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Bertrand Russell, *Philosophical papers 1896-99*, edited by Nicholas Griffin and Albert C. Lewis. Volume 2 of *The Collected Papers of Bertrand Russell*.London/Boston/Sydney/ Wellington, Unwin Hyman, 1990; available from Routledge, London.

Reviewed by

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I once heard an undergraduate philosophy instructor seriously proclaim that Bertrand Russell's reputation was so great that, if he chose to scribble notes on bathroom tissue, any distinguished philosophical journal to which he might submit those scribblings would publish them forthwith and without question. Something of the sort has come to pass. It is also popular opinion that Russell could write a paper straight through, without having to make corrections. While this may have become true through many decades of practice on Russell's part, it was certainly far from true for the pre-*Principia* Russell who had barely finished his studies at Cambridge when the materials included in the present volume were first penned.

This book is the result of a considerable amount of scholarly effort by historians of mathematics, by the secretarial and production staff of the Bertrand Russell Editorial Project (BREP) and the personnel of the Russell Archives at McMaster University in Hamilton, Ontario. It took over a decade for the preparation of this volume; work on it had already begun by Gregory H. Moore and philosopher Nicholas Griffin before I arrived for a one-year stint at BREP in June 1982. After I left, work was taken up again by Moore and Albert C. Lewis, with Griffin providing the continuity during the entire period. Was it worth the effort, the time?

The materials in this volume are divided into four parts. The first contains miscellaneous writings on philosophy of mathematics and philosophy of science, much of it intended for a project developing the "dialectic of the sciences", written during the period from 1896 to 1899 with the intent of presenting a Hegelian system of science that began with general concepts of mathematics and, working from the general to the concrete, to a consideration of physics. Some of these writings were incorporated in revised form into Chapter 4 of Russell's *My Philosophical Development*. Also included are a number of book reviews that had been published during the period, in particular the review of Arthur

Hannequin's *Essai critique sur l'hypothèse des atomes dans la science contemporaine* [*1896*] and published in the journal *Mind* that same year [Russell *1896*], from which Russell first learned about Cantor's work in set theory, and of Louis Couturat's *De l'Infini mathématique* [*1897*], also published in *Mind* [Russell *1897a*], which presented a much more accurate and favorable treatment of Cantorian set theory than is to be found in Hannequin's book. Logicians will be especially interested in these two reviews and in such papers from this section as "On the Relations of Number and Quantity" and "On Some Difficulties of Continuous Quantity" in which Russell develops his earliest treatment of set theory and foundations of analysis. In these writings, which were the basis of my studies of Russell's earliest work in set theory (especially my [*1984*; *1986*; *1986a*; *1987*; *1987a*; *1987b*]), Russell showed a complete misapprehension of set theory and analysis. He ignored, or failed to understand, for example, the distinctions between natural numbers, the rationals, irrationals, and reals, or even the distinction between natural numbers and cardinals, upon which so much of Cantor's work depends.

I would venture to say that, had it not been for his collaboration with Whitehead on the *Principia Mathematica*, Russell today would be largely forgotten, an insignificant figure in the history of the philosophy of mathematics with some competence in philosophical foundations of geometry and certain aspects of universal or "symbolical" algebra, and a thorough misunderstanding of real and infinitesimal analysis and Cantorian set theory.

In *My Philosophical Development* [*1985*, 29], Russell blamed the teaching methods at Cambridge for his revulsion with mathematics, asserting that the teaching was designed to inculcate clever tricks which would enable students to pass the examinations rather than to comprehend the mathematics itself or the logical structure of proofs. This alone does not explain why Russell would have difficulty understanding analysis and set theory, since Forsyth, Hardy and Whitehead were among those who went through the same system at about the same time. Another explanation which we could consider is the one which Russell's acquaintances at Cambridge assumed to have been the cause of Russell's difficulties with his mathematical studies in 1892: it was, Russell wrote in *My Philosophical Development* [*1985*, 29], "supposed that reading philosophy was what had spoilt my mathematics"; but Russell lays the blame there on a bout of influenza. In fact, all of Russell's writings in philosophy of mathematics were at this time colored by a strong streak of neo-Hegelianism that caused Russell to search for antinomies. In his efforts to study real and infinitesimal analysis, Russell's Hegelian distortions for once neatly fit with the well-known criticisms which George Berkeley in his *Analyst* of 1734 had levelled against the Newtonian calculus concerning fluxions which are treated on the one hand as infinitely small quantities and on the other hand as zero, the famous "ghosts of departed quantities" (see, e.g., [Smith, *1959*, 633)]. Most of the "difficulties" which Russell sees with the calculus, with numbers and with infinite sets in the first set of papers in this volume arise from the attempt to apply a finitist and discontinuous, or atomistic, philosophical view of mathematical reality to infinite sets and continuous processes. I have already detailed the specific errors to be found in Russell's treatment in the writings in the first part of this volume of both real and infinitesimal analysis and with his appraisal of Cantorian set theory [Anellis *1984*; *1986*; *1986a*; *1987a*; *1987b*] and pointed out the Hegelian basis of Russell's assertions by comparing his dialectical idealism with similar statements based on Friedrich Engels's dialectical materialism [Anellis, *1987*]. (This is not to say that the Hegelian philosophy of mathematics with which Russell approached these writings and the deliberate effort to identify or uncover antinomies were not without their advantage; for without these endeavors Russell may not have discovered the paradox that bears his name and made him famous.)

The second and third parts, as well as the "Various Notes on Mathematical Philosophy" from Part I, are starts at the writing of a book which never was completed whose aim was to provide Russell's fulfillment of his so-called "Tiergarten programme", formulated in 1895 and described [Russell *1967*, 125] as the writing of "one series of books on the philosophy of the sciences from pure mathematics to psychology" and "largely inspired by Hegelian ideas" (for more detail on the program, see [Griffin, *1988*; *1990*]). Most of the material on calculus, set theory, and number theory included in Part I was written in 1896 and 1897, whereas the material in Part II, "An Analysis of Mathematical Reasoning" was written in 1898, and the material in Part III, "The Philosophy of Mathematics" was written in 1898-1899.

Part II takes its name from the title of the book on which Russell worked during 1898. The material has been organized by the editors into three sections: manuscript material, typescript material, and fragments of early drafts. If one looks carefully, there is a hint here of the transition of Russell's interest away from providing a Hegelian synopsis of mathematics and science towards a concern for the relationship between logic and language, if not for presenting a fully developed logicism. The first glimmerings of this shift probably arose precisely out of Russell's Hegelian concern in his work (found in Part IV of this volume) on the philosophical foundations of geometry to settle the questions of the analyticity *versus* syntheticity of geometrical propositions and whether Euclid's axioms are necessarily true or empirically contingent. Thus, the first chapter of the first "Book", on manifolds, of "An Analysis of Mathematical Reasoning" opens with a discussion of "The Elements of Judgment", followed by a chapter on "Subject and Predicate" before taking up the topic of manifold. Here, a manifold is defined (p. 179) as a "collection of terms having that kind of unity and relation which is found associated with a common predicate:" therefore, a manifold is precisely Cantor's *Mannigfaltigkeit*. At this point Russell introduces the term *class* as a synonym for *manifold*. He concludes (pp. 184-185) that though "a manifold...may, for purely numerical purposes, be a mere assemblage" but that it must be coextensive with the terms of which some predicate can be asserted," so that a manifold is the extension of a concept found in traditional logic.

Having once defined manifolds, Russell was prepared in the next chapter to consider "the true topic of Symbolic Logic" – the mutual relations of addition and synthesis, after which he defines this "Logical Calculus" as the science which deals "with manifolds as such" and he leans heavily on Whitehead's *Treatise on Universal Algebra* to discuss the relations of symbolic logic, including especially equality and equivalence. The remainder of the book deals with increasingly concrete applications of the symbolic logic, moving from number to quantity. Much of the material from these chapters are missing and it is difficult to judge how much progress, if any, Russell was making towards understanding Cantorian set theory or the work in real analysis of Weierstrass and his colleagues, but the surviving paragraph of the chapter on infinity (p. 234) does not offer any suggestion of greater comprehension.

Part III contains *nachgelassene* material from two separate attempts to write a synoptic work on the foundations of mathematics. The first contains the extant parts of two chapters, the first on cardinal numbers, the second on ordinal numbers. In this treatment, cardinals are viewed as adjectives which apply to manifolds taken as a whole, not necessarily as the *"Anzahl"* (understood in the colloquial sense of the cardinality of a *Zahl*) or power of the manifold. In the next chapter, on ordinals, Russell distinguishes between cardinals and ordinals by noting (p. 251) that ordinals involve the notion of *order* and that "the ordinal numbers involve and presuppose the cardinal numbers, but the converse does not seem to be the case." Now Russell rescues his treatment of cardinals from the previous chapter by stating (p. 251) that cardinals express the sum of terms of manifolds. At last, then, Russell has apparently begun to admit the difference between a *Zahl* and its cardinality. But it is still unclear that he now understands the difference between natural numbers and cardinals. Nor is he prepared to talk about the real. He still speaks (p. 255) of the conception of "the number infinity, as the last term of a series which has no last term" in which all the terms are natural numbers. He has not yet come to recognize Cantor's transfinite; he still clings to his belief that the concept of the number infinity involves a contradiction, and he revives the old arguments which he used in the writings from 1896-1897, in which no distinction is made between the actual infinite and the potential infinite.

The material from his next attempt, "The Fundamental Ideas and Axioms of Mathematics", is the last before he embarked on writing *The Principles of Mathematics*. It is clearly somewhat more sophisticated in its recognition and treatment of number theory than any of the previous attempts. But it is still clearly deficient in its understanding of the infinite and lacks the technical acuity and breadth of familiarity with the technical literature found in the *Principles*, and the approach and concerns remain largely philosophical rather than mathematical. At this stage in Russell's thinking (as a vestige of his atomism), the natural numbers are built from the fundamental *unit*, the number one, which is primitive. We find then that "$1 + 1 \neq 2$" still means for Russell (p. 288) that if one adds one unit to one unit, we have two *units*, or that "*two* is a mere abbreviation" or synonym, for "one unit and one unit". In a fragment (p. 298) for part of the "Fundamental Ideas", he declares that "1 seems to mean exactly the same as *term* or *concept* or *logical subject*" and much of this material is devoted to examining the linguistic conditions under which a proposition of the form "1+1 = 2" is true or false and what it means ("if A is one and B is one, then A and B are two" and 1 is the logical subject, 2 is a predicate). It may be said in Russell's defense that the idea for replacing "1" as the primitive with "0" is due to Padoa, in his "Note di logica matematica" (*1899*), and that it did not takes its place in Peano's axiomatic presentation of number theory until the fifth (*1908*) edition of the *Formulario*. And in a note (p. 296) for the "Fundamental Ideas", he at last comes to accept the notion of irrationals. In the extant parts of the "Fundamental Idea", the infinite is treated only in the "Synoptic Table of Contents"; there is no surviving textual material in the *nachgelassene* parts of the manuscript that deals with the question of the infinite. However, the "antinomy" of the infinite that is found in Russell's earlier writings, that is in his misconstrued direct attempts from 1896 and 1897 to deal with real and infinitesimal analysis and Cantor's transfinite, still survives in the "Synoptic Table of Contents" (p. 267).

Over all, there is very little improvement exhibited in Russell's comprehension of set theory, number theory, and foundations of analysis. The contrast between the materials found here and the *Principles* is enormous, the improvement between these materials and the *Principles* – not to say the *Principia* – is exponential when set against the slight improvement between the writings of 1896-1897 and those of 1898-1899. It is difficult to believe that the same person who wrote the works presented in this volume is the same person who wrote the *Principles*. It is impossible to believe that the author of the material in this book could have had a hand in writing the mathematics in the *Principia*.

One sees here in these several uncompleted attempts at writing a book on mathematics for the Tiergarten programme, and particularly with the aid of the textual notes included in BREP's editorial apparatus, that Russell, certainly at least at this stage of his career, did not sit down to write at the first attempt a flawless manuscript. In the headnote to the manuscript "The Fundamental Ideas and Axioms of Mathematics", we are even told (p. 263) that

> "Fundamental Ideas" cannot be clearly separated from either "An Analysis of Mathematical Reasoning" or the 1899-1900 draft of "The Principles of Mathematics", and there is no sure way in which we can now recover the boundary Russell had in mind between these works.

It is added (p. 263) that "Russell often found it easier to rewrite material entirely rather than to revise it... ." Thus, one myth should be disspelled.

The fourth part of the book contains a collection of Russell's writings in geometry from the period 1898-1899, some of it unpublished, but also in particular English translations of his two papers on the philosophy of geometry, "Are Euclid's Axioms Empirical?" [*1898*] and "The Axioms of Geometry" [*1899*], first published in French in the *Revue de métaphysique et de morale*, that grew out of his reply to criticisms of Louis

Couturat and Henri Poincaré in response to their consideration of his book *An Essay on the Foundations of Geometry* [*1897*]. The focus of discussion is whether geometrical axioms are *a priori* (necessary) or *à posteriori* (empirical) and whether Euclidean geometry is mere convention or either true or false. The first appendix contains the original French texts of the two papers that appeared in the *Revue*. Russell's work on geometry and foundations of geometry is important in another respect as well: it helped shape his later logicist program. In particular, the existence of non-euclidean geometries led him to endeavor to see the axioms of "meta-geometry" as as the most basic principles common to all geometries, euclidean and non-euclidean alike. (In the *Essay*, he therefore argued that non-euclidean geometries are just special cases of projective geometry.) The questions raised in Russell's mind by the lack of proofs for the axioms of euclidean geometry and the creation of competing varieties of non-euclidean geometry, each in competition with euclidean geometry, provided Russell with the motivation for his work not only in set theory and foundations of analysis, but also in logic and foundations of mathematics generally (see [Anellis *1992*] on the role which the work in geometry played in the development of Russell's logical program).

Appendix II contains miscellaneous notes on mathematical physics and mathematics, including a treatment of the notion of *geometrical series*; Appendix II contains extracts from Russell's mathematical notebook of 1896, all of which, except for half a page on infinitesimals, are given over to notes on Cantor's writings in set theory. The editorial apparatus is augmented by headnotes annotations to the papers; these help identify the sources which Russell used, track down obscure references, and explain the context of Russell's ideas. The editors have done an excellent job of providing the historical and intellectual background for the materials.

The materials published in this volume, taken cumulatively, give the impression of a young man, barely out of college, who, having held his thoughts in check while in school, was now impelled by a strong burst of loosed energy too long pent up, to write down every thought he had ever had up to that time. This frenetic work pace, together with the philosophical baggage that distorted Russell's view of the work in analysis and set theory (and most of all of Cantor's work in set theory), might help to account for the egregious errors in Russell's first attempts to understand Cantorian set theory which I have enumerated in previous papers, including, for example, his failure to understand or accept the distinction between natural numbers and the reals. But I am not totally convinced that this by itself, or even in concert with Russell's "poor" mathematical education, is sufficient to explain Russell's difficulties in comprehending Cantor. I spent a year working on many of the materials found in the first part of this volume, and the longer I worked with the materials, the more I felt that there was a deeper problem than Russell's distorting philosophical prism or his poor mathematical training. This may appear to be unfair to Russell, and it leaves the problem of explaining how Russell could have been so "dense" – excuse the pun – about Cantorian set theory during this time and still have evolved in only a few short years into the titan of logic who co-authored the *Principia*. Peirce was studying Cantor's work at the same time and those of his unpublished writings on Cantor which I have seen demonstrate a clear understanding of what Cantor was trying to do; indeed, Peirce was already able to anticipate some of the results which Cantor was still developing. But Peirce was a mature and sophisticated mathematician whose life was already nearing its end. Louis Couturat, on the other hand, was Russell's contemporary and his *De l'Infini mathématique* showed a sound understanding of Cantorian set theory and gave Cantor's work an enthusiastic endorsement. There is something disturbing in the fact that Russell's acumen in set theory and logic should be detectable while he was under Whitehead's guidance, but neither before nor after he was under Whitehead's direct tutelage. The discomfiture increases in the face of the evidence uncovered by Victor Lowe [*1985*, 121-294] that Whitehead frequently found it necessary to correct or revise the many mistakes in Russell's proofs for the *Principia* and concluded [*1985*, 291-292] that the mathematics (as opposed to the philosophical contributions, namely the work on definite descriptions) in *Principia* was Whitehead's.

One might be reminded that Russell specifically admitted [*1944*, 11] that he had not even heard of Weierstrass until 1896, when in that year he visited the United States and there first learned of the nineteenth-century continental European work in real analysis. Russell made this point again in *My Philosophical Development* (*1985*, 30), adding that he soon thereafter began reading, among other works, "Dini's *Theory of Functions of a Real Variable*, and several French *Cours d'Analyse*." We cannot determine precisely when Russell read these modern works on real analysis; his reading log shows that he read Durège's *Theory of Functions* in August 1897 – that is, one full year after reading Couturat, that he read Harkness and Morley's *Introduction to Analytic Functions* in March 1899 and Dini's *Theorie der Functionen* in May 1900. In the end, it probably makes little difference whether Russell knew anything of real analysis before attempting to read Cantor, as we can only guess at whether Russell would have understood these any better than he understood Cantor at the time. Nor does Russell make it clear, after reciting [*1985*, 30] a list of mathematics books which he read after 1896, which of those books he "afterwards discovered, [were] quite irrelevant to my main purpose" (although we may assume from the context that he was looking for texts that would help him with his work on the foundations of geometry rather than works that might help him out of his difficulties with either calculus or Cantorian set theory).

In the end, our sense of disappointment at Russell's lack of mathematical acuity in these writings remains, especially as regards his treatment of set theory and foundations of analysis. We are thus inexorably led to the question of the value of the volume we have here. I found nothing to suggest that the material included in this volume has an intrinsic interest of its own. It neither portends great mathematical achievements from Russell in the future nor presents any long-lasting or significant contributions of its own. If one believes that, at least for the decade or so between 1900 or 1903 and 1913, Russell achieved the status of a mathematical genius as exemplified by his work in logic, then the primary value of this book is the inspiration it offers respecting the kind of remarkable improvement that can be made in a very short time by a clever and dedicated student – with or without the help of an outstanding mentor such as Whitehead – despite a previously inadequate education. On a more sober note, this book is worthy of the attention of philosophers of mathematics and philosophers of science who wish to trace Russell's development from his early Hegelian idealism to his later realism and learn how the early idealism shaped his thinking in philosophy of mathematics and philosophy of science. It is dubious that even the most dedicated of Russell scholars specializing in philosophy of mathematics or philosophy of science will find it worth the $150+ pricetag, however. And for the more sceptical, for those who do not believe in miracles that take one from mathematical mediocrity to genius almost overnight, the question remains, whether the unpublished materials that were included in this volume ought not remain unpublished. This was a question which I asked myself when I was working on some of the materials contained between these covers. If they are judged on their own merits alone, without any consideration of their historical context, the reply is clearly 'No.' If, on the other hand, the goal is to provide a complete and accurate portrait of Russell's intellectual development and of the true scope and intellectual level of his work – which is, after all, the purpose of the BREP that prepares for publication the series in which the present volume appears, then the reply is affirmative.

In its physical appearance, this book is magnificent, luxuriant, even ostentatious, with its more than 680 (xl + 647) high bulk, glossy pages and goldleaf trim. From this perspective, its $150+ price is readily apparent, befitting an author whose archives have taken on certain aspects of a temple shrine, but impractical in its exorbitance for the ordinary scholars who might find these materials of some use.

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