THE STRUCTURE AND DEVELOPMENT OF MATHEMATICAL THEORIES*

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В рамках структурно-номинативного подхода к научному знанию описываются основаные подсистемы математических теорий. Указывается на важность и естественность конструкции имемованного множества при описании основных элементов математических теорий. На примере теории множеств рассмотреные некоторые динамические аспекты математических теорий.

On the basis of a structural-naming reconstruction of scientific knowledge we give a description of the main subsystems of mathematical theories. The role of the theory of named sets for the exact analysis of their components is given. For the case of set theory we consider also some dynamic aspects.

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1. INTRODUCTION

Today almost everybody knows the importance of mathematics in contemporary civilization. But we have much less knowledge concerning the nature, structure and regularities of the development of mathematics, the reasons for its effectiveness, its differentiae and resemblances in relation to other sciences, etc. Answers to these questions may be given in different ways. In recent years, along with the informal analysis of mathematics that included historical studies and assessments of the great mathematicians (Struik; Renyi; Kitcher; Steiner; Wilder), precise reconstructions of mathematical theories have become more influential (Balzer; Schreiber). This direction tries to answer the above-mentioned questions by means of the analysis of the static and dynamic properties of reconstructions (models) of a theory. It takes mathematics as an extremely complex conceptual system that may be studied on the basis of mathe=matical logic, model theory, set theory, etc. Following this path with standard and structuralist approaches, it has obtained many important and interesting results. But neither those approaches nor their union (Pearce & Rantala) exhausts all sides of scientific knowledge systems. In order to develop these approaches and to overcome their limitations a structural-naming approach has been proposed (Burgin & Kuznetsov, *1986a – 1987*). In it a scientific knowledge system is taken as an ensemble of scientific theories and a scientific theory is treated as a hierarchical complex system. On the highest level of the hierarchy of the latter are situated logico-linguistic, model-representation, pragmatic-procedural, and problem-heuristical subsystems as well as the subsystem of ties between the previous subsystems. The construction and analysis of these subsystems of mathematical models for us in an essential way the tools and methods of the theory of named set (Burgin *1984*). In this paper we give only the basic concepts and results of this theory necessary for the structural-naming analysis of some features of subsystems of mathematical theories. It opens the way for the informal description and formal modeling of the principal aspects of the development of set theory.

2. The Main Constructions and Results of the Theory of Named Sets

It is necessary to stress that in the theory of named sets the concept of *name* is considered in a much broader sense than usual. Any semiotic construction which plays the role of symbol, referring expression, definition, description, model, etc., for internal or external entities of our consciousness may function as a name. A name in the sense of the theory of named sets exists only in conjunction with the entities baptized (named, modeled, defined, represented, explained, described, etc.) by this name. It does not exclude the situation of a void name, i.e., a name which does not baptize any existent object. In an extension of the concept of *entity*, we include on a par with material amd ideal objects and their properties the relations between them, operations on objects, their properties and relations, processes with objects and so forth. The opposition between a name and the entity that is named is relative. An entity baptized in some situations by means of a name may function in other situations as the name of other objects. For example, the conceptual model of an object plays the role of the specific name of this object and at the same time the model is an entity named by means of its logico-linguistic description.

The construction of a named set is a general and exact explication of the above-mentioned situations that are typical and extend to every element, level, and step in scientific knowledge and cognition. Generally speaking, scientific knowledge as a conceptual system consists of a very complex multilevel net of named sets. Above all, the process of correspondence between one basic entity and other entities which represent the first one in some way may be treated like the correspondence of entities to their names. In such a case, the problem of the analysis of the basic properties of entities is transformed into the problem of the analysis of the properties of their corresponding names. An analogous reduction of the analysis of entities to the analysis of their specific names (in a broad sense) characterizes the theoretical level of cognition in every scientific field.

William Shakespeare was absolutely correct insisting, "What's in a name? That which we call a rose by any other name would smell as sweet."

We have, however, to take into consideration that in science we must call things by their right names. Choosing the right names (in the sense of the theory of named sets) determines almost everything in scientific cognition. H. Poincaré wrote about this very vividly: "and now one is amazed at the power of a single word. There is an object about which it had been possible to say nothing before it was baptized. But we have a name to this object and a miracle happens. How does this happen? By giving a name to this object we implicitly assert that the object exists (i.e. it is free from contradictions) and that it is fully determined" (Poincaré).

For the exact definition of a named set let us fix three families of sets (or of classes): ENS, SET, and COL and their morphisms. The collections of all objects (i.e., sets and/or classes) from these families are denoted as, respectively, ObENS, ObSET, and ObCOL. The collections of all morphisms (i.e. mappings of sets (classes) or relations between them) from these families are denoted, respectively as MorENS, MorSET, and MorCOL. In addition, the following conditions are satisfied: (1) ObENS, ObSET $\subseteq$ ObCOL; (2) MorENS, MorSET $\subseteq$ MorCOL; (3) the collection MorCOL is closed relative to product, i.e., if the mappings $\alpha : A \to B$ and $\beta: B \to C$ (relations $\alpha \subseteq A \times B$, $\beta \subseteq B \times C$) belong to MorCOL, where A, B, C $\in$ ObCOL, then their product $\alpha\beta$ also belongs to MorCOL. Some subclass M $\subseteq$ MorCOL is distinguished. It may be done arbitrarily by means of different conditions on the class M to define the constructions necessary for the problems under consideration.

DEFINITION 1. A *named set* (with respect to M $\subseteq$ MorCOL) is a triple $\mathcal{X} = (X, \alpha, I)$ where $X \in$ ObENS, $I \in$ ObSET, $\alpha : X \to I$ ($\alpha \subseteq X \times I$) and $\alpha \in M$.

Well-known special cases of named sets are ordinary sets, multisets (Knuth), fuzzy sets (Zadeh), L-sets (Salii; Goguen), and so on.

The set X is called the *support of* $\mathcal{X}$ and is denoted by $S(\mathcal{X})$; the set I is called the *name set* or *set of names* of $\mathcal{X}$ and is denoted by $N(\mathcal{X})$; the set $N_f(\mathcal{X}) = \{a \in I \mid \exists x \in S(\mathcal{X})\ \&\ ((x, a) \in \alpha)\}$ is called the *set of non-void (factual) names of the named set* $\mathcal{X}$; the map (relation) α is called the *naming map* (*relation*) *of* $\mathcal{X}$ and is denoted by n($\mathcal{X}$); α(x) is called the *full name* of *x* in $\mathcal{X}$ and any a ∈ α (x) is called a (*partial*) *name* of *x* in $\mathcal{X}$. For example, for an arbitrary element *x* from the fuzzy set *A*, the name of *x* is the degree of membership $\mu_A(x)$ of the element *x* in the fuzzy set *A*, i.e. the value of the membership function μ_A on the element *x*.

A named set $\mathcal{X}$ is called: (a) *normalized* if $N_f(\mathcal{X}) = N(\mathcal{X})$, i.e. if the naming relation n($\mathcal{X}$) maps *X* onto *I*; (b) *functional* if the naming relation n($\mathcal{X}$) is a mapping; (c) *singly named* if for any x, y ∈ S($\mathcal{X}$), α(x) = α(y) and α = n($\mathcal{X}$) is a mapping; (d) *individualized* if different elements from S($\mathcal{X}$) have different full names in the named set $\mathcal{X}$, i.e., for any x, y ∈ S($\mathcal{X}$), x ≠ y implies α(x) ≠ α(y).

DEFINITION 2. A *morphism* of a named set $\mathcal{X}$ = (X, α, I) into a named set $\mathcal{Y}$ = (Y, β, J) is a pair Φ = (f,g) where *f* is a morphism in ENS from *X* into *Y*, g is a morphism in SET from *I* into *J* and αg = fβ is valid in COL, i.e. the following diagram is commutative:

$$\begin{array}{ccc} I & \xrightarrow{g} & J \\ \alpha \uparrow & & \uparrow \beta \\ X & \xrightarrow[f]{} & Y \end{array}$$

If Φ = (f, g) : $\mathcal{X} \to \mathcal{Y}$ and Ψ = (h, k) : $\mathcal{Y} \to \mathcal{Z}$ are morphisms of named sets, then the product of morphisms Φ and Ψ is defined as ΦΨ = (fh, gk) : $\mathcal{X} \to \mathcal{Z}$.

THEOREM 1. (Burgin *1984*). *If* ENS, SET, *and* COL *are categories, then the totality NSET consisting of all named sets and their morphisms is a category.*

Let $\mathcal{X}$ = (X, α, I) and $\mathcal{Y}$ = (Y, β, J) be named sets.

DEFINITION 3. If $X = Y$ and $\alpha = \beta\gamma$ where $\gamma: J \to I$, then $\mathcal{X}$ is called a *right extension* of $\mathcal{Y}$ and $\mathcal{Y}$ is called a *right restriction* of $\mathcal{X}$. If $I = J$ and $\alpha = \delta\beta$ where $\delta: X \to Y$, then $\mathcal{X}$ is called a *left extension of* $\mathcal{Y}$ and $\mathcal{Y}$ is called a *left restriction* of $\mathcal{X}$. If $X = Y$ and $I = J$ and $\alpha \subseteq \beta$, then $\mathcal{Y}$ is called an *inner extension* of $\mathcal{X}$.

DEFINITION 4. A *named set of second order* with respect to $H \subseteq$ MorNSET is a triple $\boldsymbol{\mathcal{X}} = (\mathcal{X}, A, \mathcal{Y})$ where $\mathcal{X}, \mathcal{Y} \subseteq$ ObNSET, $A \in H$.

DEFINITION 5. If $\boldsymbol{\mathcal{X}} = (\mathcal{X}, A, \mathcal{Y})$ and $\boldsymbol{\mathcal{U}} = (\mathcal{U}, B, \mathcal{W})$ are named sets of second order, then the morphism $\Phi: \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{U}}$ is a *pair of mappings* (Φ_1, Φ_2), where $\Phi_1: \mathcal{X} \to \mathcal{U}$ and $\Phi_2: \mathcal{Y} \to \mathcal{W}$ belong to the category NSET.

THEOREM 2. The named sets of the second order and their morphisms form a category N^2 SET.

Note that the named sets of more than second order are introduced analogously.

3. SOME EXAMPLES OF NAMED SETS

We shall give some examples of arithmetical named sets and relations between them.

From the time of Euclid every mathematical concept has functioned (implicitly) as a named set. Indeed, usually an important step of constructing a mathematical concept was giving a definite name to it. After this baptism (naming), the content of the concept was explicated by means of a definition. In fact, the definition is a description of a set of elements which form the support of the named set associated with the mathematical concept. Such a definition may also be viewed as a name of the concept. This is evident from Euclid's definition of the natural numbers. According to him, the unit is the unity by means of which any existing thing is considered united, and a number is a set consisting of units. In other words, a number is presented as a collection of units which has a definite denomination (name) from a distinguished class of words – the so-called numerals (for example, "two", "three", "five", and so on).

It is important that the fixing of the number of elements in a set by means of notches and the designation of this number by means of a word – a numeral – are rather different processes. In the first case, there is only one sign (for example, a notch) substituting for a single object. Hence, the main problem is the construction of a one-to-one correspondence between the number of these notches and the number of objects. In the second case, in the course of baptising a number, it is necessary to find some specific denomination for each distinguished set. This problem is more complex than the first one. Several thousands of years more were taken for its solution than for the solution of the first problem (see Klix). In fact, the solution of the second problem led to the formation of the named set corresponding to the natural number concept, i.e., a concept belonging to the abstract level of thinking. The solution of the first problem was obtained on the level of sense perception by means of constructing a one-to-one correspondence between objects given by the senses.

In addition, the totality of all numbers corresponding to a specific named set is also represented by a named set. Its support consists of collections of units, and elements of the named set are designations (names) of numbers. In turn, this named set has its own name – "the natural numbers". It is necessary to stress that in this and analogous cases the denotation of numbers through the use of numerals (for example, "ten"), simultaneously plays the roles of the name of a totality of units and of the name of the named set corresponding to a given number. The same situation characterizes other mathematical concepts.

Thus, returning to the concept of number, we have to keep in mind that there is a difference between the denotation of a number and the concept of number, viz., the denotation of a number is an element from the named set, but the concept of number is the named set which includes the class of denotations of numbers as its named set. The contemporary set-theoretic view of natural numbers presents them as specific named sets. Generally, the name of a number may be taken either from natural languages or from mathematical languages (enumerations). In the latter case the support of the corresponding named set is the class of all equivalent finite sets. As before, denotations of numbers are simultaneously names of classes of equivalent sets and names of those named sets which correspond to these numbers. Every number has many names from various languages – both from natural (English, Russian, etc.) and from artificial ones (different enumerations). In addition, different definitions of numbers are in fact specific names (in the sense of the theory of named sets) of numbers.

A marvellous example of the last situation is demonstrated by the different definitions of 'prime number'. According to D. Zagier (Zagier), a number is called 'prime' if it is not equal to 1 and is not divisible by any other natural number. This definition is proposed by specialists in number theory. However, other mathematicians sometimes use other definitions. Thus, for specialists in function theory, a prime number is an integral zero of the analytical function

$$1 - \frac{\sin \frac{\pi \Gamma(s)}{s}}{\sin \frac{\pi}{s}}$$

For an algebraist, a prime number is "the characteristic of a finite field" or "a point in the spectrum of $\mathbb{Z}$" or a "nonarchimedean valuation". For a specialist in combinatorics, prime numbers are defined by the recurrance formula

$$p_{n+1} = \left[1 - \log_2 \left(\frac{1}{2} + \sum_{r=1}^{n} \sum_{1 \leq i_1 \leq \ldots \leq i_r \leq n} \frac{(-1)^r}{2^{p_{i_1} \cdots p_{i_{r-1}}}} \right) \right],$$

where [x] is the integer part of the number x. Finally, logicians have recently defined prime numbers as the positive values of a very complex polynomial.

One of the simplest examples of morphisms of named sets is the conversion of numbers from one numeration to another one. This situation is depicted in the following diagram:

$$\begin{array}{ccc}
\text{“2”} & \longrightarrow & \text{“10”} \\
\uparrow & & \uparrow \\
\{\{a,a\}, \{b,b\}, \{a,b\}, ...\} & \longrightarrow & \{\{a,a\}, \{b,b\}, \{a,b\}, ...\}
\end{array}$$

The named set corresponding to the decimal representation of the number 2 is on the left and the named set connected with the binary representation of the number 2 is on the right in this diagram.

4. THE SUBSYSTEMS OF A MATHEMATICAL THEORY

The logico-linguistic subsystem. The basic level of this subsystem consists of the concepts connected with a mathematical theory. Their structure and properties are disclosed by means of various named sets, examples of which (for the concept of *number*) were given above. The symbolic forms of the expressions for some concepts (terms) used for the construction of alphabets (vocabularies) of theory languages are elements of the second level of the logico-linguistic subsystem. Symbolic forms have played a very important role in mathematics since the time of François Viète. For example, one of the forms for representing the number concept is connected with numerals as elements of some enumeration. The third level of the subsystem consists of the construction rules for expressions of the [theory] languages that are built from the elements of the second level. The fourth level includes various languages treated as systems of expressions built from symbols of alphabets in accordance with construction rules. In fact, any mathematical theory contains a family of languages. Ususally its presence is not taken explicitly into consideration. For example, in the theory of differentiable manifolds we find various logical languages (languages of predicate calculi of the first and higher orders, languages of set theory and category theory, natural languages including specific terms of the theory such as “real number”, “manifold”, “bundle”, etc.). The classification of languages may be grounded on their role in the main subsystems of a mathematical theory. In accordance with this, we can distinguish assertoric, model, procedural, algorithmic, axiological, erotetical, and other languages. All these languages as systems of expression are elements of the logico-linguistic subsystem, but their semantics are defined by means of other subsystems. The model-representation subsystem defines the semantics for model languages, the pragmantic-procedural subsystem for procedural, algorithmic, or axiological ones, the problem-heuristical subsystem for erotetic, heuristical, etc. On the basis of various principles, each of these languages may be subdivided into sublanguages.

The fifth level of the logico-linguistic subsystem contains rules for transforming expressions from the theory's languages. In accordance with the classification of languages there are many kinds of rules of transformation – for example, deduction, the best known of whose rules are *modus tollens* and *modus ponens*.

The sixth level – which is sometimes divided into two levels – contains formal calculi. The properties of calculi are primarily studied for assertoric languages. Any calculus is a named set $\mathcal{K} = (A, d, T)$ where A is an axiom system, T is the set of theorems of the given calculus, and d corresponds to rules which have been used in the process of deducing these theorems. Ordinary formal calculi, i.e. calculi for assertoric languages, are constructed by means of deduction rules. The seventh level of the logico-linguistic subsystem contains the tower of calculi introduced for the depiction of dynamic aspects of formal theories (Maslov).

The model-representation subsystem. Its first level consists of various names for entities from the object field of a theory. The second level contains names of properties and relations between entities studied by the theory. It also includes more complex constructions describing these properties and relations. The most important of such constructions is an abstract property. It has the form of a named set (U, p, L), where U is the universe of entities considered, p is a partial mapping, and L is the scale of a property (Burgin, *1985*). L is a partially ordered set.

It is often necessary in mathematical theories to treat on the same level both properties of entities and their names and more complex conceptual structures like properties of initial properties, properties of relations, properties of properties of properties, etc. In order to reflect this, the object field of a mathematical theory has to include names of objects, names of their properties and relations between them, and abstract properties together with their scales. The concept of a *set scale* $\gamma(X)$ provides the possibility for a precise description of this state of affairs (Burgin & Kuznetsov, *1986b*). This concept is a development of N. Bourbaki's set scale given in (Bourbaki) and of the concept of universes used in nonstandard analysis (Davis). Elements of these set scales are parts of the third level of the model-representation subsystem and are the supports of models which are constructed in this subsystem.

The fourth level of the model-representation subsystem contains basic abstract entities defined on the supports of models. The choice of basic properties depends on axiological judgements.

The fifth level consists of models. In the structural-naming reconstruction their general form takes the named set $\mathcal{M} = (\gamma\ (D), f, L)$. Here D is the set consisting of the names of the objects studied, the names of their properties and relations between them, the names of names, etc.; the names of abstract properties and relations corresponding to properties and relations of objects, to properties of their properties, etc.; and names of ideal entities like truth values. $\gamma\ (D)$ is the set scale with basis D. This set scale includes D and all its elements, functions from D into D, functions defined on these functions; the set of all subsets of D; and so on. In principle, the connections between elements from the support of $\mathcal{M}$ and properties of these elements are described by means of functions f_i defined on the basis or on a set from some higher level of the set scale. These functions take values in some partially ordered scales L_i, which are scales of properties of elements from the domains of the functions considered. As a matter of convenience, all of these are distinguished from the set scale $\gamma\ (D)$. As all functions are defined on the same support – the set scale $\gamma\ (D)$ – it is possible to substitute for them one function f taking values in the direct product of the scales $L = L_1 \times \ldots \times L_n$.

The functions included in a model may be distinguished on the ground of level of constructivity. For example, it is natural to consider functions given by means of descriptions in natural or mathematical non-algorithmic languages as descriptive ones. If the presentation of a function is realized in some algorithmic language, then its values may be found by means of a computational procedure. It is natural to call these functions computational ones. We may characterize different kinds of models according to the kinds of functions included in the models and the additional conditions on these functions. For example, if the functions included in a model satisfy the axioms of the theory, then one may talk about actual models, and so on. The complex structure of the fifth level of the hierarchy of the model-representation subsystem is reflected in the sixth and subsequent levels connected with the laws of different orders.

The problem-heuristical subsystem. Its structure is similar to the structure of the logico-linguistic subsystem. But instead of assertoric languages, it uses translational, heuristical and other languages for representing problems, questions, tasks, hypotheses, and heuristical methods of construction. It contains also calculi and algebras of problems, questions, hypotheses, etc. (Burgin & Kuznetsov, *1987*). It is necessary to stress that the problem-heuristical subsystems of the majority of mathematical theories are less developed and formalized than their logico-linguistic subsystems.

The pragmatic-procedural subsystem. It consists of two subsystems: the operation and the axiological. The concepts of *operation*, *algorithm*, *procedure*, and *process* occupy the central place in the first subsystem, as concepts of judgement and norm do in the second one. The problem-heuristical subsystem is closely connected with other subsystems. For example, only the introduction of the zero in the positional system of numeration allowed the situation to be overcome in which the representation and designation of numbers and their counting and recording were absolutely disjoint activities. Both activities and the ability to use those activities, were independent of one another (Knuth). The last analysis shows that our manner of doing arithmetical operations is closely connected with our way of representing numbers in these operations.

Another important concept from the pragmatic-procedural subsystem is that of *algorithm.* On the basis of an understanding of this concept it is possible to analyze it as a named set (X, A, Y), where *X* is the set of objects processed by the algorithm, *Y* the results produced by the algorithms (natural numbers, words, etc.), and *A* is the function (mapping) realized by the algorithm.

Note that in addition to the truth value applied to the assertions of a mathematical theory, there are other judgements in it such as the adequacy of models, the complexity of algorithms, and so forth.

The subsystem of ties. It unites all the subsystems considered above. Examples of the connections of these elements are the interpretations of languages and calculi in models; the correspondence between algorithms and procedures on the one hand and those processes on the other presented in the pragmatic-procedural subsystem; the determination of the properties of various elements from the main subsystems, and so on.

5. Some Aspects of the Development of Set Theory

We shall consider an application of the structural-naming reconstruction of a mathematical theory to an informal analysis of the structure and history of set theory. Some of its elements belong among the first mathematical concepts. For example, the concept of *natural number* appeared as an abstraction of definite properties of real families of mathematical objects. The words "set", "class", "collection", etc. were used in mathematics at all times and by all peoples. Descriptions such as "finite" and "infinite" existed in ancient Greece and were connected with problems such as the structure of space (is it infinitely divisible?) or of motion (the sophistic paradox of "the arrow"). As a result, different notions of infinity appeared. In ancient Greece, in addition, the following important problems were considered: Do infinite sets exist, and if so, then how – actually or potentially? Many procedures and algorithms known in antiquity and the middle ages have a set-theoretic character. As examples we can take the proof of the infinitude of the set of prime numbers or the exhaustion method of Eudoxus and Archimedes.

Thus, when set theory was beginning to take form as a separate mathematical field, elements of all its subsystems were already present in mathematics. The process of the creation of an explicit set theory was impossible before the main object of this theory – a set – was abstracted as a mathematical concept. That is why the starting point of set theory is the work of G. Cantor, where the first intuitive definition of a set is given, i.e., the concept of a *set* was introduced into mathematics. According to Cantor, we understand by a *set* a whole *M* bringing together definite distinguishable objects **m** of our contemplation or our thought (they will be called "elements" of the set *M*) (Cantor). By introducing the concept of *set*, the formation of the language of "naive" set theory was begun. This language, with some of its concepts and constructions changed, has been included as an informal (empirical) sublanguage in modern set theory and other mathematical theories.

Besides the concept of *set* (and the other terms "finite", "infinite", "element", etc., that were used earlier) in the linguistic resources of "naive" set theory, a number of new terms were introduced. They are symbolic forms of such concepts as "membership" ("membership relation"), "inclusion" ("inclusion relation"), "one-to-one correspondence", "power" or "cardinality", "equivalence", "countable", "uncountable", "transfinite number", and so on. The rules of construction from these and other mathematical terms involved words from natural languages and symbols for expressions from "naive" set theory as well as rules of propositional deduction in this theory. These rules that were used in fact fact already existed at the end of the nineteenth century.

From the very beginning of the development of set theory, its evolution was not restricted to the logico-linguistic subsystem but took place in all other subsystems and components as well.

Considering the model-representation subsystem, we see that its central component – models – have been taken from virtually all contemporary mathematics: sets of naturals were taken from arithmetic and number theory; sets of reals and multidimensional spaces were extracted from analysis and geometry; and so on. For Cantor, the initial models were trigonometric series and those number sets that appear in their study.

Connections between the logico-linguistic and the model-representation subsystems in the more general case are represented by named sets of models. In other words, for any axiomatic set theory **T** (for example, Zermelo-Fraenkel set theory or Gödel-Bernays set theory), it is possible to link a named set M_T to its models. Its support $S(M_T)$ is a collection of all sets that form a model $\mathcal{M}$ of the theory **T** (in the sense of mathematical logic). The set of names $N(M_T)$ of M_T consists all expressions from the language of the theory $\mathfrak{T}$ that denote sets, i.e. that name elements from $\mathfrak{M}$. For a set theory, elements from $\mathfrak{M}$ are usually regarded as sets or classes. The naming relation $n(M_T)$ attaches to elements from $\mathfrak{M}$ their names in $N(M_T)$. The axioms of $\mathfrak{T}$ either show how to build names from other names (for example, the axiom of union or the axiom of choice) or determine the relations between such names (for example Martin's axiom or the continuum hypothesis). In this way $S(M_T)$ belongs to the model-representation subsystem, $N(M_T)$ belongs to the logico-linguistic subsystem, and $n(M_T)$ is an element of the the subsystem of ties. This way of constructing models was used by K. Gödel (Gödel) for proving the consistency of the continuum hypothesis with other set-theoretical axioms and by P. Cohen (Cohen) for proving the independence of the continuum hypothesis. On the other hand, to any class $\mathfrak{M}$ of models its theory $\mathrm{Th}_{\mathfrak{L}}(M)$ can be applied. This theory consists of expressions from the logical language $\mathfrak{L}$ that are valid in all models in $\mathfrak{M}$. In this way many results in algebra can be obtained. In set theory such an approach is usually applied only in stages – first, using the axioms of $\mathfrak{T}$, the named set M_T is is constructed and only afterwards is its set of names (i.e. the axioms of $\mathfrak{T}$) extended by propositions valid in the model $S(M_T)$. It is also connected with the results of K. Gödel and P. Cohen mentioned above.

The formation of the pragmatic-procedural subsystem of set theory was followed by the construction of many new methods and algorithms for processing sets. Examples include the diagonal method of Cantor that was used to prove the uncountability of the continuum; Cantorian enumerations of finite and infinite totalities of countable sets; the procedures for establishing a one-to-one correspondence between the real line $\mathbb{R}$ and n-dimensional spaces $\mathbb{R}^n$ for an arbitrary n; the different algorithms for constructing "derivative" sets; the forcing that was proposed by Cohen for proving the independence of the continuum hypothesis and was later used for proving the [relative] independence of a number of hypotheses from transfinite arithmetic. infinitary combinatorics, general topology, measure theory, universal algebra, and model theory (Barwise).

In the process of the development of set theory, there appeared separate descriptions that belong to the axiological part of the pragmatic-procedural subsystem. To such descriptions as "finite" and "infinite" were added "equivalent", "countable", "constructive", "set of cardinality α", etc. The terms corresponding to these belong to the axiological language of set theory. The axiological part was further developed so that a strict mathematical form for existing and new descriptions was given. However it soon became clear that the values of such descriptions were not always univalent. As an example, "countability" is not an exact characterization of sets but depends upon the model of set theory that is under consideration (the so-called Skolem paradox (Skolem)). In the same way, constructibility means one thing in descriptive set theory and quite another in intuitionism and other branches of constructive mathematics, where constructivity is algorithmical. The latter kind of constructivity depends greatly on the construction used for the algorithm. For example, by means of the usual recursive algorithms (like Turing machines), sets from the arithmetical hierarchy at a higher level than $\sum_1$ are not computable and so are not constructive. Taking a class of sufficiently powerful superrecursive algorithms (like inductive Turing machines), however, we can make these sets computable, and so constructive (Burgin, *1983)*.

Active development of the problem-heuristical subsystem of set theory was inspired by the existence in set theory of a lot of unsolved problems, tasks, and questions. By the way, many statements of set theory (for example, the Cantor-Bernstein theorem and the continuum hypothesis) began as hypotheses.

One of the main characteristics of any mathematical theory is its consistency. It is generally thought that theories which do not satisfy this condition must be excluded from scientific knowledge as anomalies. That is why mathematicians reacted negatively when errors were found in "naive" set theory in the form of contradictions. These contradictions were regarded as paradoxes. All of them may be considered as problems of the existence in "naive" set theory of empty names (in the sense of the theory of named sets), i.e. expressions of the language of set theory that name (designate) sets whose existence contradicts other facts of set theory (set-theoretical paradoxes) or logical laws (semantic paradoxes).

Even at the time when the first paradoxes were found, set theory in its still incomplete and imperfect form had demonstrated its power and fruitfulness for mathematics as a whole. Hence, after finding these paradoxes many mathematicians expended great efforts to "save" set theory. To do that, they limited it in such a way as to exclude known paradoxes. Transformations connected with this effort occurred primarily in the logico-linguistic and the pragmatic-procedural subsystems of set theory. The appearance of some important trends in mathematics depended on the nature of these changes: classical or axiomatic, intuitionistic or constructive, each of them in its turn subdivided into more specialized trends and directions.

6. Main Directions in the Development of Set Theory

In classical approaches to set theory the main device chosen for excluding paradoxes was construction of strictly formalized assertoric languages for set theory and the selection of axioms on the one hand which were "foolproof" in avoiding the paradoxes and on the other hand sufficiently "powerful" to obtain the existing mathematical results in all of the main fields. In this way, axiomatic systems with their own languages were proposed. The most popular ones were the axiomatizations of Zermelo-Fraenkel, Gödel-Bernays, Russell, and von Neumann.

Note that in von Neumann's set-theoretic system (as well as in others), the central idea of a function is determined by the axioms. The possibility of considering this axiomatization as a set-theoretical one is based on the fact that in classical mathematics sets and functions are supposed to be equally fundamental. That means that each of them may be expressed in terms of the other. When the concept of a set is taken as a basis (for example in the systems of Zermelo-Fraenkel and Gödel-Bernays), then functions are represented by subsets of direct products of sets, so they became derivative constructions with respect to sets. In the case when a function is chosen as the main concept (as in the system of von Neumann), then any set may be represented by its characteristic function (the function equal to one for elements of this set and equal to zero for all other elements, i.e. those elements that do not belong to this set). It will be shown below that such kinds of relations are represented by morphisms of the named sets in models of the corresponding theories.

Afterwards, however, doubts appeared as to the possibility of the full mutual expressibility of the concepts of *set* and *function.* One of the arguments that shows the impossibility of reducing functions to sets is the following: set-theoretical representations of functions do not express "operational" or "transformational" aspects of the concept of *function* (Goldblatt). In other words, while sets express the static side of mathematical objects, functions explicate their dynamic side. Each of these sides is inseparably linked with the other. This shows, in particular, that functions do not exist without sets (functions always have domains and codomains) and sets at the same time are defined by the functions that separate those sets.

The mathematical construction that most completely reflects this inseparable unity of the static and dynamic sides of any mathematical object (and thus the unity of sets and functions) is the named set. It is important to note that ordinary sets (for example, in standard axiomatizations) are special kinds of named sets – just the singly named sets, i.e. the named sets in which all elements of the support have a common name. Thus, if we return to Cantor's definition of a set, then the name "M" of the whole set may be taken as such a common name. More strictly, each element of *M* will have the name "an element of the set *M*", i.e. all elements from *M* will possess the common name that is determined by their membership in *M*. In axiomatic systems, similar common names are constructed from a collection of initial names by means of those rules that are given in the language and that are formulated in the axioms of the system. For example, the empty set has a special (individual) name $\emptyset$ or the name $\{x \mid x \neq x\}$, i.e. the name "the set of objects that are unequal to themselves". As there are no such objects, the set of such objects is empty. Further, if *A* and *B* are names of sets, then $A \cup B$ and $A \cap B$ are the names of their union and intersection. The existence of these sets is guaranteed by the axioms of union, or subsets and of pairing in the given system (such as Zermelo-Fraenkel set theory). The name of a single element set generally has the form $\{\emptyset\}$.

In some general sense, all set-theoretical axioms may be divided into constructive (deterministic) and nonconstructive ones, depending on whether the names obtained by means of an axiom denote only one set or not. From this viewpoint, the axioms of pairing, union, power set, and subsets are constructive, while the axiom of choice will be nonconstructive because more than one set is extracted from it (named). This suffices to explain the negative attitude of some mathematicians towards this axiom and the higher value given to those proofs that do not use the axiom of choice.

The appearance of a direction such as descriptive set theory within the classical approach is connected with the above understanding of constructibility (Lusin; Ljapunov).

This approach originates in the demand to limit attention only to those sets that may be "calculated", "effectively constructed", "determined", etc. In other words, the names that are given to sets must have a procedural character, i.e. each name must uniquely determine the process of constructing the set with the assigned name (although in reality the process may be unrealizable) and thus must determine the set.

In this way, the difference between the standard classical and the descriptive approaches is determined by their orientation. The first is oriented by assertoric languages of the logico-linguistic subsystem. Meanwhile, the second (descriptive set theory) is oriented by a procedural language (in a generalized sense) that is used for describing set construction. Corresponding procedures are included in the pragmatic-procedural subsystem of set theory. Constructivism goes much farther still in its orientation to the pragmatic-procedural subsystem, the most radical versions of which do not use the concept of *set* (even as a name) at all. The main concept of this approach is an algorithm or, in other words, a constructive representation of functions (Markov & Nagorny). In fact algorithms are viewed in this treatment as the names of functions in the usual sense. That is why this approach is an analogue of those classical approaches in set theory in which the basis is taken to be the concept *function* and not a set. In constructive mathematics the term "set" is frequently not used at all, but in fact when it talks about alphabets, strings of letters (symbols), and words and about abstraction of potential realizability, the concept of *set* (more precisely of *multiset*) is used implicitly. It should be noted that in the intensional approach to understanding functions in constructive mathematics, algorithms play the role of functions. At the same time, computable (algorithmic) functions reflect the extensional approach.

In recursive mathematics, which has a constructive direction, the concepts of *function*, *algorithm* (as partial recursive function) and *set* exist in the language at the same time. For example, recursively enumerable, recursive, and productive sets and recursive, partial recursive, and primitive recursive functions are studied there.

In intuitionism the leading role is played not by the logico-linguistic subsystem but by the model-representation and the pragmatic-procedural subsystems. According to L. Brouwer (see Fraenkel & Bar-Hillel), mathematics is not a theory (in the usual way of understanding a system of propositions and the rules for constructing them, i.e., as a formal calculus from the logico-linguistic subsystem), but an important part of human activity (reflected in the pragmatic-procedural subsystem) that is concentrated on one form of our perception (represented by the model-representation subsystem), and in which the natural numbers are taken as the principal initial model.

One more direction in the development is alternative set theory. In alternative set theory, on the one hand, the expressive means of the set theory are considerably limited. These means are determined to some extent by the axioms of the theory, and in alternative set theory they are fewer and less powerful as compared to those of the usual axiomatic set theories. On the other hand, these limitations are partly compensated for by the introduction of new concepts (such as semiset, proper semiset, coding pair, codable classes, etc.) (Vopenka). An important feature of this approach is the possibility of building models for alternative set theory using only finite sets from other set theories (for example, from Gödel-Bernays or Zermelo-Fraenkel set theory). At the same time, the existence of infinite sets in those theories that were considered previously in the paper is either postulated (as in the Zermelo-Fraenkel system by the axiom of infinity) or is deduced from postulates (in intuitionism from the principle of the *a priori* existence of the infinite set of natural numbers and in constructivism from the principle of potential realizability).

All of the set-theoretical directions and approaches considered above (even those that do not explicitly include the concept of *set*) start from the two premises that follow. According to the first, *any element either belongs to a given set or does not.* The second premise states that *any two elements from one and the same set are distinguishable.* Altering the first premise led to the new rapidly-developing direction – fuzzy set theory (Zadeh). According to its founder L. Zadeh, a fuzzy subset A of the universe U is a pair $[A, M_A]$, where M_A: $U \rightarrow [0,1]$ is the membership function of the fuzzy subset.

Denial of the second premise has also led to the appearance of a new set-theoretical construction that came to be called a "multiset". It has many interesting and important applications in combinatorics and the theory of programming. By an informal definition given by Donald Knuth (Knuth), a multiset is in many ways analogous to an ordinary set, but differs from ordinary sets in that a multiset may include identical indiscernible elements. This allows a number of copies of any element to be in a multiset and so this number becomes important.

Without necessary details we note two principal points. Firstly, any ordinary set may be regarded as a special case of a fuzzy set (when the membership function takes only two values from the interval [0,1], either 1 or 0), as well as of a multiset (when it includes only one copy of any of its elements). Secondly, both fuzzy sets and multisets are particular cases of named sets. In fact, according to Zadeh, fuzzy sets result if the class (algebraic category) to which the support of the associated named set belongs consists of arbitrary sets. At the same time, the class of the sets of names consists of a single object, namely the interval [0,1]. Arbitrary maps of sets are considered as morphisms in the first class and as arbitrary binary relations on the interval [0,1] in the second class.

In accordance with Knuth's definition as given above, a multiset may be obtained if only total functions are taken as naming relations and the following axiom is added: *in an arbitrary named set* $\mathbf{X} = (X, \alpha, I)$ *any* *two elements from the support* X = S($\mathfrak{X}$) *are discernible iff they have different names in the set* N($\mathfrak{X}$) = I *of names.*

In this way, modern set-theoretical study is tied up with the subsystems of set theory. In this study, the resources of language are modified (for instance, a concept or *urelement* is introduced, i.e. of elements of sets that are not sets themselves); the notion of *membership* is made more precise or more general; or an abstraction of the identity elements is reconsidered); new operations and procedures are proposed (games and computations on sets, forcing, or ways of constructing nonstandard models); new axioms are introduced (the axiom of determinateness or Martin's axiom); and new models are elaborated (nonstandard or constructive) (Barwise).

7. Constructions from the Theory of Named Sets as a Means of Describing the Development of Set Theory

Stages in the development of set theory may be described by means of some constructions in the theory of named sets. We shall limit ourselves to an informal level of description using only definitions 2 and 3.

The passage from "naive" set theory to axiomatic systems reduced the class of names for sets, making them more formal; that is, the named sets of models of set theory were genuinely restricted. A restriction was required because of the discovery of paradoxes in "naive" set theory. These paradoxes indicated the existence in "naive" set theory of empty names, i.e. names to which, according to logical laws, no sets in the named sets of models of set theory could be applied. Hence, the restriction was introduced primarily in order to exclude such empty names, although it should be noted that in the process of restriction some admissible names were also prohibited.

The passage to descriptive set theory was tied to an internal restriction of the named sets of models. It was a consequence of a new reduction in the collection of admissible (correct) names, a result of the demands of constructivity. Even more severe conditions of constructivity or effectivity were elaborated in the constructive and intuitionistic approaches. This led to still greater internal restrictions on the named sets of models.

Any interpretation of one set theory T_1 in another one T_2 (for example, constructive or alternative theory in classical theory) or a construction of models for one of these theories by means of another one (frequently used for consistency proofs) may be represented as a morphism of named sets

$$J: M_{T_1} \to M_{T_2}$$

Passage from the theory of ordinary sets to the theory of fuzzy sets or to the theory of multisets is respectively connected with right extensions of the named set of models by the introduction of new admissible names and left extensions of the named set of models at the expense of the appearance of new models. Analogous extensions provide a passage to the theory of named sets, which has the widest object field and a more extensive class of models.

8. CONCLUSION

1. In spite of its brevity, the structural-naming analysis given above of "naive" set theory uncovered the main subsystems that must be present in any scientific theory. From this viewpoint, all of these subsystems of set theories continued their historical development. The various paths taken in its development gave support to the elaboration of one or another subsystem. Nevertheless, each proposed variant of set theory contains all of the subsystems necessary for a mathematical theory. These variants are connected by different ties and relations (historico-genetic, logical, linguistic, model-theoretic, and so on). They form a theory-net (Balzar & Sneed), in which the elements are individual set theories.

2. The concept of *set* appeared to be very complex and many-sided, leading to the construction of many set theories. Consequently the term "set theory" acquired at least two meanings in modern mathematics. In a wider

sense, set theory is viewed as a mathematical field where sets are studied as the central objects. In a narrow sense, set theory is some completely or partially formalized mathematical theory. It may be capable of saying a good deal about properties of sets, have a developed system of procedures for working with sets, and may serve as a basis for all or a specific part of mathematics. In addition, the theory must be consistent. Examples of such theories are Zermelo-Fraenkel set theory, the Gödel-Bernays theory of sets and classes, Russell's theory of types, von Neumann's axiomatics, Quine's system, Wang Hao's system, Lorenzen's system, descriptive set theory, alternative set theory, etc.

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