John N. Crossley, Jeffrey B. Remmel, Richard A. Shore, Moss E. Sweedler (eds.), *Logical Methods: In Honor of Anil Nerode's Sixtieth Birthday,* Boston, MA, Birkhäuser Boston, 1993. x + 813 pp.

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Anil Nerode was born in 1932, at about the same time as recursion theory itself came to life as an independent field. In 1992 a conference was held at Cornell University to celebrate Nerode's 60^{th} birthday. *Logical Methods* is the proceedings of that conference.

Over the years Nerode's research has reached in various directions; often, though by no means always, his work has ultimately had its roots in the area of recursive equivalence types (RET's) and isols. J. C. E. Dekker initiated the study of RET's in 1953. Two sets of natural numbers A, B are said to be *recursively equivalent* if $f(A) = B$ for some 1–1 partial recursive function f whose domain contains A. Thus *recursive equivalence types*, the equivalence classes of this relation, form a recursion-theoretic version of cardinal numbers. As with the cardinal numbers, one may define addition, multiplication, an order relation, and so on. The algebraic and order structures of the RET's are much more complex than the Cantorian originals, however.

Of particular interest are the RET's that behave like finite cardinals. If a set A is not recursively equivalent to any of its proper subsets, A is called *isolated* and its RET is called an *isol*. The isolated sets are easily seen to be precisely those sets with no infinite recursively enumerable subsets. Like the whole set of RET's, the isols form a very rich system as far as algebraic and order properties. For example, there exists an uncountable collection of mutually incomparable isols. However, isols are much more tractable than RET's in general. In particular, they may be shown to share many properties with the natural numbers.

How does one show that? In the latter half of the 1950's, John Myhill got the ball rolling in this direction with his work on combinatorial functions and combinatorial operators. Every function f from N to N

(where N is the set of natural numbers) has a unique representation in the form $f(x) = \sum a_k C(n, k)$, where the a_k are integers and the $C(n, k)$ are the usual combinatorial coefficients.

If all the a_k are nonnegative, f is called a *combinatorial function*. An exposition by Dekker of combinatorial functions appears in the inaugural issue of THIS JOURNAL [D1990]. See also Dekker's highly readable book in French [D1966].

By thinking of natural numbers as finite cardinals, one has N as a subset of the RET's. The attraction of combinatorial functions is that there exists a natural way to extend recursive combinatorial functions to functions on the whole set of RET's (with range consisting of RET's). Myhill accomplished this extension process by characterizing combinatorial functions in terms of so-called *combinatorial operators*, certain maps on the power set of N. Combinatorial operators are then used to define the RET extension of a function. (For simplicity, I am outlining the "one-dimensional" case. Combinatorial functions of several variables and their corresponding operators are developed similarly.)

Enter Nerode. His paper [N1961] introduced the notion of *frame*. The definition of frame is very simple: a frame is just a collection of finite subsets of N, the only requirement being that this collection is closed under (finite) intersection. The purpose of a frame is to provide a means of approximating infinite sets by finite ones. Nerode generalized combinatorial operators to objects called frame maps and looked beyond the recursive combinatorial functions to *almost recursive combinatorial* (a.r.c., for short) functions.

Using the frame machinery and working within this expanded setting, Nerode accomplished the following:

1. He characterized the a.r.c. functions as the appropriate ones to extend to RET's. In particular, he showed that a function is a.r.c. iff its extension to RET's is defined on all isols and maps isols to isols.

2. Most significantly, he proved the metatheorem that the a.r.c. functions on N and their extensions to isols share the same properties, at least among the properties that can be expressed by universal Horn sentences in an appropriate language. In more concrete terms, this means that various results about isol arithmetic (e.g., cancellation laws) that had previously been proved on a theorem-by-theorem basis, were now seen, at one fell uniform swoop, to be inherited from the natural numbers. To quote Thomas McLaughlin, Nerode "cleared away the primordial jungle of *ad hoc* constructions in the algebra of isols" [McL1982].

For more on the history of the development of RET theory, see [C1981a], a transcript of a roundtable discussion with participants Nerode, Dekker, McLaughlin, Alfred Manaster, and moderator John Crossley, plus later additions by Myhill.

Just as Dekker effectivized cardinal numbers with his RET's, one may effectivize ordinals by imposing a computability requirement on the order isomorphisms one works with. During the 1960's Manaster and Crossley independently pursued such ideas. In 1969 Dekker similarly introduced recursive equivalence ideas into the context of vector spaces. Crossley and Nerode, in addition to specific work on vector spaces, developed a unified treatment of recursive equivalence that could be applied to a variety of structures. Their book *Combinatorial Functors* [CN1974] extends the combinatorial operator/frame machinery to a fairly general setting.

From this, Nerode moved on to other recursion-theoretic investigations of algebraic structures, becoming a principal exponent of what became known as effective algebra. Of particular note are the papers Nerode co-wrote with George Metakides that appeared beginning in the mid-1970's. Of course, the idea of investigating the recursive content of classical algebraic results had been around for many years. What set apart the work of Metakides and Nerode was the systematic use of priority arguments in this context. At the same time, their approach was also informed by the idea of uniformly approximating the infinite by the finite—the idea which underlay not only the basic notion of frame, but the category-theoretic scaffolding of the Crossley-Nerode book as well.

More recently, Nerode's RET/isol attention shifted back to the original setting of the natural numbers. But this time, in the spirit of computational complexity, the functions involved were required to be polynomial-time computable. The theory of so-called polynomial-time equivalence types was initiated by Nerode and Jeffrey Remmel, two joint papers of theirs appearing in 1990.

The above has been a sketch of a portion of Nerode's work. A much more comprehensive survey is given by Crossley and Remmel in the first article of *Logical Methods*. Their 75-page paper discusses in more detail the topics mentioned here; it also deals with many other areas of Nerode's research, RET-related and otherwise. These areas range from automata to nonmonotonic logic to the connections between isols and nonstandard models of arithmetic. The accompanying bibliography includes an occasional surprise, such as "An algebraic proof of Kirchhoff's network theorem," a 1961 *Monthly* paper by Nerode and H. Shank. However, the bibliography does not include abstracts; several abstracts by Nerode are listed in "A bibliography of Effective Algebra" by Crossley and Sara Miranda (pp. 251–290 in [C1981b]).

Another omission in *Logical Methods* is worth noting: there is no list of Nerode's doctoral students. The Crossley-Remmel survey sometimes, in passing, identifies a logician, such as Metakides or Louise Hay, as a Nerode student. And it refers to "the 34 Ph.D. students that he has superivsed to date,"but without naming names.

The volume contains 25 other papers by various authors. These are research papers in mathematics and computer science, many dealing with areas in which Nerode has worked. (Indeed, two of the papers were co-written by Nerode.) A somewhat random sampling:

"Prime isols and the Theorems of Fermat and Wilson" by Joseph Barback;
"Extracting programs from proofs by an extension of the Curry-Howard process" by Crossley and John Shepherdson;
"A bird's-eye view of twilight combinatorics" by Dekker;
"Multiple agent autonomous control — a hybrid systems architecture" by Nerode and Wolf Kohn;
"On the strength of Fraïssé's Conjecture" by Richard Shore.

One paper deserves particular mention here, namely, the historical article "Who put the 'back' in back-and-forth?" by Jacob Plotkin (whose name, in the list of conference participants, somehow got condensed to "Jain"). Plotkin traces the history of the back-and-forth argument commonly misattributed to Cantor. In 1895 Cantor proved that every countable dense linearly ordered set without endpoints is order-isomorphic to the rationals. However, his proof is not back-and-forth, just "forth". The back-and-forth argument as we know it is due to Felix Hausdorff (1907, 1914), but a "rudimentary form" of it had been introduced by E. V. Huntington in 1905.

In summary, like most volumes of proceedings, *Logical Methods* probably has libraries, rather than individuals, as its best market niche — especially because of the breadth of topics over which the papers range. Plotkin's article is of near-universal interest; the other will appeal more or less to various constituencies. In the area of RET's and related matters, the book is particularly valuable. Despite a journal literature of reasonable size in this area, comparatively little has appeared in book form, and much of what has appeared is not always easily obtainable. Thus the overall view offered by the Crossley-Remmel article, the close-ups of current research activity provided by several of the other papers, and the bibliographic material together make *Logical Methods* a very useful source of RET information.

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"This conviction of the solvability of every mathematical problem is a powerful incentive to the worker. We hear within us the perpetual call: There is the problem. Seek its solution. You can find it by pure reason, for in mathematics there is no ignorabimus." – David Hilbert, 1900