK \leq K = z^*$ and thus $Z = (z, z^*) = (z, K) \in \operatorname{epi} H$. Since $X, Z \in \operatorname{epi} H$ with $x^*, z^* \leq K$, we have

$$(2.29) \qquad\qquad [X, Z] \subset \operatorname{epi} H,$$

from the proof in case (ii) with Z instead of Y. Similarly, since $Z, Y \in \operatorname{epi} H$ with $z^*, y^* \geq K$, we have

$$(2.30) \qquad\qquad [Z, Y] \subset \operatorname{epi} H,$$

from case (i) above with Z instead of X. From (2.28)–(2.30), we have (2.25) also in case (iii) and thus in all cases. We obtain the convexity of H in $\mathbb{R}^N$.

Since B is bounded one can choose $\rho \in (0, \infty)$ such that

$$(2.31) \qquad\qquad B \subset B_\rho(0).$$

Because $H(x) = G(x)$ for all $x \in \mathbb{R}^N$, $|x| \geq \rho$, we immediately see that $H \sim G$.

Let us prove now that H satisfies a global Δ_2 condition. First, note from its definition that $H(x) > 0$ for all $x \in \mathbb{R}^N \setminus \{0\}$. Furthermore, from the definitions of B and K, we have that $\overline{B_M(0)} \subset B$. Let $K_1 = \max\{H(x) : |x| \leq 2\rho\} \in (0, \infty)$ where ρ satisfies (2.31) and $m_1 = \min\{H(x) : M/2 \leq |x| \leq \rho\} \in (0, \infty)$.

For any $x \in \mathbb{R}^N$, we have the following possibilities: $|x| \leq M/2$, $M/2 < |x| < \rho$, and $|x| \geq \rho$.

If $|x| \leq M/2$ then $x, 2x \in B$ and $H(2x) = G_1(2x) = tK$ where $2x = tx_0$, $x_0 \in \partial B$. Since $x = tx_0/2$, we have $H(x) = G_1(x) = tK/2$. Thus $H(2x) = 2H(x)$ if $|x| \leq M/2$.

In the case where $M/2 < |x| < \rho$, we have $H(2x) \leq K_1$ and $H(x) \geq m_1$ and thus $H(2x) \leq (K_1/m_1)H(x)$.

If $|x| \geq \rho$ then $x, 2x \notin B$. In particular, $|x| \geq M$ and from (2.4), we have $H(2x) = G(2x) \leq LG(x) = LH(x)$. Combining the above cases, we get

$$H(2x) \leq \max\{2, K_1/m_1, L\}H(x), \quad \text{for all } x \in \mathbb{R}^N. \qquad\qquad \square$$

PROOF OF PROPOSITION 2.7. First, assume that G^* satisfies a global Δ_2 condition, that is,

$$(2.32) \qquad G^*(2x) \leq LG^*(x), \quad \text{for all } x \in \mathbb{R}^N,$$

for some $L > 2$. By mathematical induction, we have

$$(2.33) \qquad G^*(2^k x) \leq L^k G^*(x), \quad \text{for all } x \in \mathbb{R}^N \text{ and all } k \in \mathbb{N}.$$

As in the proof of Lemma 3.14, [16], we define the function $f \colon [1, \infty) \to [L, \infty)$ by

$$(2.34) \qquad f(r) = r[(1-r)L^{k+1} + \lambda L^{k+2}],$$

if $r = (1-\lambda)2^k + \lambda 2^{k+1}$, with $\lambda \in [0,1]$, $k \in \mathbb{N} \cup \{0\}$, i.e. $f(2^k) = 2^k L^{k+1}$ and $f(r)/r$ is affine in each interval $[2^k, 2^{k+1}]$ for $k \in \mathbb{N} \cup \{0\}$. From (2.33) and the convexity of G^*, we see that

$$(2.35) \qquad G^*(rx) \leq f(r) \, G^*(x), \quad \text{for all } x \in \mathbb{R}^N \text{ and all } r \in [1, \infty).$$

Let $r \geq 1$. From (2.35) and Young's inequality, for each $x \in \mathbb{R}^N$, there exists $y \in \mathbb{R}^N$ such that

$$\begin{aligned} G(x) + G^*(y) &= \frac{1}{f(r)} \left(\frac{f(r)}{r} x \right) \cdot (ry) \\ &\leq \frac{1}{f(r)} \left[G\left(\frac{f(r)}{r} x \right) + G^*(ry) \right] \leq \frac{1}{f(r)} G\left(\frac{f(r)}{r} x \right) + G^*(y). \end{aligned}$$

This proves that

$$(2.36) \qquad G\left(\frac{f(r)}{r} x \right) \geq f(r) \, G(x), \quad \text{for all } x \in \mathbb{R}^N \text{ and all } r \geq 1.$$

On the other hand, it follows from the definition of f that $r \mapsto f(r)/r$ is strictly increasing function from $[1, \infty)$ onto $[L, \infty)$. Let $g \colon [L, \infty) \to [1, \infty)$ be its inverse. We obtain from (2.36) that

$$G(g^{-1}(r)x) \geq f(r) \, G(x), \quad \text{for all } x \in \mathbb{R}^N \text{ and all } r \geq 1.$$

For $s \geq L$, let $s = g^{-1}(r) = f(r)/r$. We have $f(r) = sg(s)$ and the above inequality becomes

$$(2.37) \qquad G(sx) \geq sg(s) \, G(x), \quad \text{for all } x \in \mathbb{R}^N \text{ and all } s \geq L.$$

Let $u \in L_G(\Omega)$ with $\|u\|_G > L + 1$. For all $\varepsilon \in (0,1)$, we have $\|u\|_G - \varepsilon > 1$ and

$$\int_\Omega G\left(\frac{u}{\|u\|_G - \varepsilon} \right) dx > 1.$$

For almost every $x \in \Omega$, it follows from (2.37) that

$$G(u(x)) \geq (\|u\|_G - \varepsilon) \, g(\|u\|_G - \varepsilon) \, G\left(\frac{u(x)}{\|u\|_G - \varepsilon} \right),$$

and thus, for all $\varepsilon \in (0,1)$,

$$\int_\Omega G(u)\,dx \geq (\|u\|_G - \varepsilon)g(\|u\|_G - \varepsilon)\int_\Omega G\left(\frac{u}{\|u\|_G - \varepsilon}\right)dx$$
$$\geq (\|u\|_G - \varepsilon)g(\|u\|_G - \varepsilon).$$

Hence,

$$\int_\Omega G(u)\,dx \geq \|u\|_G g(\|u\|_G) \quad \text{for all } u \in L_G(\Omega) \text{ with } \|u\|_G > L+1.$$

Since $g(s) \to \infty$ as $s \to \infty$, we obtain (2.17).

Assume now that G^* only satisfies a Δ_2 condition at ∞. Since G satisfies (G2') and $G = (G^*)^*$, we see that G^* satisfies condition (G5') (cf. e.g. Theorems 2.35 and 11.8, [45]) and is thus continuous on $\mathbb{R}^N$. From Proposition 2.8, there exists a G-function $H\colon \mathbb{R}^N \to [0,\infty)$ such that H satisfies a global Δ_2 condition and $G^* \sim H$. Since both G^* and H satisfy condition (G5'), it follows from Proposition 2.4 that $G = (G^*)^* \sim H^* := G_1$. As H satisfies condition (G5'), its conjugate G_1 satisfies the coercivity condition (G2'). Also, $G_1^* = H$ satisfies a global Δ_2 condition. By the above proof, we have

$$(2.38) \qquad \lim_{u \in L_{G_1}(\Omega),\, \|u\|_{G_1} \to \infty} \frac{1}{\|u\|_{G_1}}\int_\Omega G_1(u)dx = \infty.$$

Since $\|\cdot\|_G$ and $\|\cdot\|_{G_1}$ are equivalent on $L_{G_1}(\Omega) = L_G(\Omega)$, we have $k_1\|u\|_G \leq \|u\|_{G_1} \leq k_2\|u\|_G$, for all $u \in L_G(\Omega)$ for some $k_1, k_2 > 0$. Moreover, since $G_1 \sim G$, there are $L_1, M_1 > 0$ such that

$$(2.39) \qquad G_1(L_1 t) \leq G(t), \quad \text{for all } t \in \mathbb{R}^N, \text{ with } |t| \geq M_1.$$

Since G satisfies (G5'), G^* and thus H satisfy (G2'). As a consequence, G_1 also satisfies (G5'). Hence, from (2.39), there is $C_1 > 0$ such that $G(t) \geq G_1(L_1 t) - C_1$, for all $t \in \mathbb{R}^N$. We have, for $u \in L_G(\Omega)$,

$$\frac{1}{\|u\|_G}\int_\Omega G(u)\,dx \geq \frac{1}{k_2\|u\|_{G_1}}\int_\Omega [G_1(L_1 u) - C_1]\,dx$$
$$\geq \frac{L_1}{k_2}\left[\frac{1}{\|L_1 u\|_{G_1}}\int_\Omega G_1(L_1 u)\,dx - C_1|\Omega|\right],$$

where $|\Omega|$ is the measure of Ω. Since $\|u\|_G$ is large if and only $\|L_1 u\|_{G_1}$ is, this estimate and (2.38) imply (2.17). $\qquad \square$

REMARK 2.9. Proposition 2.8 is an extension of Lemma 3.14 in [16] for N-functions to the vector case of G-functions. Its alternate proof given here also fixes a seeming gap in the proof of Lemma 3.14 in [16]. In fact, in the second part of that proof in [16] (we use the notation of [16]) when $\overline{M}$ is assumed to

have a Δ_2 property for large value of t, we have $M(st) \geq sg(s)M(t)$ only for $t \geq t_0$ and $s \geq K$. For $x \in \Omega_u$, we have $|u(x)| \geq t_0$ and thus

$$M(|u(x)|) = M\left[(\|u\|_{(M),\Omega_u} - \varepsilon)\frac{|u(x)|}{\|u\|_{(M),\Omega_u} - \varepsilon}\right].$$

In order to apply the above equality on $M(st)$, we need both $\|u\|_{(M),\Omega_u} - \varepsilon \geq k$ and $|u(x)|/(\|u\|_{(M),\Omega_u} - \varepsilon) \geq t_0$. The first inequality holds because $\|u\|_{(M),\Omega_u} > k$. However, the second inequality does not hold in general in Ω_u. The reason is that although $|u(x)| \geq t_0$ by the definition of Ω_u, $\|u\|_{(M),\Omega_u} - \varepsilon$ can be large, which makes the fraction $|u(x)|/(\|u\|_{(M),\Omega_u} - \varepsilon)$ less than t_0.

3. Equations and variational inequalities
with anisotropic principal parts

In this section, we consider precise settings for (1.1) and its weak formulations. We also prove general existence theorems and study some properties of the resolvent mapping. Assume $\Phi\colon \mathbb{R}^N \to [0,\infty)$ is a coercive G-function, i.e. Φ satisfies conditions (G1), (G2'), (G3), (G4), (G5'), (G6). As noted above, the conjugate Φ^* of Φ also satisfies these conditions. We also assume that both Φ and Φ^* satisfy Δ_2 conditions at infinity in the sense of Definition 2.1. Assume A is a Young function satisfying Trudinger's embedding theorem, i.e. (2.11)–(2.12) above. Therefore, we have (2.13) and (2.14). Let Φ_0 be an N-function (cf. [1]) such that

$$(3.1) \qquad\qquad\qquad\qquad \Phi_0 \ll A,$$

(in the sense of Young functions, we refer to [40], [26], [1], [10] for definitions and properties of Young functions and N-functions). We know that the conjugate Φ_0^* of Φ_0 is also an N-function. Furthermore, it follows that the G-function $G\colon \mathbb{R}^{N+1} \to [0,\infty]$,

$$(3.2) \qquad\qquad G(u_0, u) = \Phi_0(u_0) + \Phi(u) \quad (u_0 \in \mathbb{R},\ u \in \mathbb{R}^N),$$

also satisfies the conditions (G1), (G2'), (G3), (G4), (G5'), (G6). It is easy to prove that

$$(3.3) \qquad\quad G^*(v_0, v) = \Phi_0^*(v_0) + \Phi^*(v), \quad \text{for all } (v_0, v) \in \mathbb{R}^{N+1}.$$

For simplicity of calculations, we also assume here that

$$(3.4) \qquad\qquad\qquad \Phi_0 \text{ and } \Phi_0^* \text{ satisfies } \Delta_2 \text{ conditions at infinity.}$$

It follows from (3.2)–(3.3) that G and G^* also satisfy Δ_2 conditions at infinity. As a consequence (cf. e.g. [46] or [24]), L_G and thus $W^1L_G, W_0^1L_G$ are reflexive and moreover $(L_G)^* = L_{G^*}$ under the usual pairing given by integrals.

We study the boundary value problem (1.1) with the above setting. The weak formulation of (1.1) is

$$(3.5) \quad \begin{cases} \displaystyle\int_\Omega \nabla\Phi(\nabla u)\nabla v\,dx = \int_\Omega f(x,u)v\,dx, & \text{for all } v \in X, \\ u \in X, \end{cases}$$

where X is an appropriate function space. Because of the growth condition of the principal term and the boundary condition in (1.1), a natural choice of X is the anisotropic Orlicz–Sobolev space defined by Φ,

$$(3.6) \qquad\qquad X = H^0(\Omega, \Phi) = W_0^1 L_G,$$

where G is given by (3.1)–(3.2). The appropriateness of this choice of function space is reflected in the following property, which means that the integral in the left hand side of (3.5) is defined for all $u, v \in X$.

PROPOSITION 3.1. *Assume Φ is differentiable in $\mathbb{R}^N$. Then*

$$(3.7) \qquad\qquad \nabla\Phi(\nabla u)\nabla v \in L^1(\Omega), \quad \text{for all } u, v \in X.$$

PROOF. First, note that

$$(3.8) \qquad\qquad \Phi^*(\nabla\Phi(\xi)) \le \Phi(2\xi), \quad \text{for all } \xi \in \mathbb{R}^N.$$

In fact, since Φ is differentiable, we have $\partial\Phi(\xi) = \{\nabla\Phi(\xi)\}$, for all $\xi \in \mathbb{R}^N$. Also (cf. e.g. [3]) $\nabla\Phi(\xi)\xi = \Phi(\xi) + \Phi^*(\nabla\Phi(\xi))$ and thus $\Phi(\xi) \le \nabla\Phi(\xi)\xi \le \Phi(2\xi) - \Phi(\xi)$. Hence, $\Phi^*(\nabla\Phi(\xi)) = \nabla\Phi(\xi)\xi - \Phi(\xi) \le \Phi(2\xi) - 2\Phi(\xi) \le \Phi(2\xi)$. For almost every $x \in \Omega$, we have from Young's inequality,

$$(3.9) \qquad |\nabla\Phi(\nabla u(x))\nabla v(x)| \le \Phi(\nabla v(x)) + \Phi^*(\nabla\Phi(\nabla u(x)))$$
$$\le \Phi(\nabla v(x)) + \Phi(2\nabla u(x)).$$

Since Φ satisfies a Δ_2 condition at ∞, the integrals $\int_\Omega \Phi(2\nabla u)\,dx$ and $\int_\Omega \Phi(\nabla v)\,dx$ are finite, i.e. $\Phi(\nabla v), \Phi(2\nabla u) \in L^1(\Omega)$. (3.9) thus implies (3.7). $\qquad\square$

Let us consider the functional

$$J\colon X \to [0,\infty], \qquad J(u) = \int_\Omega \Phi(\nabla u)\,dx, \quad u \in X.$$

Since Φ is convex, so is J. Moreover, since Φ satisfies a Δ_2 condition, $D(J) = X$. A standard application of Fatou's lemma shows that J is lower semicontinuous on X. As a consequence, J is locally Lipschitz on X. However, if Φ is not differentiable on $\mathbb{R}^N$, J is not differentiable on X in general. On the other hand, if Φ has some smoothness on $\mathbb{R}^N$ then J inherits such property on X, which also shows the natural choice of X as a suitable function space for problem (1.1). We have the following result about the derivative of J on X. For a proper convex functional f from a normed vector space Z into $\mathbb{R} \cup \{\infty\}$, and for $u \in D(f)$,

$h \in Z$, we use the usual notation $Df(u)h$ for the directional derivative of f at u in the direction h.

PROPOSITION 3.2. (a) *For any $u, h \in X$, the directional derivative of J at u in the direction of h exists and is given by*

$$(3.10) \qquad DJ(u)h = \int_\Omega D\Phi(\nabla u(x))\nabla h(x)\, dx.$$

(b) *Assume Φ is Gâteaux differentiable on $\mathbb{R}^N$ and*

$$(3.11) \qquad \Phi(\xi) \neq 0, \quad \text{for all } \xi \in \mathbb{R}^N \setminus \{0\}.$$

Then J is Gâteaux differentiable on X and DJ is given by (3.10).

(c) *If Φ is of class C^1 on $\mathbb{R}^N$ and satisfies (3.11) then J is of class C^1 on X and*

$$\langle J'(u), h \rangle = \int_\Omega \nabla\Phi(\nabla u(x))\nabla h(x)\, dx.$$

PROOF. We present the proof of (a) in detail and skip those of (b) and (c) since they are similar. Since J is convex with $D(J) = X$, the directional derivative $DJ(u)h$,

$$DJ(u)h = \lim_{t \to 0^+} \frac{J(u + th) - J(u)}{t},$$

exists in $[-\infty, \infty]$ for all $u, h \in X$. Let us prove that $DJ(u)h$ is given by (3.10) in our case. For $t > 0$, we have

$$\frac{1}{t}[J(u + th) - J(u)] = \int_\Omega \frac{1}{t}[\Phi(\nabla u(x) + th(x)) - \Phi(\nabla u(x))]\, dx.$$

Since Φ is convex with $D(\Phi) = \mathbb{R}^N$ (cf. assumption (G5')), the limit

$$(3.12) \qquad \lim_{t \to 0^+} \frac{1}{t}[\Phi(\nabla u(x) + th(x)) - \Phi(\nabla u(x))] = D\Phi(\nabla u(x))(\nabla h(x))$$

exits in $[-\infty, \infty]$ for almost every $x \in \Omega$. The function $x \mapsto D\Phi(\nabla u(x))(\nabla h(x))$ is thus measurable on Ω. Moreover (cf. Proposition 4.1 in [3]),

$$(3.13) \qquad \Phi(\nabla u(x)) - \Phi(\nabla u(x) - \nabla h(x)) \leq D\Phi(\nabla u(x))(\nabla h(x))$$
$$\leq \Phi(\nabla u(x) + \nabla h(x)) - \Phi(\nabla u(x)),$$

for almost every $x \in \Omega$.

Since the function $t \mapsto [\Phi(\nabla u(x) + t\nabla h(x)) - \Phi(\nabla u(x))]/t$ is increasing on $(0, \infty)$, we have for $t \in (0, 1)$, $x \in \Omega$,

$$-\frac{1}{2}\Phi(2\nabla u(x)) - \frac{1}{2}\Phi(2\nabla h(x)) \leq \Phi(\nabla u(x)) - \Phi(\nabla u(x) - \nabla h(x))$$

$$\leq D\Phi(\nabla u(x))\nabla h(x) \leq \inf_{s>0}\left\{\frac{1}{s}[\Phi(\nabla u(x) + s\nabla h(x)) - \Phi(\nabla u(x))]\right\}$$

$$\leq \frac{1}{t}[\Phi(\nabla u(x) + t\nabla h(x)) - \Phi(\nabla u(x))]$$

$$\leq \Phi(\nabla u(x) + \nabla h(x)) - \Phi(\nabla u(x)) \leq \frac{1}{2}\Phi(2\nabla u(x)) + \frac{1}{2}\Phi(2\nabla h(x)).$$

Consequently,

$$(3.14) \quad \left|\frac{1}{t}[\Phi(\nabla u(x) + t\nabla h(x)) - \Phi(\nabla u(x))]\right| \leq \frac{1}{2}\Phi(2\nabla u(x)) + \frac{1}{2}\Phi(2\nabla h(x)),$$

for all $t \in (0, 1)$, almsot every $x \in \Omega$. Since Φ satisfies a Δ_2 condition and $\nabla u, \nabla h \in L_\Phi(\Omega)$, the function in the right hand side of (3.14) belongs to $L^1(\Omega)$. From (3.12), (3.14), and the Lebesgue dominated convergence theorem, we get

$$DJ(u)h = \int_\Omega \lim_{t\to 0^+}\left\{\frac{1}{t}[\Phi(\nabla u(x) + t\nabla h(x)) - \Phi(\nabla u(x))]\right\} dx$$

$$= \int_\Omega D\Phi(\nabla u(x))\nabla h(x)\, dx. \qquad \square$$

REMARK 3.3. Note that for any G-function Φ satisfying (G1), (G2'), (G3), (G4), (G5'), by using the construction in Proposition 2.8, we see that there always exists a G-function Φ_1 that satisfies these conditions and also (3.11). Therefore in the sequel we assume Φ satisfies (3.11).

In what follows, we are interested in nonsmooth G-functions Φ, that is Φ is assumed only to satisfy (G1), (G2'), (G3), (G4), (G5'), (G6) and thus may not be differentiable in $\mathbb{R}^N$. If $\mathcal{F}$ is given by

$$(3.15) \qquad \langle\mathcal{F}(u), v\rangle = \int_\Omega f(x, u)v\, dx.$$

then under certain growth condition of f, $\mathcal{F}(u)$ belongs to X^* for any $u \in X$. Equation (3.5) now becomes

$$(3.16) \qquad DJ(u)(v) = \langle\mathcal{F}(u), v\rangle, \quad \text{for all } v \in X.$$

In case J is not differentiable, thanks to its convexity, we can consider instead of equation (3.16), the inclusion $\partial J(u) \ni \mathcal{F}(u)$ or equivalently,

$$(3.17) \qquad \partial J(u) - \mathcal{F}(u) \ni 0 \quad \text{in } X^*.$$

Inclusion (3.17) is equivalent to the variational inequality

$$(3.18) \qquad \begin{cases} J(v) - J(u) - \langle\mathcal{F}(u), v - u\rangle \geq 0, & \text{for all } v \in X, \\ u \in X. \end{cases}$$

In what follows, we study instead of (3.18) a more general variational inequality:

(3.19)
$$\begin{cases} J(v) - J(u) - \langle \mathcal{F}(u), v - u \rangle \geq 0, & \text{for all } v \in K, \\ u \in K, \end{cases}$$

where K is a closed convex subset of X. We have the following abstract existence result for (3.19).

THEOREM 3.4. *Assume*

(3.20)
$$\mathcal{F} \colon X \to X^* \text{ is a pseudomonotone mapping,}$$

and there are $u_0 \in X$ *and* $R > \|u_0\|$ *such that*

(3.21)
$$K \cap B_R(0) \neq \emptyset,$$

and

(3.22)
$$\langle \mathcal{F}(u), u - u_0 \rangle + J(u) > J(u_0),$$

for all $u \in K$ *with* $\|u\| = R$. *Then the variational inequality* (3.19) *has solutions.*

PROOF. Theorem 3.4 is a direct consequence of Corollary 2.6 in [33]. In fact, since Φ satisfies a Δ_2 condition at ∞, we see that J is bounded on bounded sets of X and thus locally Lipschitz on X. It follows (cf. e.g. Theorem 4.2 in [3]) that $\partial J(u) \neq \emptyset$, for all $u \in X$. Furthermore,

(3.23)
$$D(\partial(J + I_K)) = K.$$

In fact, since $(J + I_K)(u) = \infty$ if $u \notin K$, we have $D(\partial(J + I_K)) \subset K$. If $u \in K$ then for $l \in \partial J(u)$, we have for any $v \in X$,

$$(J + I_K)(v) - (J + I_K)(u) \geq J(v) - J(u) \geq \langle l, v - u \rangle.$$

This shows that $l \in \partial(J + I_K)(u)$, i.e. $\partial(J + I_K)(u) \neq \emptyset$, and thus $u \in D(\partial(J + I_K))$, proving (3.23). Note that X is reflexive since both G and G^* satisfy Δ_2 conditions at infinity and thus Corollary 2.6 in [33] is applicable in our problem here. Also, (3.21)–(3.23) imply that the conditions in Corollary 2.6 of [33] are satisfied with $\psi = J$, $\phi = J + I_K$, $L = 0$, and $B = -\mathcal{F}$. According to that result, inequality (3.19) has solutions. □

As a consequence of Theorem 3.4, we have the following existence result.

COROLLARY 3.5. *Assume* $\mathcal{F}$ *is a pseudomonotone mapping such that*

(3.24)
$$\langle \mathcal{F}(u), u - u_0 \rangle \geq -\alpha_1 \|u\| - \beta_1, \quad \text{for all } u \in K,$$

for some $\alpha_1, \beta_1 \geq 0$. *Then* (3.19) *has solutions.*

PROOF. It follows from (3.24) that

$$\frac{J(u) + \langle \mathcal{F}(u), u - u_0 \rangle}{\|u\|} \geq \frac{J(u)}{\|u\|} - \alpha_1 - \frac{\beta_1}{\|u\|},$$

for all $u \in K \setminus \{0\}$. From (2.17) in Proposition 2.7, we see that

$$\lim_{\|u\| \to \infty, \, u \in K} \frac{J(u) + \langle \mathcal{F}(u), u - u_0 \rangle}{\|u\|} = \infty,$$

which implies (3.22). $\square$

It is clear that $\mathcal{F}(u) = L \in X^*$ satisfies (3.24), thus the set $S_K(L)$ of solutions to the variational inequality

(3.25)
$$\begin{cases} J(v) - J(u) \geq \langle L, v - u \rangle, & \text{for all } v \in K, \\ u \in K, \end{cases}$$

or equivalently, of the inclusion $\partial J(u) \ni L$ is nonempty.

REMARK 3.6. Note that in this particular case where $f = L \in X^*$, (3.25) is equivalent to the minimization theorem

(3.26) $$u \in K : J(u) - \langle L, u \rangle = \min_{v \in K}[J(v) - \langle L, v \rangle].$$

From the coercivity property of J given in Proposition 2.7 and its lower semicontinuity with respect to the weak* topology of X (cf. Theorem 2.1, Chapter 8, [11]), we obtain the existence of solutions of (3.26) and thus of (3.25) for any weakly* closed and convex subset K of X, with a Δ_2 condition assumed only on G^* but not necessarily on G.

The mapping $S_K \colon L \mapsto S_K(L)$ is resolvent (mapping) solution of (3.25). In the particular case where $K = X$ and Φ is smooth, S_X is the same as the inverse operator $[-\text{div}(\nabla \Phi(\nabla(\,\cdot\,)))]^{-1}$. Some properties of S_K are given in the following result.

THEOREM 3.7. (a) $u \in S_K(L)$ *if and only if* u *is a minimizer of the functional* $J - L$ *on* K.
(b) *For any* $L \in X^*$, $S_K(L)$ *is a nonempty closed convex and bounded subset of* K. *If* Φ *is strictly convex then* $S_K(L)$ *is a singleton.*
(c) *The mapping* $S_K \colon X^* \to 2^X$ *is a bounded and strong-weak upper semicontinuous, i.e. upper semicontinuous from* X^* *with the strong topology to* 2^X *with the weak topology on* X.

PROOF. The proof of (a) is straightforward.
(b) As noted above, $S_K(L) \neq \emptyset$. The convexity, boundedness, and closedness of $S_K(L)$ follows directly from the convexity, coercivity (cf. Proposition 2.7), and lower semicontinuity of J. When Φ is strictly convex on $\mathbb{R}^N$, it is straightforward

to show that J is also strictly convex on X. Therefore, $J - L$ has a unique minimizer in K, which is the unique element of $S_K(L)$.

(c) To prove the boundedness of S_K, let W be a bounded set in X^* and $M = \sup\{\|L\|_* : L \in W\}(< \infty)$. Let u_0 be a fixed element of K and put $M_0 = |J(u_0) - \langle L, u_0 \rangle|$. From Proposition 2.7, there exists $\rho > 1$ such that $J(u)/\|u\| > M + M_0$ for all $u \in X$ with $\|u\| > \rho$. Let $L \in W$ and $u \in S_K(L)$. We have $J(u) - \langle L, u \rangle \leq J(u_0) - \langle L, u_0 \rangle \leq M_0$ and thus

$$\frac{J(u)}{\|u\|} \leq \frac{|\langle L, u \rangle|}{\|u\|} + \frac{M_0}{\|u\|} \leq \|L\|_* + M_0 \leq M + M_0$$

if $\|u\| > \rho$. This implies from the choice of ρ that $\|u\| \leq \rho$. Hence $\bigcup\{S_K(L) : L \in W\} \subset \overline{B_\rho(0)}$, proving that S_K is a bounded mapping.

Let $L_0 \in X^*$. To prove the strong-weak upper semicontinuity of S_K at L_0, we assume by contradiction that there exists a weakly open set W in X and a sequence $\{L_n\} \subset X^*$ such that

(3.27) $L_n \to L_0$ in X^*, $S(L_0) \subset W$,   but $S(L_n) \not\subset W$,   for all $n \in \mathbb{N}$.

Therefore, for each $n \in \mathbb{N}$, there exists

(3.28) $$u_n \in S(L_n) \setminus W.$$

Since $\{L_n\}$ is bounded, $\{u_n\}$ is a bounded sequence in K. Because X is reflexive and K is weakly closed in X, by passing to a subsequence if necessary we can assume that

(3.29) $$u_n \rightharpoonup u_0 \in K.$$

The weak lower semicontinuity of J on X implies that

(3.30) $$J(u_0) \leq \liminf J(u_n).$$

On the other hand, (3.27) and (3.29) implies that

(3.31) $$\langle L_n, u_n \rangle \to \langle L_0, u_0 \rangle.$$

Let $v \in K$. For any $n \in \mathbb{N}$, we have $J(v) - \langle L_n, v \rangle \geq J(u_n) - \langle L_n, u_n \rangle$. Letting $n \to \infty$ and taking into account (3.30) and (3.31), we have

$$J(v) - \langle L, v \rangle = \lim[J(v) - \langle L_n, v \rangle] \geq \liminf[J(u_n) - \langle L_n, u_n \rangle]$$
$$= \liminf J(u_n) - \lim\langle L_n, u_n \rangle \geq J(u_0) - \langle L_0, u_0 \rangle.$$

This shows that $u_0 \in S_K(L_0)$. From (3.29), we see that $u_n \in W$ for all n sufficiently large, contradicting (3.28). This contradiction proves the strong-weak upper semicontinuity of S_K. $\qquad\square$

When $K = X$ and Φ is strictly convex, the continuity of S_X in Theorem 3.7(c) can be strengthened as follows.

Theorem 3.8. *Assume $K = X$ and Φ is strictly convex on $\mathbb{R}^N$. Then for each $L \in X^*$, $S_X(L) = \{u_L\}$ and the mapping $S_X \colon L \mapsto u_L$ is continuous from X^* to X.*

The proof of this theorem is based on the following lemma about a compactness property of ∂J on X which seems to have some interest itself.

Lemma 3.9. *Let $\{u_n\}$ be a bounded sequence in X, $\{\xi_n\}$ a sequence in X^*, $u \in X$, and $\xi \in X^*$ such that $\xi \in \partial J(u)$, $\xi_n \in \partial J(u_n)$, for all $n \in \mathbb{N}$. Furthermore,*

$$(3.32) \qquad \langle \xi_n - \xi, u_n - u \rangle \to 0 \quad as \ n \to \infty,$$

then $u_n \to u$ in X.

Proof. Assume u, ξ, u_n, ξ_n be as in the assumptions of Lemma 3.9. Note that X can be identified with a closed subspace $\widetilde{X}$ of $L_\Phi(\Omega)$:

$$\widetilde{X} = \{(w_1, \ldots, w_N) \in L_\Phi(\Omega) : \exists\, u \in W_0^{1,1}(\Omega) : \partial_j u = w_j \text{ a.e. on } \Omega\}.$$

Therefore, from Theorem 3.3 in [25], we have, for any $w \in X$,

$$\partial J(w) \subset \int_\Omega \partial \Phi(\nabla w(x)) \nabla(\,\cdot\,)\, dx,$$

i.e. for any $\eta \in \partial J(w)$, there exists a measurable function $\widehat{\eta} \colon \Omega \to \mathbb{R}^N$ such that

$$(3.33) \qquad \widehat{\eta}(x) \in \partial \Phi(\nabla w(x)) \quad \text{for a.e. } x \in \Omega,$$

$\widehat{\eta} \nabla v \in L^1(\Omega)$, and

$$(3.34) \qquad \langle \eta, v \rangle = \int_\Omega \widehat{\eta}(x) \nabla v(x)\, dx, \quad \text{for all } v \in X.$$

Let $\widehat{\xi}_n$, $\widehat{\xi}$ correspond to ξ_n, ξ and u_n, u as in (3.33)–(3.34). From (3.33), we have

$$\widehat{\eta}(x) \nabla w(x) = \Phi(\nabla w(x)) + \Phi^*(\widehat{\eta}(x)),$$

for almost every $x \in \Omega$. Hence, from (3.34),

$$\infty > \langle \eta, w \rangle = \int_\Omega \widehat{\eta}(x) \nabla w(x)\, dx = \int_\Omega \Phi(\nabla w(x))\, dx + \int_\Omega \Phi^*(\widehat{\eta}(x))\, dx.$$

This implies that

$$(3.35) \qquad \Phi^*(\widehat{\eta}) \in L^1(\Omega), \quad \text{i.e. } \widehat{\eta} \in L_{\Phi^*}(\Omega).$$

On the other hand, since Φ is strictly convex, $\partial \Phi$ is strictly monotone on $\mathbb{R}^N$, that is, for all $z, z' \in \mathbb{R}^N$, $z \neq z'$, all $\widehat{z} \in \partial \Phi(z)$, $\widehat{z'} \in \partial \Phi(z')$, we have

$$(3.36) \qquad \langle \widehat{z} - \widehat{z'}, z - z' \rangle > 0.$$

Therefore,

$$(3.37) \qquad \langle \widehat{\xi}_n(x) - \widehat{\xi}(x), \nabla u_n(x) - \nabla u(x) \rangle \geq 0, \quad \text{for a.e. } x \in \Omega, \text{ all } n \in \mathbb{N}.$$

Since $\langle \xi_n, v \rangle = \int_\Omega \widehat{\xi}_n \nabla v \, dx$ and $\langle \xi, v \rangle = \int_\Omega \widehat{\xi} \nabla v \, dx$, we have

$$\langle \xi_n - \xi, u_n - u \rangle = \int_\Omega (\widehat{\xi}_n - \widehat{\xi})(\nabla u_n - \nabla u) \, dx.$$

(3.37) and (3.32) imply that

(3.38) $$(\widehat{\xi}_n - \widehat{\xi})(\nabla u_n - \nabla u) \to 0 \quad \text{in } L^1(\Omega).$$

We prove that for any $\varepsilon > 0$, there exists $\delta \in (0, \varepsilon)$ such that for all $H \subset \Omega$ measurable with $|H| < \delta$ ($|H|$ is the Lebesgue measure of H), we have

(3.39) $$\int_H \Phi(\nabla u_n) \, dx < \varepsilon, \quad \text{for all } n \in \mathbb{N}.$$

In fact, given $\varepsilon > 0$, it follows from (3.38) that there exists $n_1 \in \mathbb{N}$ such that

$$\int_\Omega (\widehat{\xi}_n - \widehat{\xi})(\nabla u_n - \nabla u) \, dx < \frac{\varepsilon}{4}, \quad \text{for all } n \geq n_1.$$

Together with (3.37), this implies that

$$\int_H (\widehat{\xi}_n - \widehat{\xi})(\nabla u_n - \nabla u) \, dx < \frac{\varepsilon}{4}, \quad \text{for all } n \geq n_1,$$

for any measurable subset H of Ω. Therefore,

(3.40) $$\int_H \widehat{\xi}_n \nabla u_n \, dx < \frac{\varepsilon}{4} + \int_H \widehat{\xi}_n \nabla u \, dx + \int_H \widehat{\xi} \nabla u_n \, dx - \int_H \widehat{\xi} \nabla u \, dx$$
$$\leq \frac{\varepsilon}{4} + \int_H \widehat{\xi}_n \nabla u \, dx + \int_H \widehat{\xi} \nabla u_n dx.$$

In the next step, we present some inequalities that are useful in the next calculations. First, basic properties of convex functions give us

(3.41) $$\Phi(z) \leq \zeta z \leq \Phi(2z),$$

and

(3.42) $$\Phi^*(\zeta) \leq \Phi(2z),$$

for all $z \in \mathbb{R}^N$, all $\zeta \in \partial\Phi(z)$. On the other hand, we have from Young's inequality that for all $\rho \in (0, \infty)$, all $z_1, z_2 \in \mathbb{R}^N$,

(3.43) $$|z_1 z_2| \leq \Phi(\rho^{-1} z_1) + \Phi^*(\rho z_2).$$

Since Φ satisfies a Δ_2 condition, there are $K, b \geq 0$ such that

(3.44) $$\Phi(2z) \leq K\Phi(z) + b, \quad \text{for all } z \in \mathbb{R}^N.$$

For any $x \in \Omega$, $\rho \in (0, 1)$, we have

(3.45) $$|\widehat{\xi}_n(x) \nabla u(x)| \leq \Phi(\rho^{-1} \nabla u(x)) + \Phi^*(\rho \widehat{\xi}_n(x))$$
$$\leq \Phi(\rho^{-1} \nabla u(x)) + \rho \Phi^*(\widehat{\xi}_n(x)).$$

Similarly,

$$\text{(3.46)} \qquad |\widehat{\xi}(x)\nabla u_n(x)| \le \Phi(\rho\nabla u_n(x)) + \Phi^*(\rho^{-1}\widehat{\xi}(x))$$

$$\le \rho\Phi(\nabla u_n(x)) + \Phi^*(\rho^{-1}\widehat{\xi}(x)).$$

Also, since $\widehat{\xi}_n(x) \in \partial\Phi(\nabla u_n(x))$ for almost every $x \in \Omega$, we have from (3.44) and (3.42) that for almost every $x \in \Omega$,

$$\text{(3.47)} \qquad \Phi^*(\widehat{\xi}_n(x)) \le \Phi(2\nabla u_n(x)) \le K\Phi(\nabla u_n(x)) + b.$$

Combining (3.40), (3.41), (3.45), (3.42), (3.47), (3.46) and simplifying the resulted estimate, we get

$$\text{(3.48)} \quad [1 - \rho(K+1)]\int_H \Phi(\nabla u_n)\,dx < \frac{\varepsilon}{4} + [\rho b + b^*(\rho)]|H|$$

$$+ \int_H \Phi(\rho^{-1}\nabla u)\,dx + K^*(\rho)\int_H \Phi(2\nabla u)\,dx.$$

Choose $\rho = [2(K+1)]^{-1}$. Since Φ satisfies a Δ_2 condition and $\nabla u \in L_\Phi$, we have $\Phi(\rho^{-1}\nabla u), \Phi(2\nabla u) \in L^1(\Omega)$. Hence, there are $\delta_1, \delta_2 > 0$ such that if $|H| < \delta_1$ then

$$\int_H \Phi(\rho^{-1}\nabla u) < \frac{\varepsilon}{12}$$

and, if $|H| < \delta_2$, then

$$\int_H \Phi(2\nabla u)\,dx < \frac{\varepsilon}{12K^*([2(K+1)]^{-1})}.$$

Letting

$$\delta = \min\left\{\delta_1, \delta_2, \frac{\varepsilon}{12(b[2(K+1)]^{-1} + b^*([2(K+1)]^{-1}))}\right\},$$

we see that if $H \subset \Omega$ measurable and $|H| < \delta$ then the last three terms in the right hand side of (3.48) is less than $\varepsilon/12$ and thus $\int_H \Phi(\nabla u_n)\,dx < \varepsilon$. (3.39) is proved.

Next, we prove that

$$\text{(3.49)} \qquad \nabla u_n \to \nabla u \quad \text{a.e. in } \Omega.$$

From (3.38), there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and a subset U of Ω of measure zero such that for all $x \in \Omega\backslash U$, $\widehat{\xi}_n(x) \in \partial\Phi(\nabla u_n(x))$, $\widehat{\xi}(x) \in \partial\Phi(\nabla u(x))$, for all $n \in \mathbb{N}$ and

$$\text{(3.50)} \qquad (\widehat{\xi}_{n_k})(x) - \widehat{\xi}(x))(\nabla u_{n_k}(x) - \nabla u(x)) \to 0 \quad \text{as } k \to \infty.$$

In particular, for all $x \in \Omega \setminus U$, there is $\alpha_0 \ge 0$ (which may depend on x) such that $(\widehat{\xi}_{n_k})(x) - \widehat{\xi}(x))(\nabla u_{n_k}(x) - \nabla u(x)) \le \alpha_0$, for all $k \in \mathbb{N}$. Using again

the estimates in (3.40), (3.41), (3.45), (3.42), (3.47), and (3.46), we see that for all $x \in \Omega \setminus U$, all $k \in \mathbb{N}$,

$$[1 - \rho(K+1)]\Phi(\nabla u_{n_k}(x)) \leq \alpha_0 + \rho b + b^*(\rho) + \Phi(\rho^{-1}\nabla u(x)) + K^*(\rho)\Phi(2\nabla u(x)).$$

By choosing $\rho = [2(K+1)]^{-1}$, we see that $\{\Phi(\nabla u_{n_k}(x))\}$ is a bounded sequence for all $x \in \Omega \setminus U$. From (G2') (or even (G2)), this means that $\{\nabla u_{n_k}(x)\}$ is a bounded sequence in $\mathbb{R}^N$, which together with (3.47), implies that $\{\Phi^*(\widehat{\xi}_{n_k}(x))\}$ and thus $\{\widehat{\xi}_{n_k}(x)\}$ are bounded sequences (for all $x \in \Omega \setminus U$).

For each $x \in \Omega \setminus U$, by passing to subsequences (that may as well depend on x), we can assume for simplicity of notation that

$$(3.51) \qquad \nabla u_{n_k}(x) \to w_0 = w_0(x) \quad \text{in } \mathbb{R}^N,$$

$$(3.52) \qquad \widehat{\xi}_{n_k}(x) \to \widehat{\xi}_0 = \widehat{\xi}_0(x) \quad \text{in } \mathbb{R}^N.$$

Since $\widehat{\xi}_{n_k}(x) \in \partial\Phi(\nabla u_{n_k}(x))$, for all $k \in \mathbb{N}$, by the upper semicontinuity of the mapping $\partial\Phi \colon \mathbb{R}^N \to 2^{\mathbb{R}^N}$, $w \mapsto \partial\Phi(w)$, we see that $\widehat{\xi}_0 \in \partial\Phi(w_0)$. Let $k \to \infty$ in (3.50), we see that $(\widehat{\xi}_0 - \widehat{\xi}(x))(w_0 - \nabla u(x)) = 0$. Due to the strict monotonicity of $\partial\Phi$, we conclude that $\nabla u(x) = w_0$, i.e. $\nabla u_{n_k}(x) \to \nabla u(x)$. Since this procedure holds for any subsequence of $\{\nabla u_n(x)\}$, we obtain (3.49). From (3.49) and the continuity of Φ in $\mathbb{R}^N$, we get

$$(3.53) \qquad \Phi(\nabla u_n - \nabla u) \to 0 \quad \text{a.e. in } \Omega.$$

Let us prove that the sequence $\{\Phi(\nabla u_n - \nabla u)\}$ is uniformly integrable on Ω. In fact, for all $n \in \mathbb{N}$ and almost every $x \in \Omega$,

$$\Phi(\nabla u_n(x) - \nabla u(x)) \leq \frac{1}{2}\Phi(2\nabla u_n(x)) + \frac{1}{2}\Phi(2\nabla u(x))$$
$$\leq \frac{K}{2}\Phi(\nabla u_n(x)) + \frac{b}{2} + \frac{1}{2}\Phi(2\nabla u(x)).$$

Hence for all $H \subset \Omega$ measurable,

$$(3.54) \qquad \int_H \Phi(\nabla u_n - \nabla u)\,dx \leq \frac{K}{2}\int_H \Phi(\nabla u_n)\,dx + \frac{b|H|}{2} + \frac{1}{2}\int_H \Phi(2\nabla u)\,dx.$$

Given $\varepsilon > 0$. It follows from (3.39) that there exists $\delta_3 > 0$ such that $\int_H \Phi(\nabla u_n)\,dx < 2\varepsilon/(3K)$ whenever H is a measurable subset of Ω with $|H| < \delta_3$. On the other hand, since $\Phi(2\nabla u) \in L^1(\Omega)$, there is $\delta_4 > 0$ such that $\int_H \Phi(2\nabla u)\,dx < 2\varepsilon/3$ if $|H| < \delta_4$. Choosing $\delta = \min\{\delta_3, \delta_4, 2\varepsilon/(3b)\}$, we see from (3.54) that if $H \subset \Omega$ measurable and $|H| < \delta$ then $\int_H \Phi(\nabla u_n - \nabla u)\,dx < \varepsilon$ for all $n \in \mathbb{N}$.

According to Vitali's convergence theorem (cf. e.g. [23]), together with (3.53), this uniform integrability of $\{\Phi(\nabla u_n - \nabla u)\}$ shows that $\Phi(\nabla u_n - \nabla u) \to 0$ in $L^1(\Omega)$. Since Φ satisfies a Δ_2 condition at ∞, the norm and modular convergences are equivalent in $L_\Phi(\Omega)$. Hence $\|\nabla u_n - \nabla u\|_{L_\Phi} \to 0$, i.e. $u_n \to u$ in X.□

COROLLARY 3.10. *If Φ is strictly convex in $\mathbb{R}^N$ then $\partial J \colon X \to 2^{X^*}$ is of class $(S)_+$ in the following sense: If $u_n \rightharpoonup u$ in X and there exists $\xi_n \in \partial J(u_n)$ such that $\limsup\langle \xi_n, u_n - u\rangle \le 0$, then $u_n \to u$ in X.*

PROOF. Let $\xi \in \partial J(u)$. We have $\langle \xi, u_n - u\rangle \to 0$ and thus

$$0 \le \liminf\langle \xi_n - \xi, u_n - u\rangle \le \limsup\langle \xi_n - \xi, u_n - u\rangle = \limsup\langle \xi_n, u_n - u\rangle \le 0.$$

This shows that $\lim\langle \xi_n - \xi, u_n - u\rangle = 0$. Also, since $\{u_n\}$ is weakly convergent, it is bounded. According to Lemma (3.9), we have $u_n \to u$ in X. $\qquad\square$

REMARK 3.11. In the case of modulars given by anisotropic convex functions as considered above, Corollary 3.10 improves Theorem 3.8 in [13] in the sense that we assume here the strict convexity of Φ instead of uniform convexity conditions as in [13]. Also, we follow here a different approach to prove Corollary 3.10.

We are now ready to prove Theorem 3.8.

PROOF PROOF OF THEOREM 3.8. Since Φ is strictly convex on $\mathbb{R}^N$, J is strictly convex on X. The uniqueness of the solution of (3.25) follows from the strict convexity of $J - L$. Let $\{L_n\} \subset X^*$, $L_n \to L_0$ in X^*. Let us prove that $u_n \to u_0$ in X where $\{u_n\} = S_X(L_n)$, $n \in \mathbb{N} \cup \{0\}$. In fact, from Theorem 3.7(c), we know that $u_n \rightharpoonup u_0$ in X. As above, we have $L_n \in \partial J(u_n)$, for all $n \in \mathbb{N}$ and also $u_0 \in S_X(L_0)$, i.e. $L_0 \in \partial J(u_0)$. Since $L_n \to L_0$ in X^* and $u_n \rightharpoonup u_0$ in X, we get $\langle L_n - L_0, u_n - u_0\rangle \to 0$. It now follows from Lemma 3.9 that $u_n \to u$ in X.$\square$

REMARK 3.12. (a) Since we are interested here in variational inequalities on any closed and convex subset K of X, the Δ_2 conditions on both G and G^* are assumed here in order to apply the abstract existence results in [33] (Theorem 2.2 and Corollary 2.6, [33]) for the solvability of inclusions and variational inequalities in reflexive Banach spaces. Δ_2 conditions are assumed on both G and G^* to guarantee the reflexivity of X and thus the applicability of the above abstract theorems. As a consequence, we obtain in Theorem 3.4 and Corollary 3.5 the existence of solutions of inequalities in which the lower order terms (or right-hand-side terms) can be given by general pseudomonotone operators of u.

Another reason for such assumption about Δ_2 conditions here is that we study, in addition to the existence of solutions of variational inequality (3.19) and inclusion (3.17), properties of their solutions and their related mappings as well. For example, we need a Δ_2 condition on G^* in the proof of Proposition 2.7 about a coercivity property of J and a Δ_2 condition on G to get the $(S)_+$ property of ∂J in Corollary 3.10.

(b) Compared to the papers [17] and [22], we note that in those papers the function spaces are general anisotropic Orlicz–Musielak spaces, that is, the

generating N-functions depend also on x. It was not assumed in [17] any Δ_2 condition on the N-function M nor on its conjugate M^* as we assume here. The right hand side f was assumed in [17] to be a function on Ω which was the divergence of a vector field F in E_{M^*} which, roughly speaking, corresponds to the case $f = L \in X^*$ here. It was proved in [22] the existence and uniqueness of renormalized and, in particular cases, of weak solutions of the inclusion

$$\beta(x, u) - \operatorname{div}(a(x, \nabla u) + F(u)) \ni f \quad \text{in } \Omega,$$

where β is maximal monotone in the second variable without any further growth condition, F is locally Lipschitz, and f belongs to $L^d(\Omega)$ in the case of weak solutions (d is the dimension of Ω). The problem was studied in a general Orlicz–Musielak space where the conjugate of the generating function satisfied a Δ_2 condition. The existence results in the above works were based on truncation methods together with Minty–Browder type techniques.

Here, we get the existence of solutions of variational inequalities in anisotropic Orlicz–Sobolev spaces (generated by G-functions that do not depend on x) by a combination of variational and monotonicity approaches based on an abstract existence theorem. Compared to [17] and [22], the different approaches lead to existence results of different natures. As noted in (a), we assumed in Theorem 3.4 Δ_2 conditions on both G and G^*, but we get the existence of solutions of variational inequalities rather than of equations, with right hand sides of different nature (pseudomonotone terms that may depend on u). In the illustrative example in Section 5, the signed term h is assumed to have subcritical growth but is not necessarily monotone in u, while the "sublinear" term g depends on u and has no monotone property in general. Therefore, even in the case of equation (i.e. $K = X$, in problem with Δ_2 conditions), that particular example seems not to be completely covered by the theories in [17] and [22]. Furthermore, we are interested here in some other properties of solutions and of their related mappings.

4. Fixed point formulation and topological degree

By using the resolvent mapping S_K, we can write the variational inequality (3.19) equivalently as the fixed point inclusion

$$(4.1) \qquad\qquad u \in X : u \in S_K(\mathcal{F}(u)).$$

In cases where $\mathcal{F}$ has some compactness properties, the existence and other properties of solutions of (4.1) can be studied by fixed point or topological methods. An example of such situations is presented here.

Assume $f \colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$(4.2) \qquad\qquad |f(x, u)| \le B_0(x) + C_0[(\Phi_0^*)^{-1}\Phi_0](|u|),$$

for almost every $x \in \Omega$, all $u \in \mathbb{R}$, where $B_0 \in L_{\Phi_0^*}$ and $C_0 > 0$. Note that Φ_0 is given in (3.1)–(3.2) and condition (4.2) is an extension of the classical subcritical growth condition to our problem with anisotropic principal term.

From (3.1), the embedding $i = i_{\Phi_0} \colon X \hookrightarrow L_{\Phi_0}$ is compact, and so is its adjoint $i^* \colon L_{\Phi_0^*} (= (L_{\Phi_0})^*) \hookrightarrow X^*$. It follows from the growth condition (4.2) that $\widetilde{f}(u) = f(\,\cdot\,, u(\,\cdot\,)) \in L_{\Phi_0^*}$ for every $u \in L_{\Phi_0}$ and the mapping $\widetilde{f} \colon u \mapsto f(\,\cdot\,, u(\,\cdot\,))$ is continuous and bounded from L_{Φ_0} into $L_{\Phi_0^*}$. Let

$$(4.3) \qquad\qquad\qquad \mathcal{F} = i^* \widetilde{f} i.$$

In this case (4.1) is equivalent to a fixed point inclusion in L_{Φ_0}.

PROPOSITION 4.1. *Let $\mathcal{F}$ be given by (4.3) with f satisfying (4.2). If u is a solution of (4.1) then $w = iu$ is a solution of the fixed point inclusion*

$$(4.4) \qquad\qquad w \in L_{\Phi_0} : w \in (iS_K i^* \widetilde{f})(w).$$

Conversely, if w is a solution of (4.4) then $w \in X$ and is a solution of (4.1).

PROOF. If u is a solution of (4.1) then $w = iw \in L_{\Phi_0}$ satisfies $w = iu \in iS_K[\mathcal{F}(u)] = iS_K[i^* \widetilde{f}(iu)] = (iS_K i^* \widetilde{f})(w)$. Conversely, assume w is a solution of (4.4). Since $S_K(X^*) \subset K \subset X$, we have $w \in K \subset X$. Hence, $w = iw \in iS_K[i^* \widetilde{f} i(w)] = S_K[\mathcal{F}(w)]$, i.e. w satisfies (4.1). $\qquad\square$

Some useful properties of $iS_K i^* \widetilde{f}$ and thus of the formulation (4.4) are given in the following result.

PROPOSITION 4.2. (a) *For each $w \in L_{\Phi_0}$, $(iS_K i^* \widetilde{f})(w)$ is a nonempty, convex, compact subset of L_{Φ_0}.*

(b) *The mapping $iS_K i^* \widetilde{f}$ is upper semicontinuous and compact from L_{Φ_0} into $2^{L_{\Phi_0}} \setminus \{\emptyset\}$.*

PROOF. (a) For any $L \in X^*$, since $S_K(L)$ is nonempty, convex, closed, and bounded in X, and i is compact, it is straightforward to see that $(iS_K i^* \widetilde{f})(w)$ is nonempty, convex, and compact in L_{Φ_0}.

(b) Let $w_0 \in L_{\Phi_0}$ and U be an open set in L_{Φ_0} such that

$$(iS_K i^* \widetilde{f})(w_0) \subset U.$$

Since the embedding i is compact, $i^{-1}(U) = U \cap X$ is weakly open in X. Furthermore, $(S_K i^* \widetilde{f})(w_0) \subset i^{-1}(U) = U \cap X$. From Theorem 3.7(c), there exists an open neighbourhood W of $i^* \widetilde{f}(w_0)$ in X^* such that $S_K(W) = \bigcup_{L \in W} S_K(L) \subset U \cap X$. Since $\widetilde{f}$ is continuous from L_{Φ_0} to $L_{\Phi_0^*}$ and i^* is continuous from $L_{\Phi_0^*}$ to X^*, $V = (i^* \widetilde{f})^{-1}(W)$ is an open neighbourhood of w_0 in L_{Φ_0}. If $w \in V$ then $i^* \widetilde{f}(w) \in W$ and thus $S_K(i^* \widetilde{f}(w)) \subset U \cap X$. This means that $(iS_K i^* \widetilde{f})(V) \subset U$, proving the upper semicontinuity of $iS_K i^* \widetilde{f}$.

Assume now that W is a bounded set in L_{Φ_0}. Since $\widetilde{f}$ is a bounded mapping, $\widetilde{f}(W)$ is bounded in $L_{\Phi_0^*}$ and thus $i^*\widetilde{f}(W)$ is bounded in X^*. Because of the boundedness of S_K, we see that $S_K i^*\widetilde{f}(W)$ is bounded in X. Since i is compact, $(iS_K i^*\widetilde{f})(W)$ is relatively compact in L_{Φ_0}. This proves that $iS_K i^*\widetilde{f}$ is a (multivalued) compact mapping. $\qquad\square$

The properties of $iS_K i^*\widetilde{f}$ in Proposition 4.2 allow us to use classical topological tools such the Tikhonov's fixed point theorem for multivalued compact vector fields or variants of topological degrees of multivalued perturbations of identity to study the existence and other behaviors of solutions of (4.4), cf. e.g. [2], [43], [15]. For example, Proposition 4.2 implies that the topological degree (cf. e.g. [39], [42]) $\deg(I - iS_K i^*\widetilde{f}, U, a)$ is defined, where I is the identity mapping of L_{Φ_0}, U is an open bounded subset of L_{Φ_0} and $a \in L_{\Phi_0} \setminus (I - iS_K i^*\widetilde{f})(\partial U)$. We have therefore the following basic existence and localization theorem for the inclusion (4.4) in L_{Φ_0} and thus for the variational inequality (3.19).

THEOREM 4.3. *Assume U is an open bounded subset of L_{Φ_0} such that*

$$0 \notin u - (iS_K i^*\widetilde{f})(u), \quad \text{for all } u \in \partial U,$$

and $\deg(I - iS_K i^*\widetilde{f}, U, 0) \neq 0$. Then the inclusion (4.4) and thus the variational inequality (3.19) have solutions in $K \cap U$.

5. An example

We conclude this paper with a more concrete example and prove an existence theorem when f is the sum of a "subcritical" signed function and a "sublinear" one.

Let F be a Young function from $\mathbb{R}$ to $[0, \infty]$. We denote by $\widehat{F}$ the function from $\mathbb{R}^N$ to $[0, \infty]$ defined by

$$\widehat{F}(\xi) = F(|\xi|), \quad \text{for all } \xi \in \mathbb{R}^N.$$

It is easy to see that $\widehat{F}$ is a G-function (i.e. $\widehat{F}$ satisfies (G1)–(G6)) and if F is an N-function then $\widehat{F}$ satisfies also (G2') and (G5').

Assume

(5.1) $$f = g + h,$$

where g and h are Carathéodory functions from $\Omega \times \mathbb{R}$ into $\mathbb{R}$. The sublinearity (with respect to Φ) of g is extended from the classical case Sobolev spaces to our case as follows:

(G) There are N-functions Ψ and Ψ_0 such that

(5.2) $$\Psi_0 \ll \Psi,$$

(5.3) $$\widehat{\Psi} \prec \Phi,$$

and

(5.4) $$|g(x,u)| \leq B_1(x) + C_1[(\Psi_0^*)^{-1}\Psi_0](|u|),$$

for almost every $x \in \Omega$, all $u \in \mathbb{R}$, where $C_1 > 0$ and $B_1 \in L_{\Psi_0^*}$.

Note that condition (G) is an extension of the usual sublinear growth condition to our case. Concerning h, we assume that:

(H) h has subcritical growth (as in (4.2)):

(5.5) $$|h(x,u)| \leq B_2(x) + C_2[(\Phi_0^*)^{-1}\Phi_0](|u|),$$

for almost every $x \in \Omega$, all $u \in \mathbb{R}$, where $C_2 > 0$ and $B_2 \in L_{\Phi_0^*}$, and

(5.6) $$h(x,u)u \geq 0, \quad \text{for a.e. } x \in \Omega, \text{ all } u \in \mathbb{R}.$$

From (5.3), we see that X is continuously embedded into the Orlicz–Sobolev space $W_0^1 L_{\widehat{\Psi}} = W_0^1 L_\Psi$. From (5.2), $W_0^1 L_\Psi$ is compactly embedded into L_{Ψ_0}. Hence, the embedding $i_{\Psi_0} \colon X \hookrightarrow L_{\Psi_0}$ is compact. On the other hand, as discussed above, the embedding $i_{\Phi_0} \colon X \hookrightarrow L_{\Phi_0}$ is also compact. Furthermore, the growth conditions (5.4) and (5.5) imply that the Niemytskii operators $\widetilde{g} \colon u \mapsto g(\,\cdot\,, u(\,\cdot\,))$ and $\widetilde{h} \colon u \mapsto h(\,\cdot\,, u(\,\cdot\,))$ are bounded and thus continuous from L_{Ψ_0} into $L_{\Psi_0^*}$ and from L_{Φ_0} into $L_{\Phi_0^*}$, respectively. The mapping $\mathcal{F}$ given by (3.15) with f given by (5.1) is well defined and moreover, $\mathcal{F}$ can be written in this case as

(5.7) $$\mathcal{F} = i_{\Psi_0}^* \widetilde{g} i_{\Psi_0} + i_{\Phi_0}^* \widetilde{h} i_{\Phi_0}.$$

We have the following existence result for (3.19).

THEOREM 5.1. *Assume $\mathcal{F}$ is given by (3.15) and (5.1) with g and h satisfying* (G) *and* (H). *Then the variational inequality (3.19) has solutions.*

PROOF. We shall apply Theorem 3.4 by checking the assumptions there. First, note that since i_{Ψ_0}, i_{Φ_0} are compact and $\widetilde{g}$, $\widetilde{h}$ are continuous on their domains, the mappings $i_{\Psi_0}^* \widetilde{g} i_{\Psi_0}$, $i_{\Phi_0}^* \widetilde{h} i_{\Phi_0}$, and thus $\mathcal{F}$ are completely continuous from X into X^* (i.e. continuous from X with the weak topology to X^* with the strong topology). This immediately implies that $\mathcal{F}$ is pseudomonotone. We only need to verify the coercivity condition (3.22) with $u_0 = 0$ and R sufficiently large. In fact, we have for any $u \in X$,

(5.8) $$\langle \mathcal{F}(u), u \rangle + J(u) = \int_\Omega \Phi(\nabla u)\, dx + \int_\Omega g(x,u)u\, dx + \int_\Omega h(x,u)u\, dx$$
$$\geq \int_\Omega \Phi(\nabla u)\, dx + \int_\Omega g(x,u)u\, dx.$$

From (5.3), there exist $k_1, \alpha_0, \beta_0 > 0$ such that $\Phi(k_1 \xi) \geq \alpha_0 \widehat{\Psi}(\xi) - \beta_0 = \alpha_0 \Psi(|\xi|) - \beta_0$ for all $\xi \in \mathbb{R}^N$. Since Φ satisfies a Δ_2 condition at ∞, this implies

the existence of $\alpha_1, \beta_1 > 0$ such that

(5.9) $$\Phi(\xi) \geq \alpha_1 \Psi(|\xi|) - \beta_1, \quad \text{for all } \xi \in \mathbb{R}^N.$$

It follows from Poincaré's inequality for Orlicz–Sobolev spaces (Lemma 5.7, [16]) that

(5.10) $$\int_\Omega \Psi(|\nabla v|)\, dx \geq \alpha_3 \int_\Omega \Psi(k_2|v|)\, dx, \quad \text{for all } v \in W_0^1 L_\Psi,$$

for some constants $k_2, \alpha_3 > 0$. Since $u \in X \hookrightarrow W_0^1 L_\Psi$, (5.9) and (5.10) give

(5.11) $$\int_\Omega \Phi(\nabla u)\, dx \geq \alpha_1 \int_\Omega \Psi(|\nabla u|)\, dx - \beta_1 |\Omega| \geq \alpha_4 \int_\Omega \Psi(k_2|u|)\, dx - \beta_4.$$

On the other hand, (5.4) and Young's inequality imply

(5.12) $$\left| \int_\Omega g(x, u) u\, dx \right| \leq \int_\Omega B_1 |u|\, dx + C_1 \int_\Omega (\Psi_0^*)^{-1} \Psi_0(|u|)|u|\, dx$$

$$\leq C_3 \|u\|_{L_{\Psi_0}} + C_1 \int_\Omega \left\{ \Psi_0^*[(\Psi_0^*)^{-1}\Psi_0(|u|)] + \Psi_0(|u|) \right\} dx$$

$$\leq C_4 \|u\| + 2C_1 \int_\Omega \Psi_0(|u|)\, dx.$$

Given any $\varepsilon > 0$, from (5.2) there exists $\beta_5 > 0$ such that

(5.13) $$\Psi_0(|s|) \leq \varepsilon \Psi(k_2|s|) + \beta_5, \quad \text{for all } s \in \mathbb{R}.$$

Combining (5.12), (5.13) with (5.11) yields

(5.14) $$\left| \int_\Omega g(x, u) u\, dx \right| \leq C_4 \|u\| + 2\varepsilon C_1 \int_\Omega \Psi(k_2|u|)\, dx + 2C_1 \beta_5 |\Omega|$$

$$\leq C_4 \|u\| + \frac{2\varepsilon C_1}{\alpha_4} \int_\Omega \Phi(\nabla u)\, dx + \left(2C_1 \beta_5 |\Omega| + \frac{2\varepsilon C_1 \beta_4}{\alpha_4} \right).$$

Choosing $\varepsilon = \alpha_4/(4C_1)$, we obtain from (5.8) and (5.14) that

$$\langle \mathcal{F}(u), u \rangle + J(u) \geq \frac{1}{2} \int_\Omega \Phi(\nabla u)\, dx - C_4 \|u\| - \left(2C_1 \beta_5 |\Omega| + \frac{\beta_4}{2} \right).$$

As a consequence of Proposition 2.7, the right hand side of this inequality is positive for all $u \in X$ with $\|u\|$ sufficiently large. Therefore, both conditions (3.21) and (3.22) hold for R sufficiently large. According to Theorem 3.4, (3.19) has solutions. $\square$

REMARK 5.2. (a) We can get a similar result to Theorem 5.1 by using alternatively topological degrees and Theorem 4.3. As noted in Remark 3.6, by following this approach we only need to assume that G^* satisfies a Δ_2 condition and thus can extend several results in [36] to variational inequalities in anisotropic Orlicz–Sobolev spaces.

(b) We present here a monotonicity and a topological/fixed point method to study the existence of solutions of (1.1); other approaches such as variational,

topological, and sub-supersolution ones on problems with other boundary conditions will be the subjects of some forthcoming projects. Regarding to the relaxation of the Δ_2 conditions, we hope that the arguments used in problems without such conditions in (isotropic) non-reflexive Orlicz–Sobolev spaces that we studied before in e.g. [14], [27]–[32], [34]–[38] could be extended to problems in anisotropic Orlicz–Sobolev spaces or Orlicz–Musielak–Sobolev spaces. Such extensions are not within the scope of this introductory paper and would be investigated in some future works.

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