Review of *A Peircean Reduction Thesis: The Foundations of Topological Logic* by Robert W. Burch. Lubbock, Texas Tech University Press, 1991.

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Burch's book will be of interest to historians of logic primarily because of its avowed goal of completing a project which was inaugurated by Charles S. Peirce and continued by Peircean scholars, but will teach them little about the history of algebraic logic in general, or about Peirce's work in particular. The author in fact denies that his book is a piece of Peirce scholarship *per se*, stating that it is instead a contribution to original research in an area begun by Peirce. The core of the book is Burch's proof of a formalized version of Peirce's Reduction Thesis, according to which all polyadic relations with more than three terms can be reduced to combinations of monadic, dyadic, and triadic relations. In order to carry out his proof, Burch painstakingly and cleverly develops a formal axiomatic system, dubbed "Peircean Algebraic Logic" or PAL, which is described (p. 5) as "an attempt to amalgamate various systems of logic that Peirce developed over his long career." It takes as its syntactical cue the graphical representation of relational logic that Peirce worked out as the analogues of chemical valency and which were inspired in part by the work of Peirce's British colleague, Alfred Bray Kempe (1849–1922), in applications of algebraic logic to geometry and the work in topology of Johann Benedikt Listing (1808–1882). The topological model finally settled on is Peirce's "existential graphs" for first-order quantificational systems. Burch also makes a connection to more recent work by designing PAL in such a way that some of the features of PAL specifically correspond to features of Paul Bernays' algebraic logic as presented in Bernays' (*1959*) paper *Über eine natürliche Erweiterung des Relationenkalküls*. Except for systems of logics developed by Peirce scholars and designed specifically to prove the Reduction Thesis, however, Burch largely ignores current research in algebraic logic.

The best example of a system system designed by Peirce scholars for proving the Reduction Thesis, and the one upon which Burch most heavily depends, is Herzberger's (*1981*) "bonding" algebra, which, however, substitutes for strictly algebraic proof a physical dependence upon "hooks" of relations which allow those relations to grab hold of one another in accordance with the analogy of chemical valency. PAL is thus a system combining the (*1981*) Herzberger formalization of Peirce's bonding algebra with Peirce's graph-theoretical representation of his algebra.

Burch states (p. 1) that Peirce was the "creator of the sort of generalized algebraic logic that has become well known in this century," and in particular points to the work of Tarski, Lyndon, Henkin, Bernays, Halmos, and others as examples. He cites in particular the work of Tarski, Lyndon, and Henkin on representation for relation algebras and cylindric algebras. Except for this brief mention, however, Burch makes no attempt whatever anywhere in his book to examine the work of Tarski, Lyndon, Henkin, or others, let alone the much more recent work of Tarski and Givant, whose (*1987*) book *A formalization of set theory without variables* can be seen as a comprehensive and systematic summary of the results of some four-and-a-half decades of work by Tarski and his students on the question of the representability of relation algebras. Nor does Burch consider how the work of Tarski and his students impacts his own case. Instead, he measures the consistency of the Reduction Thesis for PAL with Löwenheim's results of 1915 and with Quine's results, in particular with the work in Quine's (*1954*) paper on reduction of formulae of an interpreted theory Θ formulated in FOL into an interpreted theory Θ' formulated in FOL and containing a single dyadic predicate. Burch argues that Quine's reduction involves the notion of "Thirdness", and is therefore fully consistent with the Reduction Thesis for PAL.

The heart of Burch's book, the Reduction Thesis for PAL, according to which there is a term t of PAL which corresponds to a term w of standard classical first-order predicate logic with identity (FOL) and which is a translation of w, is found in the Representation Theorem for PAL (*Theorem* 8.1, p. 102). The term w of FOL corresponds to the term t of PAL in the sense that there is an "enterpretation" or "extensional interpretation" of PAL in which the terms of PAL are classes of n-tuples over a domain D that is a set, a function * from terms of PAL to finite sequences of classes of n-tuples over D. Then the pair $(D, *)$ is the enterpretation of PAL in FOL. If t is a translation of w in PAL and $M = (D, F)$ is any interpretation (in the standard model-theoretic sense) of FOL, then for $(D, *_{fD},)$ the corresponding enterpretation of PAL, we have $f(w) \approx *_{fD} (t)$. The corollary to the Representation Theorem for PAL states (*Corollary* 8.1.1, p. 103) that any well-formed formula of FOL may be translated into a term t of PAL such that $f(w) \approx *_{fD} (t)$. There follows the Extensional Reduction Theorem for PAL (*Theorem* 9.1, p. 109), together with related theorems for "intensional" interpretation in PAL and Herzberger's reduction theorem for bonding algebras, which together are presumably equivalent to Peirce's original claim, as that claim is understood from the perspective of Burch's PAL, that all polyadic relations with more than three terms can be reduced to combinations of monadic, dyadic, and triadic relations.

Burch's creation is neither a new rocket waiting to be launched nor a history of flight. Continuing the metaphor, it is best likened to a computerized model of Leonardo da Vinci's flying machine, as based upon Leonardo's sketches and a papier mâché model. As such it is a museum curiosity and is likely to be ignored by most active researchers in algebraic logic. Despite the assertion that Peirce was the creator of the algebraic logic that has been pursued in recent times by Tarski and his students, Burch makes no efforts to engage the work of contemporary algebraists, either to utilize their results or to compare their results with his own. This is rather a shame, since Tarski began his work precisely from his endeavor to answer the same questions that Peirce's Reduction Thesis has posed, and certainly at least to deal with closely related questions. Tarski and Givant (*1987*, p. xv–xvi) began by noting that Schröder was the first to ask whether all elementary statements about relations are expressible as equations in the calculus of relations. Schröder thought the answer to be positive, but Löwenheim critiqued Schröder's proposed solution and included in his own (*1915*) paper Alwin Korselt's negative solution to Schröder's question. Tarski (*1941*) extended Korselt's result and then asked whether there is an algorithm for deciding in every case whether an elementary statement about relations is expressible (as an equation) in a given equational theory of abstract relation algebras without variables, quantifiers or sentential connectives, and which is equipollent to a particular fragment of first-order logic having one binary connective and containing just three variables. Around 1971, Michael Kwatinetz showed that the answer to Tarski's question is negative; in (*1981*), Kwatinetz published his proof that the set of elementary statements of the calculus of relations which can be equivalently formulated using just three variables is not recursive. If Burch then ignores the work of Tarski and his students, it is perhaps because his main result appears to contradict the results established by Tarski and his students. In (*1987*), Tarski and Givant present their proof of the Mapping Theorem, according to which the calculus of relations is equipollent (in means of expression and proof) to a three-variable fragment of FOL with a binary relation, and present a fourteen page proof (*1987*, 110–124). Earlier Henkin (*1973*) proved that, although the associativity of relative product can be expressed by an FOL-formula with binary relation symbols and although that FOL-formula has only three variables, it nevertheless cannot be expressed in the usual FOL-axioms without using four variables. A relation algebra is *representable* if it is isomorphic to a proper relation algebra, that is, to a relation algebra all of whose equations are expressible in the three-variable fragment of FOL. The literature abounds in examples of nonrepresentable relation algebras, and much of the energy of Tarski and his students since 1941 has been devoted to determining which algebras are representable and which are not. Histories of their work are given in (Maddux *1991*, especially pp. 445–449) and (Anellis *1994*). Burch fails to take any of this work into account.

An historically interesting question concerns the basis for any possible connections between the work of Peirce on the one hand and of Tarski and his colleagues and followers on the other. In particular, it is striking that so much of the work in algebraic logic of Tarski, his colleagues, and followers, from the 1940's through the 1980's, should be devoted to determination of the representability of relation algebras and to some extent to the determination of which relation algebras are capable of expressing all equations concerning relations in a three-variable fragment of FOL, even if it should be that the modern algebraic forms of these questions turn out not to be equivalent to or directly connected with the modern equivalent of Peirce's claim or conjecture that all polyadic relations could be reduced to combinations of monadic, dyadic, and triadic relations.

Burch's work falls outside of the main lines of contemporary research in algebraic logic because he makes no serious attempt to explore the connections of his results with those of Tarski and Tarski's followers. Burch simply fails to engage the contemporary concerns of researchers. Indeed, his work is totally out of touch with current research in algebraic logic.

On the surface, Peirce's Reduction Thesis, stated in its classical Peircean terms, is unrelated to the work of Tarski and his students on representable relation algebras, since Peirce's Reduction Thesis concerns the properties of relations, and specifically whether every polyadic relation of adicity greater than three is analyzable as a combination of monadic, dyadic, and triadic relations, while the question for the Tarskians concerns the expressive power of the three-variable fragment of FOL through its ability to express equations of relation algebras involving more than three variables in a sentence of the three-variable fragment of FOL. However, Burch himself shows the relevance of the work of the Tarskians by making heavy use of his Representation Theorem for PAL, which makes the same claims of isomorphism between PAL and FOL that Tarski's Mapping Theorem makes for representable relation algebras and FOL. If, then, Burch's PAL turns out to be equivalent in a nontrivial way either to modern proper relation algebras or to any representable relation algebra, then his entire enterprise is in serious danger, and in particular, he is faced, as I point out in (*1994*) (in which I trace the steps that led from Peirce's work to that of Tarski and his students), with the possibility of numerous counterexamples — such as the one noted by Henkin — to the Reduction Theorem for PAL which he defends. In fact, Burch claims (p. 105) that the representation theorem for PAL "shows that, when triadic relations are included in the resources of PAL, these resources are adequate for expressing all relations" in the particular sense "of the translatability of well-formed formulae of quantifica-tional logic into terms of PAL." Moreover, this implies that "all relations may be expressed as constructions from relations exclusively of adicities 1, 2, and 3," while the reduction theorems for PAL explicate and define the conditions for this construction.

Burch has, however, left himself an "escape route" from the criticisms of contemporary algebraic logicians, arguing (pp. vii–viii) that there are several senses of "reduction", in particular Peirce's intensional sense, and the extensional sense shared by nearly everyone else, by those who work within the framework of "exact logical terms" — to use Burch's expression. The extensionalist interpretation sees relations in terms of sets of *n*–tuples, Burch follows Peirce's intensional conception, according to which relations are "something *sui generis*, they were relations *as such*, relations *simpliciter*, relations *period*," (although they might also on occasion be analyzed by Peirce in terms of the *n*–tuples that satisfy those relations). Despite the fact that Burch claims (p. viii) that his Reduction Thesis might incorporate results about sets of *n*-tuples, Burch's "escape" from the results of Tarski and contemporary algebraists is at least partially obviated by his having given an extensional reduction theorem for PAL (p. 109) as well as an intensional reduction theorem for PAL (p. 113), along with extensional and intensional versions of Herzberger's reduction theorem. Here then is a very crucial sense in which the correctness of Burch's reduction theorem proofs must yet be determined, even apart from the question of whether Tarski's irreducible sentences count as counter-examples to their proofs.

It is regrettable from the historian's point of view that neither Burch nor most Peirce scholarship have explored in detail Peirce's own sketch of a proof of his Reduction Thesis. Although that sketch was included in a letter to William James of August 1905 which was published in part by Carolyn Eisele (*1976*, especially pp. 832–833), it appears to have been ignored — or perhaps unrecognized — by Peirce scholars such as Herzberger who have sought to defend and proof the Peircean Reduction Thesis. It is likewise not mentioned by Burch. This is especially disappointing, since Burch's proof appears (at least on a cursory inspection) to closely follow Peirce's in its general outline and structure. The disappointment is heightened by Burch's expression of confidence (p. viii) that "a close study of Peirce's logical writings will indeed tend to confirm that most of this work's details are either accurate or at least approximate representations of Peirce's thinking." A detailed reconstruction of this proof will, however, be found in (Anellis *1994*). The difficulty in the use, whether by Peirce (for example in (*1897*) or by present-day Peircean scholars (especially Herzberger (*1981*) and Burch in this book, of such so-called "valency proofs" for the reducibility of tetradic and polyadic relations to products of monadic, dyadic, and triadic relations, which are diagrammatic, is that they have not yet been shown to be algebraically justified. In this sense, too, the correctness of Burch's reduction theorem proofs must yet be determined, even apart from the question of whether Tarski's irreducible sentences count as counter-examples to Burch's proof.

The "museum curiosity" character of this book will make it of mild interest to historians of logic. And as a piece of original research, it is nevertheless so far from the mainstream of current research on representable relation algebras and so far removed in style and spirit from this contemporary research that it is likely to be of only slight interest to working mathematicians. Indeed, I predict that Burch's work will receive as much attention from workers on representable relation algebras as Burch has afforded the results of those workers.

To summarize, if Burch's work is to have any relevance for current researchers, he must first translate sentences of FOL into equations of PAL and then show that the resulting translations are equivalent to the original sentences of FOL, since it does not follow that the translations in fact are equivalent to the original FOL-sentences. Moreover, if there is no such equivalence, then his reduction theorem fails to apply. In fact, however, Burch simply claims, but makes no effort to prove, equivalence. Once this is done, Burch would have to show that PAL is equivalent to a representable relation algebra if his reduction theorem is to have any application to the work of Tarski and his followers. But Burch has simply ignored this question altogether — as indeed he must — since the existence of irreducible sentences as shown by Tarski and Givant contradicts Burch's main results in case PAL is a representable relation algebra, whereas his work lacks any value whatever for current research if PAL does not belong to the family of relation algebras that is currently being studied.

There are two formatting flaws in this book which will be immediately noticeable. One, which is a minor irritation, is that the formulae, including statements of theorems, etc., are not set in italics, and neither are the terms of Burch's PAL syntax. A more serious flaw from the standpoint of scholarship concern the incomplete references given in the bibliography. In particular, pagination is not given for articles appearing in books. In addition, Burch's bibliography gives the subtitle of Jean van Heijenoort's anthology *From Frege to Gödel*, but not its title, and the year of its publication is incorrectly listed in the bibliography (as well as in the endnotes at pp. xi, 122) as 1966 rather than 1967. It should also be noted that Burch gives the reference to Quine's (*1954*) in its reprinted version, rather than its first publication, whereas Tarski's (*1941*) is cited in its original publication rather than in Givant and McKenzie's edition of Tarski's collected works. Inclusion of an index of numbered formulae and of an index of symbols would have been useful.

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