ℵ Modern Logic ω

# SMALL MODELS

George WEAVER

Department of Philosophy
Bryn Mawr College
Bryn Mawr, PA 19010-2899, USA

ABSTRACT. This paper surveys some problems in the model theory of uncountable first order languages. These problems were first raised in Mal'cev [*1959*]. Their solutions involve the construction of infinite models which are "small" relative to the cardinality of their language. The most important of these problems concern extending the Upward Löwenheim-Skolem Theorem for uncountable languages. The earliest results relevant to this problem used the ultraproduct constructlon to obtain "small" elementary extensions. Stronger results have been obtained more recently by other methods. These are used to construct "small" elementary subsystems.

AMS (MOS) 1991 subject classifications:
03C15, 03C75; 03C07, 03C20, 03C50; 01A60

This paper surveys some problems in the model theory of uncountable first order languages. These problems were first raised in Mal'cev [*1959*]. Their solutions involve the construction of infinite models which are "small" relative to the cardinality of their language. The most important of these problems concern extending the Upward Löwenheim-Skolem Theorem for uncountable languages. The earliest results relevant to this problem used the ultraproduct constructlon to obtain "small" elementary extensions. Stronger results have been obtained more recently by other methods. These are used to construct "small" elementary subsystems.

Let $K$ be a non-empty set of objects whose members are called non-logical constants. $L_K$ denotes the first-order language with equality over $K$. $S_K$ denotes the set of sentences in $L_K$. Interpretations for $L_K$ are ordered pairs $\mathfrak{A} = (A, f_{\mathfrak{A}})$ where $A$ is a non-empty set (the domain of $\mathfrak{A}$) and $f_{\mathfrak{A}}$ is a function defined on $K$ taking values in the usual way. $T_K$ denotes the proper class of interpretations for $L_K$. For any set $B$, $|B|$ denotes the cardinal number of $B$, and $|\mathfrak{A}|$ denotes $|A|$. In the following, the cardinality of $L_K$ will be $|S_K|$.

For $\mathfrak{A} \in T_K$, $\mathfrak{A}$ is *small* provided $|\mathfrak{A}| < |S_K|$. Thus, all finite interpretations are small; and, when $K$ is uncountable, $T_K$ contains small infinite interpretations. In the following, attention is restricted to small infinite interpretations, and so to uncountable languages. The "small model" terminology is from Mal'cev [*1959*]. Mal'cev is credited with having initiated the study of uncountable languages some twenty years earlier (cf. Mal'cev [*1936*], [*1941*]). These papers represent a step toward viewing languages as abstract mathematlcal objects intended to model either natural or artificial languages (cf. Vaught [*1974*], p. 164). Even on this "abstract" view, uncountable languages are "imaginary" in that there are no "real" languages to which they correspond (cf. Vaught [*1973*], p. 7), and it has been argued that using the term "language" in referring to such an uncountable collection is misleading (cf. Church [*1956*], p. 52).

Most of the results below have appeared in the literature. However, some of the proofs have not. Some of the results concerning small models have been discussed elsewhere in the literature. The discussions are usually part of a survey of applications of the ultraproduct construction and focus on the work of Rabin rather than that of Mal'cev (e.g., Keisler [*1965*], Chang [*1967*]). The majority of results are not widely known and there is no single place in the literature where they have been assembled and discussed.

For $\mathfrak{A} \in T_K$, $\phi \in S_K$, $\mathfrak{A} \models \phi$ indicates that $\mathfrak{A}$ is a model of $\phi$. For $S \subseteq S_K$, $\mathfrak{A} \models S$ indicates that $\mathfrak{A}$ is a model of $S$. When $\phi(x_1, \ldots, x_n)$ is a formula in $L_K$ and $a_1, \ldots, a_n$ are in $A$, $\phi(x_1, \ldots, x_n)\ [a_1, \ldots, a_n]$ indicates that $\phi(x_1, \ldots, x_n)$ is satisfied by $(a_1, \ldots, a_n)$ in $\mathfrak{A}$. For $\mathfrak{B} \in T_K$, $\mathfrak{A}$ i*s a subsystem of* $\mathfrak{A}$ ($\mathfrak{B}$ *is an extension of* $\mathfrak{A}$) provided (1) $A \subseteq B$ and (2) for all $\phi(x_1, \ldots, x_n)$ atomic formulae in $L_K$ and all $a_1, \ldots, a_n$ in $A$, $\mathfrak{A} \models \phi(x_1, \ldots, x_n)\ [a_1, \ldots, a_n]$ iff $\mathfrak{B} \models \phi(x_1, \ldots, x_n)[a_1, \ldots, a_n]$. $\mathfrak{A} \subseteq \mathfrak{B}$ indicates that $\mathfrak{A}$ is a subsystem of $\mathfrak{B}$. $\mathfrak{A}$ *is an elementary subsystem of* $\mathfrak{A}$ ($\mathfrak{B}$ *is an elementary extension of* $\mathfrak{A}$) provided (1) $A \subseteq B$; and (2) for all $\phi(x_1, \ldots, x_n)$ formulae in $L_K$ and all $a_1, \ldots, a_n$ in $A$, $\mathfrak{A} \models \phi(x_1, \ldots, x_n)\ [a_1, \ldots, a_n]$ iff $\mathfrak{B} \models \phi(x_1, \ldots, x_n)\ [a_1, \ldots, a_n]$. $\mathfrak{A} \leq \mathfrak{B}$ indicates that $\mathfrak{A}$ is an elementary subsystem of $\mathfrak{B}$. $\mathfrak{A}$ *is a proper elementary subsystem of* $\mathfrak{B}$ ($\mathfrak{B}$ is a *proper elementary extension of* $\mathfrak{A}$) provided $\mathfrak{A} \leq \mathfrak{B}$ and $B \neq A$.

The focus of the following is on two related subjects: (1) interpretations with small elementary extensions and (2) theories with small infinite models. The discussion of the first is centered on the following relation on infinite cardinals. For $\alpha$, $\beta$, $\kappa$ infinite cardinals, $\alpha \ {}_{\kappa}\!\!\mapsto \beta$ iff (1) $\alpha < \beta$ ; and (2) for all $K$ such that $|S_K| = \kappa$ and $\mathfrak{A} \in T_K$ such that $|\mathfrak{A}| = \alpha$, then there is a $\mathfrak{B} \in T_K$ such that $\mathfrak{A} \leq \mathfrak{B}$ and $|\mathfrak{B}| = \beta$. In the following $\alpha$, $\beta$, and $\kappa$ will range over infinite cardinals.

THEOREM 1 [Tarski-Vaught (*1957*)]: *If* $\alpha < \beta$ *and* $\kappa \leq \beta$, *then* $\alpha \ {}_{\kappa}\!\!\mapsto \beta$.

Theorem 1 is just the Upward Löwenheim-Skolem Theorem. Notice that when $\kappa$ is countable and $\alpha < \beta$, then $\alpha \ {}_{\kappa}\!\!\mapsto \beta$. However, when $\alpha < \beta$, and $\beta < \kappa$, as can happen when $\kappa$ is uncountable, Theorem 1 gives no information. In this case, showing that $\alpha \ {}_{\kappa}\!\!\mapsto \beta$ involves the construction of small elementary extensions. This construction is not entirely trivial in the sense that it is easily obtained by some modification of any of the standard arguments for Theorem 1. To see this, consider the compactness argument for Theorem 1. Given $\alpha < \beta$, $K$ such that $|S_K| = \kappa$ and $\mathfrak{B} \in T_K$ where $|A| =$

$\alpha$, let $K' = \{k_a : a \in A\}$ be a set of individual constants disjoint from $K$ and let $K'' = \{k_{\Theta'} : \Theta < \beta\}$ be another set of individual constants disjoint from $K \cup K'$. In $L_{K \cup K'}$, form the complete diagram of $\mathfrak{A}$ and let T be the complete diagram of $\mathfrak{A}$ together with $\{k_\Theta \neq k_{\Theta'} : \Theta \neq \Theta', \Theta, \Theta' < \beta\}$. T is a set of sentences in $L_{K \cup K' \cup K''}$ and $|S_{K \cup K' \cup K''}| = \alpha + \kappa + \beta$. When $\beta \leq \kappa$, we have $\alpha + \kappa + \beta = \beta$. Every finite subset of T has a model. Hence, by the com-pactness theorem for $L_{K \cup K' \cup K''}$, T has a model of cardinality $\leq \beta$; and this model, when restricted to an interpretation for $L_K$, is an elementary extension of $\mathfrak{A}$. But, since T contains $k_\Theta \neq k_{\Theta'}$, the cardinality of thls model is exactly $\beta$. When $\beta \leq \kappa$, the cardinality of $L_{K \cup K' \cup K''}$ is $\kappa$, hence $\mathfrak{A}$ has an elementary extension of cardinality at least $\beta$ and no greater than $\kappa$, and the above compactness argument fails.

LEMMA 1 [Rabin (*1959*)]: *If* $\alpha = \alpha^{\aleph_0}$ *and* $\mathfrak{A} \in T_K$ *is such that* $|\mathfrak{A}| = \alpha$, *then there is a* $\mathfrak{B}$ *in* $T_K$ *such that* $|\mathfrak{B}| = \alpha$ *and* $\mathfrak{B}$ *is a proper elementary extension of* $\mathfrak{A}$.

This lemma was proved by Rabin using an ultraproduct construction. Lemma 1 and the elementary chain theorem yield the following:

THEOREM: *If* $\alpha = \alpha^{\aleph_0}$, *then for all* $\kappa$, $\alpha \;{}_\kappa\!\mapsto \alpha^+$.

PROOF: Let $K$ be such that $|S_K| = \kappa$ and let $\mathfrak{A} \in T_K$. Suppose that $|\mathfrak{A}| = \alpha$ and $\alpha = \alpha^{\aleph_0}$. Construct the elementary chain $\{\mathfrak{A}_\lambda : \lambda < \alpha^+\}$ where $\mathfrak{A}_0 = \mathfrak{A}$, $\mathfrak{A}_{\lambda+1}$ is a proper elementary extension of $\mathfrak{A}_\lambda$, and for all $\lambda$, $|\mathfrak{A}_\lambda| = \alpha$. If $\mathfrak{A}_{\alpha^+}$ is the union of this chain, then $\mathfrak{A} \precsim \mathfrak{A}_{\alpha^+}$ and $|\mathfrak{A}_{\alpha^+}| = \alpha+$.

The following is immediate from Theorem 2.

COROLLARY 1:

(1) $2^\alpha \;{}_\kappa\!\mapsto (2^\alpha)^+$ ; *and*

(2) $\beth_{\theta+1} \;{}_\kappa\!\mapsto (\beth_{\theta+1})^+$ , *for any ordinal* $\theta$.

LEMMA 2 [Mal'cev (1959)]: *If* $\mathfrak{A} \in T_K$ *and* $|\mathfrak{A}| = \alpha$, *then there is a* $\mathfrak{B} \in T_K$ *such that* $|\mathfrak{B}| = \alpha^{\aleph_0}$ *and* $\mathfrak{B}$ *is a proper elementary extension of* $\mathfrak{A}$.

While Mal'cev stated the above, he proved only that $|\mathfrak{B}| \leq \alpha^{\aleph_0}$. However, Lemma 2 is easily established (cf. Lemma 6). Notice that Lemma 2 implies Lemma 1. In addition, the following is immediate.

THEOREM 3: *If* $\alpha < \alpha^{\aleph_0}$, *then* $\alpha \;{}_{\kappa}\!\!\mapsto \alpha^{\aleph_0}$ *for all* $\kappa$.

Thus, $\aleph_0 \;{}_{\kappa}\!\!\mapsto 2^{\aleph_0}$, so, assuming the Continuum Hypothesis, $\aleph_0 \;{}_{\kappa}\!\!\mapsto \aleph_1$. An example of Mal'cev [*1959*] shows that the assumption of the Continuum Hypothesis here is essential. In particular, $\aleph_0 \;{}_{\kappa}\!\!\not\mapsto \beta$ for all $\kappa$ and $\beta$ where $\kappa \geq 2^{\aleph_0}$ and $\aleph_1 \leq \beta < 2^{\aleph_0}$. Let $K = \{f_h : h \in {}^{\aleph_0}2\} \cup \{k_g : h \in \cup\, {}^{n}2\}$ $(n \geq 1)$, where $f_h$ is a functional constant of degree 1 and $k_g$ is an individual constant. Let T be the set of sentences in $L_K$ of the form

$$\forall x\Big(\mathop{\&}_{\substack{g \in \cup\, {}^{\mu}2 \\ 1 \leq \mu \leq n}} x \neq k_g \supset f_h(x) \neq f_{h'}(x)\Big)$$

where $h \restriction n \neq h' \restriction n$. Then T has a countable model. Let $A = \cup\, {}^{n}2$ $(n \geq 1)$, $f_{\mathfrak{A}}(k_g) = g$ and $f_{\mathfrak{A}}(k_h)\,(g) = h \restriction m$ iff $g \in {}^{m}2$. $(A, f_{\mathfrak{A}}) \models T$. Notice that no countable model of T has a proper elementary extension of cardinality $\beta$ where $\aleph_1 \leq \beta < 2^{\aleph_0}$.

The following result was also noticed by Mal'cev.

THEOREM 4:

(1) *If* $2^{\alpha} = \alpha^{+}$, then $\alpha \;{}_{\kappa}\!\!\mapsto \alpha^{+}$ *for all* $\kappa$; *and*

(2) *Assuming the Generalized Continuum Hypothesis, if* $\alpha < \beta$, *then* $\alpha \;{}_{\kappa}\!\!\mapsto \alpha^{+}$, *for all* $\kappa$.

PROOF: (2) is immediate from (1) and the elementary chain theorem. (1) follows from Theorems 2 and 3 and a little cardinal arithmetic. Suppose that $2^\alpha = \alpha^+$. Either $\alpha^{\aleph_0} = \alpha$ or $\alpha < \alpha^{\aleph_0}$. If $\alpha^{\aleph_0} = \alpha$, apply Theorem 2. Suppose $\alpha < \alpha^{\aleph_0}$. By Theorem 3, $\alpha \;{}_{\kappa}\!\!\mapsto \alpha^{\aleph_0}$ and, by supposition, $2^\alpha = \alpha^+$. Thus $\alpha^{\aleph_0} = \alpha^+$.

Theorem 4(2) is also immediate from the following lemma.

LEMMA 3: *If for all cardinals* $\lambda < \alpha$, $2^\lambda \leq \alpha$, *then* $\alpha \;{}_{\kappa}\!\!\mapsto \beta$ *for all* $\kappa$.

Lemma 3 is established by an ultraproduct construction (cf. Weaver [*1992*]). The following is also a consequence of Lemma 3.

THEOREM 5: $\beth_\theta \;{}_{\kappa}\!\!\mapsto \beth_{\theta+1}$ *for all* $\theta$ *and* $\kappa$, *where* $\theta$ *is a limit ordinal.*

The following lemma is stronger than Lemma 3, and can be established without the construction of an ultraproduct.

LEMMA 4: *If* $|\mathfrak{A}| = \alpha$ *and* $\beta \geq 2^\alpha$, *then there is a* $\mathfrak{B}$ *such that* $\mathfrak{A} \leq \mathfrak{B}$ *and* $|\mathfrak{B}| = \beta$.

PROOF: Given $K$, $\mathfrak{A} \in T_K$ and infinite cardinals $\alpha$ and $\beta$, suppose that $|\mathfrak{A}| = \alpha$ *and* $\beta \geq 2^\alpha$. For $n \geq 1$ let $\vec{x}$ be $x_1, \ldots, x_n$, a sequence of $n$ distinct variables. Let $\phi(\vec{x})$ and $\psi(\vec{x})$ be formulas in $L_K$ whose free variables are exactly $x_1, \ldots, x_n$. Let $\phi(\vec{x}) \sim \psi(\vec{x})$ iff $\mathfrak{A} \models \forall x(\phi(\vec{x}) \equiv \psi(\vec{x}))$. $\sim$ is an equivalence relation; and, for each $n$ there are at most $2^\alpha$-many equivalences classes of formulas in free variables $x_1, \ldots, x_n$. By Theorem 1, there is an infinite cardinal $\lambda$, and a $\mathfrak{C} \in T_K$ such that $\beta \leq \lambda$, $|\mathfrak{C}| = \lambda$ and $\mathfrak{A} \leq \mathfrak{C}$. If $\lambda = \beta$, we are finished. Suppose $\beta < \lambda$. We claim that there is a $\mathfrak{B}$ such that $\mathfrak{A} \leq \mathfrak{B}$, $\mathfrak{B} \leq \mathfrak{C}$, and $|\mathfrak{B}| = \beta$. Let $B'$ be any subset of $C$ such that $A \subseteq B'$ and $|B'| = \beta$. Notice that $\phi(\vec{x}) \sim \psi(\vec{x})$ iff $\mathfrak{C} \models \forall\vec{x}(\phi(\vec{x}) \equiv \phi(\vec{x}))$. Let $f$: $C^n \rightarrow C$, where $f$ is a Skolem function for $\exists y\phi(\vec{x},y)$ in $\mathfrak{C}$ iff for all $d_1, \ldots, d_n \in C$, if $\mathfrak{C} \models \exists y\phi(\vec{x},y)\ [d_1, \ldots, d_n]$, then $\mathfrak{C} \models \phi(\vec{x},y)\ [d_1, \ldots, d_n\, f(d_1, \ldots, d_n)]$. When $\phi(\vec{x},y) \sim \psi(\vec{x},y)$, there is an $f$ which is a Skolem function for both $\exists y\phi(\vec{x},y)$ and $\exists y\psi(\vec{x},y)$ in $\mathfrak{C}$. Hence, there are at most $2^\alpha$-many Skolem functions needed for formulas in $L_K$. Let $\mathfrak{B}$ be that subsystem of $\mathfrak{C}$ obtained by closing $B'$ under these Skolem functions in $\mathfrak{C}$. $\mathfrak{B} \precsim \mathfrak{C}$, $\mathfrak{A} \subseteq \mathfrak{B}$, and $|\mathfrak{B}| = \beta$. Further, since $\mathfrak{A} \precsim \mathfrak{C}$, then $\mathfrak{A} \precsim \mathfrak{B}$.

A stronger version of Lemma 4 can be found among the exercises in Chang and Keisler [*1974*] (cf. exercise 6.4.6, p. 377). Notice that in the proof of Lemma 4, $\mathfrak{C}$ is any elementary extenslon of $\mathfrak{A}$. Thus, a similar proof can be employed to yield results about small elementary subsystems, in particular, the following lemma.

LEMMA 5: *For* $\mathfrak{B} \in T_K$, *if* $\alpha < |\mathfrak{B}|$ *and there is an* $\mathfrak{A} \in T_K$ *such that* $|\mathfrak{A}| = \alpha$ *and* $\mathfrak{A} \precsim \mathfrak{B}$, *then for all* $\beta$ *such that* $2^\alpha \leq \beta < |\mathfrak{B}|$, *there is a* $\mathfrak{D} \in T_K$ *such that* $\mathfrak{A} \precsim \mathfrak{D}$, $\mathfrak{D} \precsim \mathfrak{B}$ and $|\mathfrak{D}| = \beta$.

If $2^\alpha = \alpha^+$ , then the above holds for all $\beta$ such that $\alpha < \beta < |B|$. When $\alpha = \aleph_0$, the assumption of the Continuum Hypothesis is essential. To see this, consider the theory T above. T has a countable model $\mathfrak{A}$. Thus, $\mathfrak{A}$ has an elementary extension $\mathfrak{B}$ such that $|B| \geq 2^{\aleph_0}$. But $\mathfrak{B}$ has no elementary subsystem of cardinality $\beta$ where $\aleph_1 \leq \beta < 2^{\aleph_0}$.

The following theorem is immediate from Lemma 4.

THEOREM 6: (1) *If* $\beta \geq 2^\alpha$, *then* $\alpha \;{}_{\kappa}\!\mapsto \beta$ *for all* $\kappa$;

(2) If $\alpha^+ = 2^\alpha$, *then if* $\alpha < \beta$, *then* $\alpha \;{}_{\kappa}\!\mapsto \beta$ *for all* $\kappa$.

Note that Theorem 6(2) is stronger than Theorem 4(2), and that the proof of Theorem 4(2) does not use the elementary chain theorem. Lemma 4 can be strengthened further.

LEMMA 6: *If* $|\mathfrak{A}| =$ $\alpha$ and $\beta \geq \alpha^{\aleph_0}$, *then there is a* $\mathfrak{B} \in T_K$ *such that* $|\mathfrak{B}| =$ $\beta$ *and* $\mathfrak{B}$ is a proper elementary extension of $\mathfrak{A}$.

PROOF: Let $\mathfrak{A} \in T_K$ where $|\mathfrak{A}| = \alpha$. Let $\beta \geq \alpha^{\aleph_0}$. Let $\mathfrak{A}^*$ be a Skolem expansion of $\mathfrak{A}$, in $L_{K \cup K^*}$. Let $K' = \{k_a : a \in A\}$ where $k_a$ is an individual constant and $K \cap K' \cap K^* = \Lambda$. Let T* be the complete diagram of $\mathfrak{A}^*$ in $L_{K \cup K^* \cup K'}$. Let $D$ be any countably infinite subset of $A$. For $\tau(x_1, \ldots, x_n)$ any term in $L_{K \cup K^* \cup K'}$, there is a function $f_\tau$: $D^n \rightarrow A$ such that $(\mathfrak{A}^*, a)_{a \in A} \models \{\tau(x_1,\ldots,x_n) = x_{n+1}\}[a_1,\ldots, a_n, f_\tau(a_1, \ldots, a_n)]$ for $a_1, \ldots, a_n \in D$. Equate $\tau(x_1,\ldots,x_n)$ and $\tau'(x_1,\ldots,x_n)$ iff $f_\tau = f_{\tau'}$. For each $n$, there are, at most, $\alpha^{\aleph_0}$-many equivalence classes of $n$-ary terms. Let $K'' = \{k_\theta : \theta < \beta\}$ be a set of individual constants disjoint from $K \cup K^* \cup K'$. Let $T^{*\prime}$ be $T^* \cup \{k_\theta \neq k_a : a \in A, \theta < \beta\} \cup \{k_\theta \neq k_{\theta'} : \theta, \theta' < \beta, \theta \neq \theta'\} \cup \{\tau(k_{\theta 1},\ldots,k_{\theta n}) = \tau'(k_{\theta 1},\ldots,k_{\theta n})$, where $f_\tau = f_{\tau'}$ and $k_{\theta i} \in K''\}$. Every finite subset of $T^{*\prime}$ has a model. Thus, by compactness, $T^{*\prime}$ has a model. Let $\mathfrak{C}^*$ denote that model; let $C = \{f_{\mathfrak{C}^*}(k_\theta)\ \theta < \beta\}$ and let $\mathfrak{C}^*$ be the Skolem hull of $C$ in $\mathfrak{B}^*$. $|\mathfrak{C}^*| = \beta$, and $\mathfrak{A} \leq \mathfrak{C}$ where $\mathfrak{C}$ is the reduct of $\mathfrak{C}^*$ to the interpretation for $L_K$.

Lemma 6 is formulated and proved in Morley [*1968*] where the result is attributed to Keisler (cf. Chang and Keisler [*1974*], Corollary 6.5.12, p. 391). Notice that Lemma 6 implies both Lemma 1 and Lemma 2; hence, neither of the results is an essential application of the ultraproduct construction (cf. Keisler [*1965*] and Chang [*1967*]). Note the differences between the proof of Lemma 4 and that of Lemma 6. In the second proof, an elementary subsystem is constructed from a particular elementary extension of $\mathfrak{A}$. Thus, the argument for Lemma 6 does not yield an analogue of Lemma 5, i.e., if $\mathfrak{B}$ has a subsystem of cardinality $\alpha$, then $\mathfrak{B}$ has elementary subsystems of cardinality $\beta$ where $\alpha^{\aleph_0} \leq \beta \leq |\mathfrak{B}|$. When $2^\alpha = \alpha^+$, this later result is immediate from Lemma 4. Lemma 6 yields the following.

THEOREM 7: (1) *If* $\alpha < \beta$ *and* $\alpha^{\aleph_0} \leq \beta$, *then* $\alpha \;{}_\kappa\!\!\leftrightarrow \beta$ *for all* $\kappa$;

(2) *If* $\alpha^{\aleph_0} \leq \alpha^+$ *and* $\alpha < \beta$ , *then* $\alpha \;{}_\kappa\!\!\leftrightarrow \beta$ *for all* $\kappa$.

The example from Mal'cev [*1959*] discussed above shows that Theorem 7(1) is the best result obtainable for all infinite cardinals $\alpha$. Results of Keisler [*1963*] show that 7(1) is the best result obtainable (without assuming $\alpha^+ = 2^\alpha$) for $\alpha$ not measurable. $\mathfrak{A} \in T_K$ is *complete* provided for all $n \geq 1$ and $R \subseteq A^n$, there is a $P \in K$ such that $f_{\mathfrak{A}}(P) = R$. Note that when $\mathfrak{A}$ is complete, $|S_K| \geq 2^{|\mathfrak{A}|}$.

LEMMA 7 [Keisler (*1963*)]: *If* $\mathfrak{A}$ *is complete and* $|\mathfrak{A}|$ *is not measurable, then if* $\mathfrak{B}$ *is a proper elementary extension of* $\mathfrak{A}$, $|\mathfrak{B}| \geq \alpha^{\aleph_0}$.

Keisler's proof of Lemma 7 used limit ultraproducts. Chang [*1965*] contains a proof of Lemma 7 which does not mention ultraproducts. The following theorem is immediate.

THEOREM 8: *If* $\alpha$ *is not measurable*, $\kappa \geq 2^\alpha$ *and* $\beta$ *is such that* $\alpha < \beta < \alpha^{\aleph_0}$, *then* $\alpha \not\rightarrow_{\kappa} \alpha^{\aleph_0}$.

Theorem 7 implies that any theory having a model of cardinality $\alpha$ has a model of cardinality $\beta$ for $\beta \geq \alpha^{\aleph_0}$ no matter what the cardinality of the language. There is another result about models of theories which likewise is independent of the cardinality of the language.

THEOREM 9 [Mal'cev (1959)]: *For all* $K$, $S \subseteq S_K$,

(1) *if* $S$ *has arbitrarily large finite models,* $S$ *has a model of cardinality* $2^{\aleph_0}$; *and*

(2) *if* $\{\lambda_n : n \in \aleph_0\}$ *is a strictly increasing sequence of infinite cardinals such that for all n,* $S$ *has a model of cardinality* $\lambda_n$, *then* $S$ *has a model of cardinality* $\Pi\lambda_n$.

Theorem 9(2) is a stronger version of a result of Mal'cev [*1959*]. Mal'cev showed that $S$ has a model of cardlnality $\beta$ such that $\Sigma\lambda_n \leq \beta \leq \Pi\lambda_n$. Theorem 8 is obtainable by an ultraproduct construction (cf. Weaver [*1992*]). Mal'cev [*1959*] contains an example of a theory with arbitrarily large finite models which has no models of cardinality $< 2^{\aleph_0}$. Notice that it follows from Theorem 7 that if $S$ has arbitrarily large finite models, then $S$ has models of cardinality $\beta$ for all $\beta \geq 2^{\aleph_0}$. This later result can be proved directly without the construction of an ultraproduct.

LEMMA 8: *For all $K$, $S \subseteq S_K$, if $S$ has arbitrarily large finite models, then for all $\beta \geq 2^{\aleph_0}$, $S$ has a model of cardinality $\beta$.*

PROOF: For each $n$, let $\mathfrak{A}_n \models S$ where $\mathfrak{A}_n$ is finite and $|\mathfrak{A}_n| \leq |\mathfrak{A}_{n+1}|$. For $\phi(x_1, ..., x_m)$ a formula in free variables $x_1, ..., x_m$, there is a function $f_\phi : \aleph_0 \rightarrow \cup \{P(A_n^m) : n \in \aleph_0\}$ such that $f_\phi(n) = \{(a_1, ..., a_m) : \mathfrak{A}_n \models \phi(x_1, ..., x_m) [a_1, ..., a_m]\}$. Note that there are, at most, $2^{\aleph_0}$-many such functions. Let $S' = S \cup \{\forall x_1, ..., x_m(\phi(x_1, ..., x_m) \equiv \psi(x_1, ..., x_m)): f_\phi = f_\psi\ m \geq 1\}$. Notice that $\mathfrak{A} \models S'$ for all $n$. Thus, $S'$ has arbitrarily large finite models. Let $\beta \geq 2^{\aleph_0}$. There is a $\lambda \geq \beta$ such that $S'$ has a model of cardinality $\lambda$. If $\lambda = \beta$, we are finished. Suppose $\lambda > \beta$. Let $\mathfrak{B} \models S'$ and $|\mathfrak{B}| = \lambda$. Let $C \subseteq B$ such that $|C| = \lambda$. Using the argument of the proof of Lemma 6, there is an $\mathfrak{C} \in T_K$ such that $\mathfrak{C} \leq \mathfrak{B}$ and $|\mathfrak{C}| = \lambda$.

The above argument also yields an extension of Theorem 9(2).

LEMMA 9: *For all $K$, $S \subsetneq S_K$, if $\{\lambda_n: n \in \aleph_0\}$ is a strictly increasing sequence of infinite cardinals such that for all $n$, $S$ has a model of cardinality $\lambda_n$* and $2^{\lambda_n} \leq \lambda_m$ *for some $m > n$, then for all $\beta \geq \Pi\lambda_n$, $S$ has a model of cardinality $\beta$.*

Whether or not Theorem 9(2) essentially depends on the ultraproduct construction appears to be an open question. Assuming that $2^{\lambda_n} = \lambda^+$ for each $n$, Theorem 9(2) follows immediately from Lemma 9. Without this assumption, it can be shown (without an ultraproduct construction) that for $\beta \geq \Pi\lambda_n$, $S$ has a model of cardinality $\beta$.

There is another three-place relation on infinite cardinals suggested by Lemma 5. For $\alpha$, $\beta$, $\kappa$ infinite cardinals, $\alpha \ {}_\kappa\!\!\models\!\!\Rightarrow \beta$ iff $\alpha < \beta$ for all $K$ such that $|K| = \kappa$, and all $\mathfrak{A}$, $\mathfrak{B} \in T_K$ with $|\mathfrak{A}| = \alpha$, $|\mathfrak{B}| > \beta$, and $\mathfrak{A} \leq \mathfrak{B}$, then there is a $\mathfrak{D} \in T_K$ such that $\mathfrak{A} \leq \mathfrak{D}$, $\mathfrak{D} \leq \mathfrak{B}$, and $|\mathfrak{D}| = \beta$. The following is immediate from Lemma 5.

THEOREM 10: (1) *If $\beta \geq 2^\alpha$, then $\alpha \ {}_\kappa\!\!\models\!\!\Rightarrow \beta$, for all $\kappa$.*

(2) *If* $2^{\alpha} = \alpha^{+}$, *then for all* $\beta > \alpha$, $\alpha\ {}_{\kappa}\!\!\models\!\!\!\Rightarrow \beta$, *for all* $\kappa$.

Note that the assumption that $2^{\alpha} = \alpha^{+}$ is essential. The example from Mal'cev discussed above implies that if $\kappa \geq 2^{\aleph_0}$ and $\aleph_0\ {}_{\kappa}\!\!\models\!\!\!\Rightarrow \beta$, then $\beta \geq 2^{\aleph_0}$. Further, it is immediate from the Downward Löwenheim-Skolem Theorem that if $\beta > \alpha$ and $\kappa \leq \beta$, then $\alpha\ {}_{\kappa}\!\!\models\!\!\!\Rightarrow \beta$. Finally, it follows from Lemma 7, that if $\kappa \geq 2^{\alpha}$, then $\alpha\ {}_{\kappa}\!\!\models\!\!\!\Rightarrow \beta$ and $\alpha$ is not measurable, then $\beta \geq \alpha^{\aleph_0}$.

Is Theorem 10(1) the strongest result obtainable? In particular, are there cardinals $\alpha$, $\kappa$ such that if $\alpha\ {}_{\kappa}\!\!\models\!\!\!\Rightarrow \beta$, then $\beta \geq \alpha^{\aleph_0}$? $\aleph_0$ and $2^{\aleph_0}$ provide one such pair of cardinals. Other pairs are provided by the following:

THEOREM 11: *If, for all cardinals* $\lambda < \alpha$, $2^{\lambda} \leq \alpha$, $\alpha$ *is not measurable and* $\alpha$ *is of cofinality* $\omega$, *then if* $\kappa \geq 2^{\alpha}$ *and* $\alpha\ {}_{\kappa}\!\!\models\!\!\!\Rightarrow \beta$, *then* $\beta \geq 2^{\alpha}$.

PROOF: Let $\alpha$ be such that for all cardinals $\lambda < \alpha$, $2^{\lambda} \leq \alpha$ and $\alpha$ is of cofinality $\omega$. Then, $2^{\alpha} = \alpha^{\aleph_0}$. The theorem is then immediate from Lemma 7.

In each of the above examples $2^{\alpha} = \alpha^{\aleph_0}$. Are there examples where $\alpha^{\aleph_0} < 2^{\alpha}$ ? It is shown below that this question has an affirmative answer under the assumption that there is an uncountable measurable cardinal. Whether or not this assumption is essential appears to be an open question. In particular, the question of whether or not an affirmative answer can be obtained under the weaker assumptlon that there is an uncountable inaccessible cardinal appears to be open.

THEOREM 12: *If* $\alpha$ *is a measurable cardinal*, $\kappa \geq 2^{\alpha}$ *and* $\alpha\ {}_{\kappa}\!\!\models\!\!\!\Rightarrow \beta$, *then* $\beta \geq 2^{\alpha}$.

PROOF: Suppose that $\alpha$ is a measurable cardinal, $\kappa \geq 2^{\alpha}$ and $\alpha\ {}_{\kappa}\!\!\models\!\!\!\Rightarrow \beta$. We want to show that $\beta \geq 2^{\alpha}$. Since $\alpha$ is measurable, there is an $\alpha$-complete, nonprincipal ultrafilter **U** on $\alpha$. Let $K$ be a set of non-logical constants of cardinality $\kappa$. $K$ contains at least $\alpha$-many individual constants, at least $2^\alpha$-many unary functional constants, and at least one binary relational constant.

Let $\mathfrak{A} \in T_K$ be such that $\alpha$ is the domain of $\mathfrak{A}$, each member of $\alpha$ is named in $\mathfrak{A}$ by an individual constant, each unary function is the extension in $\mathfrak{A}$ of a unary functional constant and the natural well ordering on $\alpha$, < is the extension in $\mathfrak{A}$ of the binary functional constant $\bar{<}$. When $a$ is a member of $\alpha$, $\bar{a}$ is that individual constant which names $a$ in $\mathfrak{A}$; and when $h$ is a unary function on $\alpha$, $\bar{h}$ is that unary functional constant whose extension in $\mathfrak{A}$ is $h$.

Since $\alpha$ is inaccessible, there is a $\Delta \subseteq \alpha$ such that $|\Delta| = 2^\alpha$, and for all $h$, $h'$ different members of $\Delta$, there is a $\lambda < \alpha$ s.t for all $\theta \geq \lambda$, $h(\theta) \neq h'(\theta)$. Let $\lambda(h, h')$ be the least such member of $\alpha$. For $h$, $h'$ different members of $\Delta$, the sentence

$$\forall x(\overline{\lambda(h, h')} \mathrel{\bar{\leq}} x \supset \bar{h}(x) \neq \bar{h}'(x))$$

is true on $\mathfrak{A}$.

Consider $\mathfrak{B} = \mathfrak{A}^\alpha/\mathbf{U}$. Let $d$ be the canonical elementary embedding of $\mathfrak{A}$ into $\mathfrak{B}$. $\mathfrak{B}$ is well ordered by $f_{\mathfrak{B}}(\bar{<})$. And, since $\mathbf{U}$ is non-principal, $d$ is not onto $B$. Further, for any member $b$ of $B$ not in the image of $\alpha$ under $d$, $(d(a), b) \in f_{\mathfrak{B}}(\bar{<})$ for all $a$ in $\alpha$ (cf. Chang and Keisler [*1973*], Lemma 4.2.15, p. 185). Thus, for $b \in B - d\,[a]$,

$$|\{f_{\mathfrak{B}}((\bar{h}))(b) : h \in \Delta\}| = 2^\alpha.$$

Thus, $\mathfrak{A}$ has an elementary extension of cardinality at least $2^\alpha$ such that no elementary subsystem properly containing $\mathfrak{A}$ is of cardinality $< 2^\alpha$.

REFERENCES

C. C. CHANG. 1965. *A simple proof of the Rabin-Keisler theorem*, Bulletin of the American Mathematical Society **71**, 642-643.

–. 1967 *Ultraproducts and other methods of constructing models*, J. N. Crossley (editor), *Sets, models and recursion theory* (North Holland, Amsterdam), 85-121.

C. C. CHANG & H.J. KEISLER. 1973 *Model theory*, North Holland, Amsterdam.

A. CHURCH. 1956. *Introduction to mathematical logic*, Princeton University Press, Princeton.

H.J. KEISLER. 1963. *Limit ultrapowers*, Transactions of the American Mathematical Society **107**, 382-408.

–. 1965. *A survey of ultraproducts*, Y. Bar.-Hillel (editor), *Logic methodology and philosophy of science* (North Holland, Amsterdam), 112-126.

A.I. MAL'CEV. 1936. *Untersuchungen aus dem Gebiete der mathematischen Logik*, Математический Сборник (нов. сер.) **1** (43), 323-336. Reprinted in English as *Investigations in the realm of mathematical logic*, [*1971*], 1-14.

–. 1941. *Об одном общем методе получения локальных теорем теории групп*, Ученые записки Иванов. Педагогическ. Инст. (Физ.-мат. Факультет) **1** (нр. 1), 3-9. Reprinted in English as *A general method for obtaining local theorems in group thoery*, [*1971*], 15-21.

–. 1959. *О малых моделя*, Доклады Акад. Наук СССР **127**, 258-261. Reprinted in English as *Small models*, [*1971*], 114-118.

–. 1971. *The metamathematics of algebraic systems* (B. F. Wells, III, editor and translator), North Holland, Amsterdam.

M. MORLEY. 1968. *Partitions and models*, M.H. Löb (editor), *Proceedings of the Summer School in Logic Leeds, 1967* (Berlin, Springer-Verlag), 109-158.

M. RABIN. 1959. *Arithmetical extensions with prescribed cardinality*, Indag. Math., **21**, 439-446.

A. TARSKI & R. L. VAUGHT. 1957. *Arithmetical extensions of relational systems*, Comp. Math. **13**, 82-101.

R. L. VAUGHT. 1973. *Some aspects of the theory of models*, The American Mathematical Monthly **80**, 3-37.

– 1974. *Model theory before 1945*, L. Henkin, J. Addison, C. C. Chang, W. Craig, D. Scott, R. Vaught (editors) *Proceedings of the Tarski Symposium* (Providence, American Mathematical Society), 153-172.

G. WEAVER. 1992. *Henkin-Keisler models* (to appear).