

The Apprenticeship of a Mathematician, by André Weil. Translated from the French by Jennifer Cage. Basel, Birkhäuser Verlag, 1992. 197 pp.

Reviewed by

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This is a volume of recollections by one of the most important mathematicians of our time, whose life spans almost the entire century. Partly an intellectual autobiography, it relates his formative years (in Paris, Rome, Göttingen) as well as his wandering years (from India and Russia to Brazil and the United States). This story of a mathematician's life stops in the middle of its course — “*al mezzo del cammin di mia vita*,” in the author's words. The unusually eventful life of a “quiet” mathematician who was imprisoned for the wrong reasons and almost killed inadvertently is told in a candid and direct style. Despite his fascination for beauty in art and his long-lasting interest in languages and poetry (Indian and Spanish among many), André Weil is no friend of rhetoric and sentimental effusion. The death of his sister, the great philosopher Simone Weil, whom he loved dearly, is recalled in a passage of restrained language and condensed emotion.

André Weil is not a man of many words. His recollection of Paul Valéry, who was reigning as poet laureate and academician at a time could be summarized as “31 is a nice prime number.” André Weil was 31 when he met Paul Valéry. Among mathematicians, he was on more equanimous terms with Jacques Herbrand, Elie Cartan, Henri Cartan, Jacques Chevalley, Jean Dieudonné, C. Ehresmann, C.L. Siegel, E. Artin and many important mathematicians of his time (among them Mordell, with whom he had an insignificant encounter).

For the student in foundations of mathematics, it is noticeable that Hilbert is remembered mainly for his dignity and his caustic wit, while Brouwer is quoted for his polemical attitude. Even within the Bourbaki circle, of Hilbertian spirit as far as the structuralist-formalist attitude is concerned, Hilbert's foundational stance does not seem to have played a major role, except maybe for the insistence on the axiomatic method. The ideas of structure and, more importantly, of isomorphism would not ripen before the concepts of category and morphism, but it seems that it is Bourbaki who is responsible for the precise meaning of isomorphism as structure-preserving bijection. Weil also says that he has introduced the notation for the null set thanks to his knowledge of Norwegian (the

letter “ø” is also found in Danish). As for structure, it was borrowed probably from the nascent structuralist linguistics (of Benveniste). Bourbaki began as an informal, although closely-knit group (somewhat reminiscent of the group around Max Dehn in Frankfurt).

As far as philosophy is concerned, André Weil does not think much of it, since one can practise it without knowing anything about it, or so it seemed to the young student he once was. On the other hand, his sister, an able philosopher, would get terrible headaches trying to read her brother’s mathematics.

Ordinary logicians will not find much to their taste (and to their height) in Weil’s work — nor does the ordinary mathematician make a quick profit of Weil’s classic of algebraic number theory, *Basic Number Theory*. But there is a mine for the constructivist logician in Weil’s results in number theory and algebraic geometry. Since the book under review does not tell the full story of Weil’s achievements in those fields, it might be worthwhile to give a brief account here.

In 1922, L.J. Mordell was able to prove that the rank of an algebraic (elliptic) curve of genus 1 over the field of rational numbers is finite. The problem was inspired by Poincaré’s paper “Sur les propriétés arithmétiques des courbes algébriques”, although the original idea goes back to Kronecker and to the work of Hilbert and Hurwitz on Diophantine equations of genus 0. In 1928, Weil succeeded in generalizing Mordell’s result to an arbitrary algebraic number field. Both results make use of Fermat’s method of infinite descent, which has a constructive character but does not always produce effective results (explicit bounds). In a comment (1979) on his 1929 paper “Sur un théorème de Mordell”, Weil complains about the fact that there is no effective method to decide if the equation

$$y^2 = P(x),$$

where P is a fourth degree polynomial, has a determinate rational solution; here, on a rare occasion, Weil appeals for aid from mathematical logic. Unfortunately, Matiyasevich’s result in 1970 on the unsolvability of Hilbert’s tenth problem — the inexistence of a general algorithm for Diophantine equations — has no bearing on this particular problem. Infinite descent is a finite form of the induction principle and it embodies a method of indirect proof (“*reductio ad absurdum*”), but since it is finitistic, it does satisfy the constructivist criterion of restricting the “*tertium non datur*” to the finite case.

Weil had the hope of proving Mordell’s conjecture, which says that on any algebraic curve of genus ≥ 2 , the number of rational points is finite. It was only in 1983 that G. Faltings could prove it using, besides descent theorems, many nonconstructive tools from the algebraic geometry developed by Weil, Grothendieck, Deligne and many others. The fusion of number theory and algebraic geometry in this century is due to a large extent to André Weil’s endeavor. In his 1949 paper “Numbers of solutions of equations in finite

fields", Weil had conjectured that the results he had obtained for curves (the Riemann hypothesis for nonsingular curves) could be extended to algebraic varieties of higher dimension. The number of solutions of equations in finite fields is equivalent to the number of rational points of an algebraic curve (variety). Here transcendental methods are used freely and in 1973 Deligne succeeded in proving Weil's conjectures, thus concluding a major chapter of contemporary algebraic geometry. It may be interesting to note that the junction of algebraic geometry and number theory has given rise more recently to what is called now arithmetic geometry, no doubt to put the emphasis on the arithmetical content of "abstract" algebraic geometry as illustrated for instance in the work of Paul Vojta on Diophantine inequalities. It is this arithmetical spirit, from Gauss and Fermat to Kronecker and Hilbert, that A. Weil has perpetuated. Kronecker is the foremost modern proponent of the arithmetic ideal. His *Grundzüge einer arithmetischen Theorie der algebraischen Grössen* purports to show that algebraic, real and transcendental numbers are but an extension of arithmetic ("*eine Gebietserweiterung der Arithmetik*"); such extensions can be dealt with in terms of indeterminates ("*Unbestimmte*") that can be reduced to a pure arithmetic (algebraic) calculus. Kronecker also introduced the notion of "*Rationalitäts-Bereich*", "domain of rationality," because he felt that "*Körper*" ("corps" in French and "field" in English) was too materially laden. André Weil has recognized the pioneering work of Kronecker (see his 1950 paper "Number theory and algebraic geometry"), as he has emphasized the importance of Fermat's invention, among others things, of the method of infinite or indefinite descent, as Fermat says — see Weil's book *Number Theory: An Approach Through History* (Boston/Basel/Stuttgart, Birkhäuser Verlag, 1984).

The clear distinction between elementary (arithmetic or algebraic) methods which are — often — of constructive import and transcendental (analytic) methods is a main feature of modern number theory. Even before the revolutionary results of Selberg in 1948 on elementary proofs of the prime number theorem and Dirichlet's theorem on the infinity of primes in an arithmetic progression, Weil had stressed the benefits of purely arithmetical or algebraic methods without discarding, of course, the transcendental ones. Constructive proofs lend more information, they can furnish effective bounds and, philosophically speaking, they are more reliable if one counts on secure grounds or what Hilbert referred to as "*Sicherung*". And as far as the arithmetical ideal or the ideal of the arithmetician is concerned, André Weil can be seen as the heir to the tradition represented by Fermat, Lagrange, Gauss and Kronecker.

I have reviewed mainly the mathematical stream of André Weil's recollections. The best companion to this part of the book is certainly Weil's *Œuvres scientifiques: Collected Papers*, vol. 1: (1926-1951) (New York/Heidelberg/Berlin, Springer-Verlag, 1979), followed by the author's commentary in French of all the papers of that period. It is from this work that I have borrowed most of the information on Weil's mathematical work. One recommendation that Weil has repeatedly made is to read the original work of mathe-

maticians, rather than read about it. It is certainly a recommendation that can be applied to the work of André Weil.

Raymond M. Smullyan, *Gödel's Incompleteness Theorems*, New York, New York, Oxford University Press, Inc., 1992. xiii + 139 pp.

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Certainly, Gödel's work on incompleteness enjoys one of the highest profiles in mathematical logic. Not only are his First and Second Incompleteness Theorems familiar to every logician, but through various popularizations this material has managed to impinge on the general consciousness. (And without even needing cute computer-generated pictures!)

Smullyan's *Gödel's Incompleteness Theorems* is an introduction, but — unlike several other of his books — not a popularization for the public at large. To quote from the Preface, the book is intended "for the general mathematician, philosopher, computer scientist and any other curious reader who has at least a nodding acquaintance with the symbolism of first-order logic...and who can recognize the logical validity of a few elementary formulas. A standard one-semester course in mathematical logic is more than enough [background]." On the other hand, again quoting from the Preface, "There is a good deal in [Chapter VII] that should interest the expert as well as the general reader." Smullyan lives up to his aims. The book provides a highly accessible, user-friendly introduction to incompleteness. At the same time the treatment is rigorous and contains material that even a professional logician can find informative and interesting.

Smullyan goes right to the heart of the matter in Chapter I by stripping incompleteness to its essentials. What basic features does a language need for an incompleteness theorem? Using these features, how does one prove such a theorem via diagonalization? In a sense, much of the rest of the book consists of an elaboration of the first chapter, examining how the abstract incompleteness scenario plays itself out in progressively more sophisticated contexts. Smullyan discusses, in turn: