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George Tourlakis
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REVIEW

ROBERT MIGNONE

Volume I: Mathematical Logic, of George Tourlakis' two volume *Lectures in Logic and Set Theory*, started out as a preliminary section to the volume on set theory that had been in preparation for some time. The author intended for the preliminary section to contain "absolutely essential topics in proof, model and recursion theory" (page ix). However, the section began to take on its own life as the author attempted to "say something about one of the most remarkable theorems of logic—arguably *the* most remarkable—about the limitations of formalized theories: Gödel's second incompleteness theorem" (page x).

An unspoken code among mathematicians is that one cannot in good conscience reference or include a result in one's own work without understanding why it is true. Another is that when presenting the proof of Gödel's incompleteness theorem cutting corners with, or avoiding completely, the "gory details" is okay. Tourlakis is true to the first code but violates the second. Volume I contains a *complete* proof of Gödel's second incompleteness theorem from Peano's axioms, "gory details and all." But despite his formalist approach, the author has an engaging style and an infectious passion for the subject that helps make this book a highly worthwhile addition to the literature.

Part I (Basic Logic)

The author begins Chapter I with an insightful and highly readable explanation of what the book sets out to accomplish. Section 1 introduces first order languages, where logical and nonlogical symbols are given along with some enlightened heuristics concerning the relationship between formal systems and mathematics. The notion of well-formed formulas formed from what the author calls an "alphabet" is given, along with the notions of "bound" and "free" variables in formulas.

Next, informal induction and recursion are introduced in Section 2, axioms and rules of inference in Section 3 and what the author calls basic metatheorems in Section 4. The section on semantics (I.5) begins in a straightforward way and is both concise and highly readable. The soundness theorem and its implications are clearly stated, the proofs are uncluttered and straightforward, with useful comments throughout.

Henkin's method is used for the proof of Gödel's completeness theorem: first by proving the consistency theorem showing that any first order theory that is consistent has a model, then realizing the completeness theorem and the compactness theorem as corollaries. This is first done for countable first order languages. Next, it is extended to any infinite first order language. The proof progresses through a series of lemmas. This takes the reader, via a series of easily manageable steps, through the proof to a satisfying conclusion. The remarks on *Truth* at the end of the section are well stated and give the reader some bearings for what has been accomplished thus far.

Moving into model theory, the book lays the groundwork for the Löwenheim-Skolem theorems. The proof of the downward Löwenheim-Skolem theorem has the tone and wordiness of a friendly lecture. It works nicely.

Next, the upward Löwenheim-Skolem theorem is established along with other model theoretic results related to chains and elementary chains of structures and categoricity. The section ends with an application of the upward Löwenheim-Skolem theorem that establishes the existence of a *non-standard* model of the real numbers. Over ten pages are devoted to the development and rudimentary understanding of the non-standard model. It provides a dramatic example of the power of the simply stated and readily understood upward Löwenheim-Skolem theorem.

The steady march toward laying the foundations for a frontal attack on Gödel's second incompleteness theorem continues. Approximately ten pages are devoted to "Defined Symbols" where exhaustive justifications are given for abbreviating "Undecipherably long formal texts, thus making them humanly understandable" (page 112). One quickly gets the gist of what is to follow, which is best glanced over quickly and left as a reference. Spending time on the details is about as illuminating as expanding a trinomial to the tenth power.

Recursion theory is introduced using Kleene's inductive number-theoretic approach in order to develop the necessary machinery for the arithmetization of the partial recursive functions. This is effectively done with enough explanation to enable the sophisticated reader lacking formal training in recursion theory to keep pace. Included here is an enlightened and useful discussion of the difference between the intuitive notion of a "computable" function and the exact definition of a recursive function. This section includes a nice treatment of the *halting problem* and a complete description of the arithmetic hierarchy.

The next section is devoted to a comprehensive proof of Gödel's first incompleteness theorem. It begins with a technique for transforming symbolic expressions in a language into the realm of the number-theoretic recursion theory developed so far. The coding method used is a version of Gödel's beta function, as described in Shoenfield's 1967 book, *Mathematical Logic*. The page or two devoted to a brief background explanation of the various approaches to coding that can be employed at this juncture are highly worthwhile. They give the reader some historical bearing and show respect for the reader's intelligence by giving some justification for this particular approach.

Once the language for arithmetic is successfully *arithmetized* and the notion of *definability* for relations over the natural numbers is put forward, the formal universe of arithmetic is divided into those sets that are *constructively arithmetic*, and hence definable over the natural numbers, and those that are not.

The notion of a formal arithmetic is presented as any first order theory over a basic language for arithmetic that contains at a minimum a specific list of non-logical axioms describing the characteristics of the successor, addition and multiplication functions and the *less than* relation. Once this theory is set out, the author next describes how it will be "recursively axiomatized". This immediately leads to what the author describes as "... at the root of Gödel's incompletableness result" (page 176): the set of theorems of any recursively axiomatized formal arithmetic is semi-recursive (whereas the set of all *true* sentences is not even arithmetic) (see page 178).

At this point the stage is set for the semantic version of Gödel's first incompleteness theorem, which states: *For a theory which is a recursively axiomatized formal arithmetic there exists sentences over its language that cannot be decided by the theory.*

After some observations, the syntactic version of Gödel's first incompleteness theorem is attacked. This begins with a definition of what it means for a relation on the natural numbers to be "formally definable" in a theory of arithmetic. The author explains how this differs from *definable* in the way that provability differs from truth.

The definition is then extended to "strongly formally definable" and it is shown how every recursive predicate is strongly definable in any consistent extension of the specific theory for arithmetic originally set out.

This section and the first part of the book conclude with Church's theorem (which states that the set of all theorems of a consistent extension of this specific theory for arithmetic is not recursive) and the syntactic version of the Gödel-Rosser first incompleteness theorem (which states that there are undecidable sentences in any consistent recursive formal arithmetic).

The ensuing discussion provides both historical perspective and mathematical insight to the first part of this epic journey. After fourteen pages of exercises the second part of the book commences.

Part II (The Second Incompleteness Theorem)

Part II begins with a statement of the qualitative difference between what was accomplished in the presentation here of Gödel's first incompleteness theorem and what is about to be accomplished in the second incompleteness theorem.

First, the axioms of Peano arithmetic are given. These are followed by a technical discussion of the induction axiom and the introduction of the "less-than-or-equal-to" predicate with its properties carefully developed. Next, the characteristic function for a formula is given, as well as proper definitions for the subtraction, remainder and quotient functions. Their respective properties are developed and the precise meaning of an extension of Peano arithmetic is defined. With the addition of these newly defined (but familiar) functions, such extensions of Peano arithmetic are seen to be recursive extensions and number theory exists within this formalized system.

A method of coding is introduced for this context. All of the necessary apparatus is formally introduced: subtraction, multiplication, division, relative primality, the coding of sequences using Gödel's beta function and primitive recursion. Bold face sigma is introduced and it shown that any bold face sigma-one sentence in a language for an extension of the standard structure for a recursive extension of Peano arithmetic that is true in this structure is provable in the recursive extension.

The arithmetization of formal arithmetic continues. The objective, in the author's words, is to be able to "... test for properties of Gödel numbers..." and to "... enable the theory to 'reason about' formulas (using Gödel numbers as aliases of such formulas)" (page 265).

The concluding section of the book begins with the highly technical verification of claims made about the coding developed in the previous section. The verification takes place in an "extension by definitions" of Peano arithmetic. Finally, the derivability conditions and the fixed-point theorem are presented, enabling Gödel's famous version of the Liar's Paradox: *I am not a theorem*. These establish the syntactical version of his second incompleteness theorem.

The book attempts to anticipate points of confusion and gives practical, intuitive and heuristic explanations to aid the novice. However, sometimes a laconic definition is given and left without further explanation, where one would be desirable for someone new to logic, such as defining a theory as inconsistent when the set of all things provable in the theory is all well-formed formulas. Just a few more words here would not insult the reader's intelligence. There are instances in the book where further elaboration is given despite it not seeming necessary and other instances where seemingly identical definitions are given in different context, leaving the reader puzzled over why space and time are devoted to a demonstration of why the same things are the same. An example of this are the definitions of well-formed formulas (Definition I.1.8) on page 15 and propositional formulas (Definition I.3.2) on page 28, along with the proof that well-formed formulas and propositional formulas are the same (Metatheorm I.3.3).

The notation becomes somewhat tedious to keep track of. Unless you plan to read this book in one sitting, keeping track of the symbols becomes annoyingly time-consuming, given that the symbol list in the back of the book is incomplete.

Some of the asides (a.k.a. "important passages") are illuminating; others (there, apparently, to satisfy the expert) seem unnecessary and may confuse and bewilder the novice. An example of this is in the explanation for the book's particular approach to the assignment of semantics given after the introduction of *diagrams* in I.6.

However, among the periodic pointers on the proper use of terminology, one particularly noteworthy example can be found on the bottom of page 128, where the author admonishes: "One occasionally sees terminology such as 'computable partial functions' or 'recursive partial functions'. Of course, 'partial' qualifies 'functions' (not 'recursive' or 'computable'): therefore one hopes never to see 'partially recursive functions' or 'partially computable functions.'"

Effective communication is a balancing act. Getting everything said precisely and symbolically correct sometimes needs to be traded off with a clear, verbal description that uses heuristics to stimulate the intuition of the reader. This book at times struggles to keep its balance, especially in the beginning.

However, the book ultimately succeeds as a thorough and complete resource for "one of the most remarkable theorems of logic" (page x). I have already used the book successfully with a talented fourth year undergraduate and it stands among the dozen or so books that will remain within easy reach from my desk. George Tourlakis' *Mathematical Logic* is a highly worthwhile addition to the literature.

References

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The College of Charleston, Charleston, SC 29424
E-mail address: MIGNONER@COFC.EDU