

Constructive Logic and the Paradoxes

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§0. INTRODUCTION

Heyting's formalization of intuitionistic mathematics started many discussions about the meaning of the logical connectives in terms of proof and construction. We focus on the ideas and results related to the interpretation of implication and on formal systems that have different rules for implication. Some of these systems are not intended to contribute to the discussions mentioned above, but are related to the Basic Calculus introduced in §§3 and 4.

The set-theoretic paradoxes of the turn of the century shocked many mathematicians into realizing that their simple intuitions about sets and logic were inconsistent. Constructive mathematics along the lines of Brouwer, Markov, and Bishop is not intended to resolve this issue, and doesn't. The most common solutions favored by mathematicians involve reducing one's attention to a hierarchical class of sets, thereby excluding the paradoxical ones. A few mathematicians and logicians kept searching for the Holy Grail of set theory with full comprehension by changing the rules of equality or logic. Of special interest to the Basic Calculus and the set theory F of §5 is Fitch's system with the additional implication hierarchy introduced by Myhill. It seems that this approach replaces a hierarchy of sets by a (simpler) hierarchy of implications.

In an attempt to find a non-circular proof interpretation for the logical connectives, we change from Heyting's axiomatization to a subsystem of intuitionistic logic with a limited *modus ponens*: Basic Calculus. In set theory with full comprehension over this subsystem, Russell's Paradox turns into a proof of Löb's Rule, a rule that is relatively inconsistent with *modus ponens*.

§1. THE PROOF INTERPRETATION

L. E. J. Brouwer's introduction of intuitionistic mathematics was not a reaction to the paradoxes, although its influence may have been felt; it offered an alternative to the formalist and logicist approaches. Consequently, a naïve extension of intuitionistic mathematics to set theory with full comprehension does not solve the paradoxes. Brouwer's Ph.D. thesis of 1907, and later work, expounded the intuitionist's point of view (see [Brouwer 1975]). One aspect of this point of view was a proscription of the use of logical principles as a guide to mathematics, the most well-known among these being the Principle of the Excluded Middle. The

Principle of the Excluded Middle holds in verifiable 'finite' situations, but cannot be generalized to a rule of mathematics. Brouwer even introduced new principles that contradict Excluded Middle; they imply, among other things, that all real-valued functions are continuous. These additional principles have been criticized by other constructivists (see, for example, [Bishop, Bridges 1985, p. 9]). Brouwer avoided the use of formal language, perhaps not because of its unreliability, but as a matter of personal style [Brouwer 1981, p. xi].

In 1927 the Dutch Mathematical Society published a prize question concerning a formalization of intuitionism. Brouwer's student A. Heyting wrote an essay on it and was awarded the prize in early 1928. It appeared in the *Sitzungsberichte der preußischen Akademie von Wissenschaften* as [Heyting 1930], [Heyting 1930a] and [Heyting 1930b], although it was originally intended to appear in *Mathematische Annalen* [Smoryński 1988, p. 48]. In these same papers Heyting introduced Heyting Arithmetic HA , the intuitionistic equivalent of Peano Arithmetic, and an incomplete axiomatization of analysis, the theory of choice sequences.

In modern notation, using sequents, the axiomatization of Intuitionistic Predicate Calculus IQC can be stated as follows. We use Latin letters to refer to substitution places for formulas, and Greek letters for actual formulas of a first-order language with logical constants \top, \perp ; binary connectives $\wedge, \vee, \rightarrow$; and quantifiers \forall, \exists . We also assume our language to have an equality predicate $=$ (there are only a few occasions where one wishes to do without $=$). Negation $\neg A$ is introduced as an abbreviation for $A \rightarrow \perp$; bi-implication $A \leftrightarrow B$, as an abbreviation for $(A \rightarrow B) \wedge (B \rightarrow A)$. IQC is given by sequent rules and sequent axioms. For the rules a thin horizontal line means that if the sequents above the line hold, then so do the ones below the line. A fat line means the same, but in both directions.

$$\frac{A \vdash A \quad A \vdash B \quad B \vdash C}{A \vdash C}$$

$$\frac{A \vdash \top}{A \vdash B \quad A \vdash C} \quad \frac{\perp \vdash A}{B \vdash A \quad C \vdash A}$$

$$\frac{A \wedge B \vdash C}{A \vdash B \rightarrow C}$$

$$\frac{A \vdash Bx}{A \vdash Bt} \dagger$$

$$\frac{A \vdash Bx}{A \vdash \forall x Bx} \dagger \quad \frac{Bx \vdash A}{\exists x Bx \vdash A} \dagger$$

$$\top \vdash x = x$$

$$x = y \vdash Ax \rightarrow Ay.*$$

We allow the substitution of new variables for bound variables. In case †, the variable x is not free in A and the term t does not contain a variable bound by a quantifier of B ; in cases ‡, the variable x is not free in A ; and in case *, the variables x, y are not bound by a quantifier of A .

The subsystem of *IQC* without quantifiers, terms, or equality is Intuitionistic Propositional Calculus *IPC*. Extensions of *IQC* may be constructed by adding additional rules and sequent axioms. Extensions are called theories. All theories are inductively defined unless explicitly presented otherwise (see *PIPC* below). A theory T is called an *extension* of a theory S if T satisfies all rules and sequents of S . Thanks to *modus ponens*, additional sequents may be assumed to be of the form $\top \vdash A$, usually abbreviated as $\vdash A$ or, if no confusion is possible, as A . Examples are Classical Predicate Calculus *CQC* equals *IQC* extended with $\vdash A \vee \neg A$, and *CPC*, which is the similar extension of *IPC*. The theory *PIPC* of propositional *prime* extensions of *IPC* is the extension of *IPC* with the rule

$$\frac{\vdash A \vee B}{\vdash A \text{ or } \vdash B}$$

The ‘or’ in the new rule makes the definition of *PIPC* non-inductive. Note that *IPC* doesn’t satisfy this rule: In its extension *CPC* both $\vdash A$ and $\vdash \neg A$ fail for some A . On the other hand, if $\varphi \vdash \psi$ holds in *PIPC*, then it also holds in *IPC* [Gödel 1932]. So *PIPC* is a proper extension of *IPC* because *PIPC* has fewer extensions. There exists a theory *PIQC* of prime extensions of *IQC* similar to *PIPC* if sufficiently many constant symbols are available: *PIQC* extends *IQC* and *PIPC* with the non-inductive rule

$$\frac{\vdash \exists x Ax}{\vdash A(c) \text{ for some constant } c}$$

Let Γ be a set of formulas, sequents, and rules and φ be a formula. We write $\Gamma \vdash \varphi$, Γ *proves* φ , if there exists a finite subset $\Delta \subseteq \Gamma \cup \{\top\}$ such that $\bigwedge_{\delta \in \Delta} \delta \vdash \varphi$ is a consequence of the system *IQC* plus the additional sequents and rules of Γ . So $IPC \vdash \varphi$ if and only if $PIPC \vdash \varphi$, for all φ .

In a letter to Oskar Becker, Heyting explains his discovery of the axiomatization of (a system equivalent to) *IQC* by going through *Principia Mathematica* [Whitehead, Russell 1925] and making a new system out of the acceptable axioms and rules.

Brouwer did not explicitly give interpretations for the logical connectives, so Heyting and others had to discover an interpretation for them independent of Brouwer or to extract their meaning from their use in Brouwer’s papers. Heyting, and independently A. N. Kolmogorov, developed a proof interpretation for the logical operators [Heyting 1934], [Kolmogorov 1932]. Their interpretations are essentially equivalent. Following [Sundholm 1983] it is more appropriate to speak of ‘explanation’ rather than of ‘interpretation.’ Unfortunately, ‘interpretation’ has become widely used, so we will adhere to that terminology.

A statement φ is *true* only if we have a proof p for it satisfying the requirements mentioned below. We assume the quantifiers \exists and \forall to range over a sufficiently simple domain.

- (1) p proves $\varphi \wedge \psi$ just in case p consists of a pair q, r of proofs of φ and ψ .
- (2) p proves $\varphi \vee \psi$ just in case p consists of a pair n, q such that either $n = 0$ and q proves φ or $n = 1$ and q proves ψ .
- (3) p proves $\varphi \rightarrow \psi$ just in case it provides a construction q that transforms proofs s of φ into proofs qs of ψ .
- (4) p proves $\exists x\varphi x$ just in case p consists of a pair q, r such that q is a construction that yields an element c such that r is a proof of φc .
- (5) p proves $\forall x\varphi x$ just in case it provides a construction q such that for all c in the domain, qc is a proof of φc .

Negation $\neg\varphi$ is an abbreviation for $\varphi \rightarrow \perp$; there is no proof for \perp . The interpretation is known as the Brouwer-Heyting-Kolmogorov (BHK) interpretation [Troelstra, van Dalen 1988, pp. 9–10].

The proof interpretation is not reductive: It doesn't express the interpretations of implication and universal quantification in simpler terms.

S. C. Kleene's realizability [Kleene 1945], although not conceived for that purpose, can be considered a formal justification of the proof interpretation with respect to intuitionistic arithmetic HA . Kleene's motivations came from the finitistic interpretation of the connectives in [Hilbert, Bernays 1934] (see [Kleene 1973]), but it takes little to understand the realizability interpretation of HA -formulas as modeling the BHK interpretation.

Consider the first-order language of arithmetic including a (primitive) recursive pairing function $\langle x, y \rangle$ with projections p_1x and p_2x , and the partial function $\{x\}y$, the Kleene bracket expression of applying the x^{th} partial function to y . We write $!\{x\}y$ as an abbreviation for $\exists z(\{x\}y = z)$. Number realizability is a translation $A \mapsto xrA$ of formulas A not containing x , inductively defined by

$$\begin{aligned} xr\top &= \top; \\ xr\perp &= \perp; \\ xr(t = u) &= (t = u), \quad t, u \text{ terms}; \\ xr(A \wedge B) &= p_1xrA \wedge p_2xrB; \\ xr(A \vee B) &= (p_1x = 0 \rightarrow p_2xrA) \wedge (p_1x \neq 0 \rightarrow p_2xrB); \\ xr(A \rightarrow B) &= \forall y(yxrA \rightarrow !\{x\}y \wedge \{x\}yxrB); \\ xr(\exists yAy) &= p_2xrA(p_1x); \quad \text{and} \\ xr(\forall yAy) &= \forall y(!\{x\}y \wedge \{x\}yxrAy). \end{aligned}$$

If it were the case that A is *true* in HA if and only if nrA is *true* for some number n , then this translation could be considered an explication of the proof interpretation, as such an n encodes the reasons why A is *true* along lines in accord with the proof interpretation. However, we may be able to show nrA for some n without being able to prove A within HA .

We call a formula *almost negative* if it is built up from formulas of the form $\exists x(t = u)$ using \wedge , \rightarrow , and \forall . The axiom schema ECT_0 , the *extended Church's thesis*, states

$$\forall x(Ax \rightarrow \exists yB(x, y)) \vdash \exists z\forall x(Ax \rightarrow !\{z\}x \wedge B(x, \{z\}x)),$$

where A is almost negative. We may interpret ECT_0 as saying that all functions of arithmetic are recursive. By [Troelstra 1973, p. 196], we have

$$HA + ECT_0 \vdash A \leftrightarrow \exists x(xrA)$$

and

$$HA + ECT_0 \vdash A \text{ if and only if } HA \vdash \exists x(xrA).$$

So nrA says that if we limit ourselves to a recursive universe, then n encodes evidence for the truth of A . Many constructivists suspect that everything that one will ever encounter in constructive arithmetic is recursive; however, the *proposition* that everything in constructive arithmetic is recursive is not constructive and therefore not acceptable.

The following variation on number realizability, q -realizability, circumvents the limitations of a recursive universe. The translation $A \mapsto xqA$ for formulas not containing x is defined inductively by

$$\begin{aligned} xq\top &= \top; \\ xq\perp &= \perp; \\ xq(t = u) &= (t = u), \quad t, u \text{ terms}; \\ xq(A \wedge B) &= p_1xqA \wedge p_2xqB; \\ xq(A \vee B) &= (p_1x = 0 \rightarrow A \wedge p_2xqA) \wedge (p_1x \neq 0 \rightarrow B \wedge p_2xqB); \\ xq(A \rightarrow B) &= \forall y((A \wedge yqA) \rightarrow !\{x\}y \wedge \{x\}yqB); \\ xq(\exists yAy) &= A(p_1x) \wedge p_2xqA(p_1x); \quad \text{and} \\ xq(\forall yAy) &= \forall y(!\{x\}y \wedge \{x\}yqAy). \end{aligned}$$

q -realizability doesn't have the straightforward connection with a formal system that number realizability has with $HA + ECT_0$. In fact, q -realizability is not closed under deduction [Troelstra 1973, p. 205]. The expression nqA only provides, following [Kleene 1973], missing information about a proof of A from HA . For example, using q -realizability, we can show that if $HA \vdash \forall x(Ax \rightarrow \exists yB(x, y))$ with A almost negative, then $HA \vdash \forall x(Ax \rightarrow !\{e\}x \wedge B(x, \{e\}x))$ for some e . As an explanation for the proof interpretation q -realizability fails for another reason as well. The translations for \vee , \rightarrow , and \exists refer to proofs of HA and therefore the explanation is not reductive. This final argument especially applies to the variant of q -realizability in [Troelstra, van Dalen 1988, p. 243].

One problem with the *BHK* proof interpretation is its circularity in the explanation of implication, as observed by G. Gentzen. This argument [Gentzen 1936,

§11], set in the context of HA , involves finitist interpretations of the connectives and is directed towards a consistency proof of arithmetic. It applies more generally. Gentzen notes that if we want to explain the meaning of ‘ p is a proof of $\varphi \rightarrow \psi$ ’ in the sense of the BHK proof interpretation, then implication introduction

$$\frac{A \wedge B \vdash C}{A \vdash B \rightarrow C}$$

is perfectly permissible as $A \wedge B \vdash C$ merely expresses that a proof of C from B , given A , is available. The restricted application of *modus ponens*

$$\frac{\vdash A \quad A \vdash B}{\vdash B}$$

is in harmony with the BHK interpretation since $A \vdash B$ expresses that we have an actual proof for B if we assume A . Restricted *modus ponens* is a special case of transitivity. On the other hand, implication elimination, the reverse of implication introduction, is equivalent to the full *modus ponens* axiom

$$A \wedge (A \rightarrow B) \vdash B.$$

In this case, the existence of a proof of B from A is expressed in terms of \rightarrow by the assumption $A \rightarrow B$, hence the explanation for implication is circular.

Negation $\neg A$ is equivalent to $A \rightarrow \perp$, so its BHK interpretation suffers from a drawback similar to that for implication. Additionally, there is the problem of interpreting the meaning of \perp . Replacement of \perp by the statement $0 = 1$ resolves the issue in the special case of HA ; the problem remains for the general situation.

Universal quantification is considered in [Gentzen 1936, §10]. According to Gentzen, a sentence $\forall x Bx$ may be understood finitistically as ‘ $B0$ and $B1$ and $B2$ and ...’ as long as Bx is fairly simple. Moreover, in the rule

$$\frac{A \vdash Bx}{A \vdash \forall x Bx}$$

with x not free in A , the justification of the rule is elementary only if the formula A is quantifier free. Unfortunately, Gentzen’s approach to universal quantification is insufficient as a justification of the BHK interpretation, as we will make abundantly clear when discussing BQC below.

M. Okada attempts to resolve the circularity in the axiomatization of \rightarrow by considering subsystems of intuitionistic logic with weakened implication introduction [Okada 1987], [Okada 1988]. The propositional logic WLJ of [Okada 1987] is equivalent to the intuitionistic propositional calculus IPC except that the rule

$$\frac{A \wedge B \vdash C}{A \vdash B \rightarrow C}$$

holds only for A that are conjunctions of implications $D_i \rightarrow E_i$. This is unexpected, particularly because Gentzen suggests problems with implication elimination, not implication introduction. Okada’s approach may produce a subsystem

for which a Gentzen-style consistency proof works, but it lacks a sound philosophical motivation. Okada introduces a validity concept for sequents and shows that all sequents derivable in *WLJ* are valid, thereby justifying the validity of *WLJ* under his constructive semantics. Valid sequents, however, need not be derivable in *WLJ*.

In [Okada 1988] we find a first-order extension *RLJ* of *WLJ* with unrestricted universal quantification. Gentzen's semantics [Gentzen 1936] gives a justification for a complete set of inference rules for *RLJ*. A justification along the lines of the *BHK* proof interpretation fails.

In [Kreisel 1962] we find the following suggested modifications of (3) and (5) of the *BHK* proof interpretation:

(3') p proves $\varphi \rightarrow \psi$ just in case p consists of a pair q, r such that q is a construction that transforms proofs s of φ into proofs qs of ψ and such that r is a proof that q is such a construction.

(5') p proves $\forall xAx$ just in case p consists of a pair q, r such that for all c in the domain, qc is a proof of Ac , and such that r is a proof that q is so.

Kreisel's version has been treated as a viable alternative to Heyting's proof interpretation (see, e.g., [Troelstra 1977] and [van Dalen 1982]); more recently it seems to have slid into the background [Troelstra, van Dalen 1988, pp. 9, 31].

The revised interpretation in [Kreisel 1962] comes from an attempt to set up a formal system in terms of which the formal rules of intuitionistic predicate calculus can be interpreted. According to Kreisel an intuitionistic statement A is understood if we have a construction r_A that decides for each construction c whether or not it is a proof of A . Moreover it is assumed that we recognize a proof when we see one.

Kreisel introduces the relation $\Pi(c, A)$ as formal notation for

Construction c proves A .

The decidability of $\Pi(c, A)$ requires the additional clauses of (3') and (5'). Expressions $\Pi(c, A)$ themselves are treated as formulas in the same way as A , allowing for composite expressions like $\Pi(d, \Pi(c, A))$ [Sundholm 1983].

P. Aczel observed that we ought to add that Kreisel's system tacitly assumes the existence of a universe to which everything belongs [Sundholm 1983]. In particular, the existence of the constructions r_A seems to assume the existence of a universe of all possible constructions because the r_A universally quantify over all such constructions. G. Sundholm states in [Sundholm 1983, p. 155] that it is the equivalence

A if and only if $\exists p(p \text{ is a proof of } A)$

that presupposes the existence of a universe of 'everything.' This argument seems slightly weaker since the quantifier may range over an incomplete universe.

In [Beeson 1985, pp. 403ff], M. J. Beeson illustrates the difficulties one encounters when trying to formalize Kreisel's predicate $\Pi(c, A)$ using a simple straightforward approach; his resulting system \mathbf{C} is conservative over HA , but refutes the

decidability of Π . In a subsequent discussion Beeson suggests that one possible cause is trying to unite two incompatible intuitive concepts—a universe and decidability of Π .

In [Bishop 1970] E. Bishop calls the ‘numerical’ meaning of implication the most urgent foundational problem of constructive mathematics. In [Bishop, Bridges 1985] there is an interpretation of implication $P \rightarrow Q$ that differs somewhat from the *BHK* definition: ‘... the validity of the computational facts implicit in the statement P must insure the validity of the computational facts implicit in the statement Q ...’, but Bishop expresses dissatisfaction with this. Fortunately, in [Bishop, Bridges 1985], in each *instance* where (even nested) implication is used, the ‘numerical’ meaning is clear, although there is no *general* interpretation for implication.

As a point of departure for finding a general interpretation for implication, Bishop, in examining some of his theorems in constructive analysis that involve implication, notices the following pattern. A *complete* mathematical statement—a theorem including all its prerequisites such as definitions and proofs of theorems on which it depends—essentially asserts that a given constructive function f with constructive domain S vanishes, that is, $fx = 0$ for all $x \in S$. Bishop’s theorems involve *incomplete* mathematical statements, statements that assert that there exists a constructive set T such that if we construct an element $y \in T$, then $P(y)$ is a complete statement. Thus incomplete statements are of the form $\exists y \forall x A(x, y)$ where A is a decidable predicate. If this is taken as a rule, then implications are of the form

$$(1) \quad \exists y \forall x A(x, y) \rightarrow \exists v \forall u B(u, v).$$

The similarity with [Gödel 1958] is obvious, and Bishop bases his argument on Gödel’s interpretation. In the formal system of [Gödel 1958] it is possible to convert a proof of (1) into a proof of

$$\exists \bar{v} \exists \bar{x} \forall y \forall u (A(\bar{x}(y, u), y) \rightarrow B(u, \bar{v}(y))).$$

Bishop speculates that one day this ‘numerical’ implication may replace ordinary implication. Unfortunately, his later studies of implication are left as only fragmentary notes [Bishop, Bridges 1985, p. 13].

§2. PARADOXES

G. Frege’s naive logical notion of set is inconsistent [Frege 1893], [Frege 1903]. B. Russell showed in 1901 that in Frege’s system the set $R = \{x \mid x \notin x\}$ is an element of itself if and only if it is not. This argument is elementary and needs only a small fragment of Frege’s system: The ‘property’ R is a set since $\{x \mid \varphi(x)\}$

is a set for all φ ; and $a \in \{x \mid \varphi(x)\}$ holds if and only if $\varphi(a)$ holds. So if $\varphi(x)$ is the formula $x \notin x$, then $R \in R$ if and only if $R \notin R$.

The correspondence between Russell's Paradox and the Liar Paradox is well-known. J. van Benthem points out the logical nature of Russell's Paradox [van Benthem 1978]; Russell's R cannot be a set because of the tautology

$$\neg \exists x \forall y (Pyx \leftrightarrow \neg Pyy).$$

This tautology is essentially non-propositional.

The most popular solutions to the Russell Paradox use hierarchical models like Russell's type theory or Zermelo-Fraenkel set theory with choice ZFC . In these systems sets come after their elements. So R cannot be a set.

Russell's Paradox is related to G. Cantor's paradox on the entity of all sets. In [Cantor 1891] Cantor showed that the power set of a set is bigger than the original set in the sense that there exists no map from a set S onto the power set $\mathcal{P}S$. For suppose $f: S \rightarrow \mathcal{P}S$ is onto. Let $R = \{x \in S \mid x \notin fx\}$. Since f is onto, $R = fv$ for some v . So $v \in fv$ if and only if $v \notin fv$.

If, by the assumption of Cantor's Paradox, there exists a set V of all sets, then $y \in V$ if and only if $y \subseteq V$. So the identity maps V onto $\mathcal{P}V$, a contradiction.

In Cantor's view the paradoxes concerning the collection of all sets or the collection of all ordinals (Burali-Forti Paradox, 1896/1897) show only that certain constructs $\{x \mid \varphi(x)\}$ do not represent sets. The same idea is reflected in the modern set/class distinction.

There is a version of λ -notation similar to set notation where we write $\lambda x.f(x)$ for $\{x \mid f(x)\}$. The λ -calculus dates back to the combinatory algebras of [Schönfinkel 1924] and [Curry 1930] and the 'extended' λ -calculus of [Church 1932], [Church 1933]. Kleene and J. B. Rosser showed that A. Church's system is inconsistent, essentially by deriving Richard's Paradox [Kleene, Rosser 1935]. Modern pure λ -calculus is a consistent derivative of Church's system [Barendregt 1984, I.1.1]. The typed combinatory algebra (it is similar for the typed λ -calculus) has basic entities I_A , $K_{A,B}$, and $S_{A,B,C}$ for all A, B, C . Each object (function) has a type $B \Leftarrow A$, and if f and g are functions of types $C \Leftarrow B$ and $B \Leftarrow A$ respectively, then fg has type $C \Leftarrow A$. Application represents *modus ponens*: If f has type $B \Leftarrow A$ and a has type A , then $f(a)$ has type B . In [Curry, Feys 1958] it is pointed out that the types of I_A , $K_{A,B}$, and $S_{A,B,C}$, being $A \Leftarrow A$, $(A \Leftarrow B) \Leftarrow A$, and $((A \Leftarrow C) \Leftarrow (B \Leftarrow C)) \Leftarrow ((A \Leftarrow B) \Leftarrow C)$, respectively, constitute a complete set of axioms for intuitionistic implicational logic.

Russell's Paradox then can be presented in the form

$$\begin{aligned} &\lambda X. \neg X(X)(\lambda X. \neg X(X)) \\ &\text{if and only if} \\ &\neg \lambda X. \neg X(X)(\lambda X. \neg X(X)). \end{aligned}$$

In [Curry 1942] H. B. Curry interprets inconsistency as the ability to derive all statements. This enables him to remove negation as an ingredient of Russell's Paradox: For arbitrary B , let F be the term $\lambda X.(X(X) \rightarrow B)$. Then $F(F)$ equals $F(F) \rightarrow B$. Following the notation of [van Benthem 1978, p. 54] Curry argues that $F(F) \rightarrow F(F)$ holds, and thus $F(F) \rightarrow (F(F) \rightarrow B)$. By absorption we have $F(F) \rightarrow B$ which equals $F(F)$. So by *modus ponens*, B .

Van Benthem mentions two reactions to Curry's Paradox. P. T. Geach suspects the special application of the absorption rule $A \rightarrow (A \rightarrow B) \vdash A \rightarrow B$ [Geach 1972, pp. 210–211], but presents insufficient motivation, while F. B. Fitch blames *modus ponens*. But most mathematicians consider *modus ponens* 'the logician's best friend' ([van Benthem 1978, p. 55]; J. Myhill in a private discussion in 1983). In this paper we attempt to show that *modus ponens* need not be our best friend.

Curry sought to solve the paradoxes by adding illative notions to pure combinatory logic [Curry, Feys 1958], [Curry et al. 1972]; both Curry and Aczel add a 'proposition' condition to the objects, making paradoxical objects non-propositions [Aczel 1980] (see [Barendregt 1984] for more references). Extensions of [Aczel 1980], and consistency for these extensions, are presented in [Flagg, Myhill 1987], [Flagg, Myhill 1987a], and [Flagg 1987].

Fitch has a system that is related to the approaches followed by Curry and Aczel. It avoids Curry's Paradox, and obtains consistent systems, by reducing the rules for implication [Fitch 1950], [Fitch 1952]. Fitch's earlier systems are weak in that many properties can only be expressed in an 'external' way; e.g., being a function from the reals to the reals must be expressed in the metalanguage [Myhill 1975, p. 181]. The weak implications that are added later on are cumbersome and, according to Myhill, philosophically unnatural [Myhill 1975, p. 182]. Myhill proposes another solution to the Curry Paradox while remaining close to Fitch's system. Essentially, Myhill's solution entails introducing indexed implications $\rightarrow_1, \rightarrow_2, \rightarrow_3, \dots$ in place of a simple implication \rightarrow . The resulting axiom system consists of a set of axioms and rules T_0 that includes neither introduction or elimination rules nor axioms for implication. The system T_0 may include set-theoretical axioms including full comprehension and other axioms involved in some of the traditional set-theoretic paradoxes. We have an ascending sequence $T_0 \subseteq T_1 \subseteq T_2 \subseteq \dots$ of extensions such that T_n satisfies the additional introduction rule

$$\frac{A \wedge B \vdash C \text{ follows from } T_{n-1}}{A \vdash B \rightarrow_n C}$$

and the elimination rule

$$\frac{A \vdash B \rightarrow_m C \text{ with } m \leq n}{A \wedge B \vdash C}$$

Myhill's system is similar to the union of the systems T_n . The system in [Myhill 1975] includes a special treatment of negation: System T_n includes the axioms

$$A \wedge \neg B \vdash \neg(A \rightarrow_n B) \quad \text{and} \quad \neg(A \rightarrow_n B) \vdash A \wedge \neg B.$$

If T_0 is a sufficiently rich language, then we can construct the object $R = \{x \mid x \in x \rightarrow_1 B\}$ for some B . With extensionality we derive

$$R \in R \vdash R \in R \rightarrow_1 B,$$

and thus

$$R \in R \vdash B$$

in T_1 but not in T_0 . So only

$$\vdash R \in R \rightarrow_2 B$$

holds and no Russell Paradox results.

In [Löb 1955] M. H. Löb answered a question of L. Henkin on the provability in Peano Arithmetic PA of sentences that assert their own provability. Let $\Box A$ be short for $\text{Prov}_{PA}(\ulcorner A \urcorner)$. Then does $PA \vdash A \leftrightarrow \Box A$ imply $PA \vdash A$? Löb's affirmative answer is a corollary of the remarkable *Löb's Rule*. Provability \Box satisfies the axiom schemas

- (1) $\Box(A \rightarrow B) \vdash \Box A \rightarrow \Box B$;
 (2) $\Box(\Box A \rightarrow A) \vdash \Box A$ (Löb's Axiom);

and the rule

$$(3) \quad \frac{\vdash A}{\vdash \Box A}.$$

The schema

$$(4) \quad \Box A \vdash \Box \Box A$$

and Löb's Rule

$$\frac{\vdash \Box A \rightarrow A}{\vdash A}$$

follow from these axioms and rule and Intuitionistic Propositional Calculus IPC . This invites us to define a modal logic PrL (by [Smoryński 1985]; G. Boolos calls it G [Boolos 1979]) by extending the system CPC with the axioms and rule (1), (2), and (3) for the modal operator. We may replace Löb's Axiom by Löb's Rule without changing the strength of PrL ; the proof of the equivalence even works when we replace CPC by IPC .

R. M. Solovay discovered that the axioms for PrL are complete in the sense described below [Solovay 1976]. Each map Φ that maps atoms p of the language of PrL to sentences Φp of the language of PA can be extended to a map on the collection of all formulas of PrL by induction: $\Phi \top = \top$; $\Phi \perp = \perp$; $\Phi(A \circ B) = \Phi A \circ \Phi B$ for $\circ \in \{\wedge, \vee, \rightarrow\}$; and $\Phi(\Box A) = \text{Prov}_{PA}(\ulcorner \Phi A \urcorner)$. Then

$$PrL \vdash \varphi \text{ if and only if } PA \vdash \Phi \varphi \text{ for all } \Phi.$$

From the Explicit Fixed Point Theorem for PrL ([Boolos 1979, p. 141], [Smor-
yński 1985, p. 79]) it follows that for every B there exists A such that

$$(5) \quad \vdash A \leftrightarrow (\Box A \rightarrow B).$$

holds in PrL . Löb, conversely, uses (5) to derive his Rule: (5) holds in PA by the Fixed Point Lemma for PA . Axioms and rule (1), (3), (4), and (5) are given as directly following from PA . Assume $\vdash \Box B \rightarrow B$. Then (5) and (3) give us $\vdash \Box(A \rightarrow (\Box A \rightarrow B))$ for some A , and thus $\vdash \Box A \rightarrow (\Box \Box A \rightarrow \Box B)$. So $\vdash \Box A \rightarrow \Box B$ and, using the assumption, $\vdash \Box A \rightarrow B$. By (5), $\vdash A$; and thus $\vdash B$. This proof shows the immediate connection with Curry's Paradox; the referee to [Löb 1955] points out that if we replace \Box by 'true', then this argument implies that every sentence is true.

§3. BASIC PROPOSITIONAL CALCULUS

Intuitionists generally consider intuitionistic logic to be a description of regularities that are observed in intuitionistic mathematical practice. It is also generally maintained that first-order intuitionistic calculus IQC is a proper reflection of these regularities; Heyting developed the formalization and proof interpretation, and Brouwer appreciated this clarification of intuitionism. We believe, however, that a 'truly' constructive logic should have an interpretation that is non-circular and constructive in itself. We have not found an interpretation for IQC satisfying this constraint. One way to obtain a system with non-circular interpretation is by reversing the approach usually taken: Start with the proof interpretation and derive logical rules that are acceptable following this interpretation.

We assume the existence of a universe of proofs U . The term universe is not to be understood as meaning 'set'. We have only limited knowledge about what constitutes a proof, and therefore only limited knowledge about the complexity of U . A sequent $\varphi \vdash \psi$ expresses that there exists a proof that derives ψ from the assumption φ . So proofs depend on assumptions φ as much as they prove statements ψ . Next we present examples of some existence and closure properties of U . If we assume φ , then we accept this very assumption as a proof of φ . So $\varphi \vdash \varphi$ holds for all φ . Axiom schemas like $A \vdash A$ and $A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$ express the existence of certain proofs in U . If p is a proof of $\varphi \vdash \psi$, and q is a proof of $\psi \vdash \theta$, then there exists a composition pq that constitutes a proof of $\varphi \vdash \theta$. So rules like

$$\frac{A \vdash B \quad B \vdash C}{A \vdash C}$$

or

$$\frac{A \vdash B \quad A \vdash C}{A \vdash B \wedge C}$$

express that U satisfies certain closure rules.

And now about implication. A formula $\varphi \rightarrow \psi$ expresses that there exists a proof in U of $\varphi \vdash \psi$. The occurrence of \rightarrow in a sequent means that the sequent makes a statement about the existence of proofs. Therefore $\varphi \vdash \psi \rightarrow \theta$ follows from $\varphi \wedge \psi \vdash \theta$. The reverse closure rule, equivalent to *modus ponens*, fails. It implies $\top \rightarrow \varphi \vdash \varphi$ which says that from the *assumption* of a proof of φ in U we derive the existence of an *actual* proof of φ in U . This is different from a statement $(\top \vdash \varphi) \vdash \varphi$ which says that if there were an *actual* proof of φ , then we have an actual proof of φ . Axioms $\varphi \vdash \psi$ where φ contains implication, may change the meaning of \rightarrow relative to \vdash . For example, when we consider intuitionistic logic with modus ponens, then a proof p of an implication $\varphi \rightarrow \psi$ includes a construction q that converts proofs s of φ into proofs qs of ψ . In basic logic, p should only include a proof q of ψ using φ as an assumption. Closure rules for U give rise to sequents involving implication. A valid sequent like $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \theta) \vdash \varphi \rightarrow \theta$ says that if we assume the existence of proofs of $\varphi \vdash \psi$ and $\psi \vdash \theta$, then we may assume the existence of a proof of $\varphi \vdash \theta$.

We start by constructing a propositional logic BPC . The interpretations of disjunction and conjunction are considered straightforward and beyond question. Following [Gentzen 1936], we also favorably regard implication introduction as saying that $A \wedge B \vdash C$ merely expresses that we can construct a proof of C from B if we are given A . This results in the following axioms and rules being acceptable.

$$\frac{A \vdash A \quad A \vdash B \quad B \vdash C}{A \vdash C}$$

$$\frac{A \vdash \top \quad A \vdash B \quad A \vdash C}{A \vdash B \wedge C} \quad \frac{\perp \vdash A \quad B \vdash A \quad C \vdash A}{B \vee C \vdash A}$$

$$\frac{A \wedge B \vdash C}{A \vdash B \rightarrow C}$$

$$A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$$

These axioms and rules form a system BPC_0 . The distributivity axiom schema is essential since we don't have *modus ponens*. The constants \top and \perp are included but are not essential to our system: \top is equivalent to all sequents of the form $A \rightarrow A$, while in many systems we have a natural candidate to replace \perp by (for example by $0 = 1$ in arithmetic). In a forthcoming paper we show that for some model-theoretic results it is essential to exclude \perp as part of the axiom system.

To get BPC we must now add implications $A \rightarrow B$ in the proper way whenever we can derive $A \vdash B$; implication must reflect provability as tightly as possible without introducing a circular argument. If, for example, we have a (derived) rule

$$\frac{\sigma_1 \vdash \tau_1 \quad \sigma_2 \vdash \tau_2}{\sigma \vdash \tau},$$

then we must add the axiom

$$(\sigma_1 \rightarrow \tau_1) \wedge (\sigma_2 \rightarrow \tau_2) \vdash \sigma \rightarrow \tau.$$

For each (derivable) sequent $\sigma \vdash \tau$ we must add the schema $A \vdash \sigma \rightarrow \tau$; this, however, immediately follows from the rule for \wedge , transitivity of \vdash , and implication introduction. So once BPC_0 has been built, we only add axioms for \rightarrow that are associated with rules. For example, for transitivity, conjunction introduction, and disjunction introduction we add the axiom schemas

$$\begin{aligned} & (A \rightarrow B) \wedge (B \rightarrow C) \vdash A \rightarrow C \\ & (A \rightarrow B) \wedge (A \rightarrow C) \vdash A \rightarrow (B \wedge C) \\ & (B \rightarrow A) \wedge (C \rightarrow A) \vdash (B \vee C) \rightarrow A \end{aligned}$$

The analogous axiom schemas that accompany the remaining rules of BPC_0 follow from BPC_0 extended with these three axiom schemas. For example, $B \vdash A \rightarrow A$ follows from $B \wedge A \vdash A$ using implication introduction; and $(B \vee C) \rightarrow A \vdash B \rightarrow A$ follows from $\vdash B \rightarrow (B \vee C)$ and the added transitivity axiom for \rightarrow . It suffices to add new axioms for the defining rules of BPC_0 only, as all derived rules then follow by transitivity. This extended system we call *Basic Propositional Calculus BPC*. The extension BPC involves new axioms and no new rules, so there is no need to repeat the process of looking for additional axioms for \rightarrow . Intuitionistic Propositional Calculus IPC is equivalent to BPC extended with implication elimination

$$\frac{A \vdash B \rightarrow C}{A \wedge B \vdash C}.$$

It turns out that the system BPC is equivalent to A. Visser's system BPL [Visser 1981]. A preprint of that paper initiated the author's original research into BPC , resulting in [Ruitenburg 1984].

From the construction of BPC we derive

3.1 PROPOSITION. Let T be a theory consisting of additional sequents only. Then

$$T \cup \{\sigma_1 \vdash \tau_1, \dots, \sigma_n \vdash \tau_n\} \vdash (\sigma \vdash \tau)$$

implies

$$T, (\sigma_1 \rightarrow \tau_1) \wedge \dots \wedge (\sigma_n \rightarrow \tau_n) \vdash \sigma \rightarrow \tau. \quad \dashv$$

Note that if we wish to add new rules to T , then, to preserve the availability of Proposition 3.1, we are required to add matching sequents to T .

[Visser 1981] mentions two substitution methods; a substitution rule and a substitution sequent. Both hold for BPC . Using Proposition 3.1, the substitution sequent

$$A \leftrightarrow B \vdash C[A] \leftrightarrow C[B]$$

follows immediately from the substitution rule

$$\frac{A \vdash B \quad B \vdash A}{C[A] \vdash C[B] \text{ and } C[B] \vdash C[A]}$$

3.2 PROPOSITION (FUNCTIONAL COMPLETENESS). Let T be a theory consisting of additional sequents only. If T satisfies

$$T, (\vdash \varphi) \vdash (\sigma \vdash \tau),$$

then

$$T, \varphi \wedge \sigma \vdash \tau.$$

PROOF: A straightforward induction on proof complexity. \dashv

Consistency of BPC is straightforward as it is a subsystem of intuitionistic propositional calculus IPC . Moreover, BPC does not satisfy *modus ponens* (or equivalently: implication elimination):

3.3 PROPOSITION. $BPC + (\vdash A \rightarrow B)$ is consistent.

PROOF: Replace all occurrences of implication in the axiomatization of BPC by \top . The resulting axiom system still is a subsystem of intuitionistic logic. \dashv

So BPC doesn't even satisfy the rule

$$\frac{\vdash \top \rightarrow B}{\vdash B}$$

In the system of Proposition 3.3 implication \rightarrow fails to reflect derivability of that system; \rightarrow does not faithfully reflect \vdash . A theory T is *faithful* if T satisfies the reverse of Proposition 3.1, that is,

$$T, (\sigma_1 \rightarrow \tau_1) \wedge \dots \wedge (\sigma_n \rightarrow \tau_n) \vdash \sigma \rightarrow \tau,$$

implies

$$T, (\sigma_1 \vdash \tau_1), \dots, (\sigma_n \vdash \tau_n) \vdash (\sigma \vdash \tau),$$

Obviously, all extensions of IPC are faithful. Proposition 3.7 below shows that BPC is faithful too.

BPC satisfies a weaker version of *modus ponens*:

3.4 PROPOSITION. BPC satisfies the rule

$$\frac{A \vdash B \rightarrow C}{A \wedge B \vdash \top \rightarrow C}$$

and, equivalently, the axiom

$$A \wedge (A \rightarrow B) \vdash \top \rightarrow B.$$

PROOF: If $A \vdash B \rightarrow C$, then $A \wedge B \vdash (\top \rightarrow B) \wedge (B \rightarrow C)$. \dashv

So *IPC* is equivalent to *BPC* plus the schema $\top \rightarrow A \vdash A$. Thus *IPC* satisfies the conditions for *T* in Propositions 3.1 and 3.2.

The paper [Visser 1981] includes a class of Kripke models and a completeness theorem. For completeness's sake we include these results, since though the general theory of Kripke models itself is not constructive, the Completeness Theorem 3.6 enables us to get a better understanding of *BPC* relative to other propositional logics, as it provides simple proofs for some properties of *BPC* without the necessity of having to go through proof theoretical technicalities.

A (generalized) Kripke model of *BPC* consists of a tuple $\mathbf{K} = \langle \mathbf{P}^{\mathbf{K}}, I^{\mathbf{K}} \rangle$ where

- (1) $\mathbf{P}^{\mathbf{K}} = \mathbf{P} = (P, \prec)$ consists of a set of nodes P with a transitive binary relation \prec on P , that is, if $\alpha \prec \beta \prec \gamma$ then $\alpha \prec \gamma$.
- (2) $I^{\mathbf{K}} = I$ assigns to each atom p of *BPC* a subset $I_p \subseteq P$ that is upward closed, that is, if $\beta \succ \alpha \in I_p$, then $\beta \in I_p$.

We write $\alpha \Vdash p$ for $\alpha \in I_p$. The canonical extension of \Vdash to all formulas of *BPC* is inductively defined by

$$\begin{aligned} \alpha \Vdash \top; \\ \alpha \Vdash \varphi \wedge \psi &\iff \alpha \Vdash \varphi \text{ and } \alpha \Vdash \psi; \\ \alpha \Vdash \varphi \vee \psi &\iff \alpha \Vdash \varphi \text{ or } \alpha \Vdash \psi; \text{ and} \\ \alpha \Vdash \varphi \rightarrow \psi &\iff \beta \Vdash \varphi \text{ implies } \beta \Vdash \psi \text{ for all } \beta \succ \alpha. \end{aligned}$$

We write \preceq for the reflexive closure of \prec . We extend \Vdash to sequents by defining

$$\alpha \Vdash (\varphi \vdash \psi) \iff \beta \Vdash \varphi \text{ implies } \beta \Vdash \psi \text{ for all } \beta \succeq \alpha.$$

A Kripke model \mathbf{K} satisfies a rule

$$\frac{\sigma_1 \vdash \tau_1 \quad \dots \quad \sigma_n \vdash \tau_n}{\sigma \vdash \tau},$$

if for all nodes α , if $\alpha \Vdash (\sigma_i \vdash \tau_i)$ for all i , then $\alpha \Vdash (\sigma \vdash \tau)$. We write $\mathbf{K} \models (\varphi \vdash \psi)$ if $\alpha \Vdash (\varphi \vdash \psi)$ for all nodes α . Obviously, if $\beta \succ \alpha \Vdash (\varphi \vdash \psi)$, then $\beta \Vdash (\varphi \vdash \psi)$. We easily verify

3.5 PROPOSITION [Visser 1981]. If $\beta \succ \alpha \Vdash \varphi$, then $\beta \Vdash \varphi$. \dashv

So $\alpha \Vdash \varphi$ if and only if $\alpha \Vdash (\top \vdash \varphi)$.

Let Σ be a set of sequents and rules. Then for all Kripke models \mathbf{K} we write $\mathbf{K} \models \Sigma$ if $\mathbf{K} \models (\sigma \vdash \tau)$ for all $\sigma \vdash \tau \in \Sigma$, and \mathbf{K} satisfies all rules of Σ . We write $\Sigma \models (\varphi \vdash \psi)$ if $\mathbf{K} \models \Sigma$ implies $\mathbf{K} \models (\varphi \vdash \psi)$, for all \mathbf{K} .

3.6 THEOREM (COMPLETENESS THEOREM) [Visser 1981]. Let $\Gamma \cup \{\varphi \vdash \psi\}$ be a set of sequents. Then $\Gamma \vdash (\varphi \vdash \psi)$ if and only if $\Gamma \models (\varphi \vdash \psi)$

PROOF: We present a sketch of the proof. For details, see [Visser 1981]. Soundness is proved by a straightforward induction on the complexity of proofs. For the completeness part, suppose $\Gamma \not\vdash (\varphi \vdash \psi)$. It suffices to construct a model \mathbf{K} such that $\mathbf{K} \models \Gamma$ and $\mathbf{K} \not\models (\varphi \vdash \psi)$. Let *PBPC*, *prime BPC*, be the extension of *BPC* with the rule

$$\frac{\vdash A \vee B}{\vdash A \text{ or } \vdash B}$$

This new rule makes *PBPC* non-inductive. A theory Δ is called closed if all sequents derivable from Δ are elements of Δ . As set of nodes P we choose the collection of all consistent closed theories extending *PBPC* + Γ ; there exists a Δ such that $(\varphi \vdash \psi) \notin \Delta$ [Visser 1981], so this collection is nonempty. We set $\Delta < \Delta'$ if $\lambda \wedge \mu \vdash \nu \in \Delta'$ whenever $\lambda \vdash \mu \rightarrow \nu \in \Delta$. For atoms p , set $I_p = \{\Delta \mid (\vdash p) \in \Delta\}$. By induction on the complexity of sequents one shows that $\Delta \Vdash (\sigma \vdash \tau)$ if and only if $\sigma \vdash \tau \in \Delta$. Thus $\mathbf{K} \models \Gamma$ and $\mathbf{K} \not\models (\varphi \vdash \psi)$. \dashv

Visser uses a derived completeness theorem to show that *BPC* is decidable; *BPC* $\vdash (\varphi \vdash \psi)$ if and only if $\mathbf{K} \models (\varphi \vdash \psi)$ for all finite \mathbf{K} of a size limited by the number of subformulas of $\varphi \wedge \psi$.

Although *modus ponens* fails even in the limited way described above, *BPC* does reflect the fact that \rightarrow embodies provability, as the second claim below illustrates.

3.7 PROPOSITION.

- (1) $\vdash \sigma \vee \tau$ holds in *BPC* if and only if $\vdash \sigma$ or $\vdash \tau$ hold. So *PBPC* $\vdash (\varphi \vdash \psi)$ if and only if *BPC* $\vdash (\varphi \vdash \psi)$.
- (2) *BPC* is faithful.

PROOF: Obviously, if *BPC* $\vdash \varphi$ or *BPC* $\vdash \psi$, then *BPC* $\vdash (\varphi \vee \psi)$. Conversely, suppose $\mathbf{K}_1 \not\models \varphi$ and $\mathbf{K}_2 \not\models \psi$ for some models \mathbf{K}_1 and \mathbf{K}_2 . Construct a new model \mathbf{K} by adding a new node α to the disjoint union of the models \mathbf{K}_1 and \mathbf{K}_2 , with $\alpha < \beta$ for all nodes β from \mathbf{K}_1 and \mathbf{K}_2 , and $\alpha \notin I_p$ for all atoms p . Then $\alpha \not\models \varphi$ and $\alpha \not\models \psi$, so $\alpha \not\models \varphi \vee \psi$. Thus *BPC* $\not\vdash \varphi \vee \psi$.

Suppose $(\sigma_1 \rightarrow \tau_1) \wedge \dots \wedge (\sigma_n \rightarrow \tau_n) \vdash \sigma \rightarrow \tau$ holds in *BPC*, and let \mathbf{K} be a model such that $\mathbf{K} \models (\sigma_i \vdash \tau_i)$ for all i . Form a new model \mathbf{L} by adding a new bottom node α to \mathbf{K} with $\alpha < \beta$ for all nodes β of \mathbf{K} , and $\alpha \notin I_p$ for all p . Then $\alpha \Vdash (\sigma_1 \rightarrow \tau_1) \wedge \dots \wedge (\sigma_n \rightarrow \tau_n)$, and so $\alpha \Vdash \sigma \rightarrow \tau$. Thus $\mathbf{K} \models (\sigma \vdash \tau)$. \dashv

Examples: Let $\top \rightarrow \varphi \vdash \sigma \rightarrow \tau$ hold in *BPC*. By Proposition 3.7, $(\vdash \varphi) \vdash (\sigma \vdash \tau)$ holds. So by Proposition 3.2 *BPC* satisfies $\varphi \wedge \sigma \vdash \tau$.

BPC satisfies the first two of the Curry-Feys axioms: $A \vdash A$ and $A \vdash B \rightarrow A$. The axiom $C \rightarrow (B \rightarrow A) \vdash (C \rightarrow B) \rightarrow (C \rightarrow A)$ fails since it implies $C \rightarrow (B \rightarrow A) \vdash B \rightarrow (C \rightarrow A)$. The following is a counterexample to that sequent. Let \mathbf{K} be the Kripke model with three nodes $\alpha < \beta < \gamma$ and $<$ is anti-reflexive. Set $I_p = \emptyset$, $I_q = \{\beta, \gamma\}$, and $I_r = \{\gamma\}$. Then $\mathbf{K} \models r \rightarrow (q \rightarrow p)$ and $\mathbf{K} \not\models q \rightarrow (r \rightarrow p)$.

Besides *IPC* there is another *BPC*-extension of note: Formal Propositional Calculus *FPC* (*FPL* in [Visser 1981]), obtained by adding the rule

$$\frac{A \wedge (\top \rightarrow B) \vdash B}{A \vdash B} \quad (\text{L\"ob's Rule}).$$

Obviously, *IPC* and *FPC* are relatively inconsistent. We easily show that *FPC* is equivalent to the system *BPC* augmented by the axiom schema

$$(\top \rightarrow A) \rightarrow A \vdash \top \rightarrow A \quad (\text{L\"ob's Axiom}):$$

use the transitivity of \rightarrow to obtain $((\top \rightarrow A) \rightarrow A) \wedge (\top \rightarrow (\top \rightarrow A)) \vdash \top \rightarrow A$ and then apply L\"ob's rule. Conversely, suppose $A \wedge (\top \rightarrow B) \vdash B$. Then $A \vdash (\top \rightarrow B) \rightarrow B$ and, applying L\"ob's Axiom, $A \vdash \top \rightarrow B$. Hence $A \vdash A \wedge (\top \rightarrow B)$, so transitivity of \vdash gives us $A \vdash B$. So *FPC* satisfies the conditions for *T* in Propositions 3.1 and 3.2.

A striking property of *FPC* is

3.8 THEOREM (EXPLICIT FIXED POINT THEOREM). For all $A[p]$, *FPC* satisfies the sequents

$$A[A[\top]] \vdash A[\top] \quad \text{and} \\ A[\top] \vdash A[A[\top]].$$

PROOF: See [Visser 1981]. \dashv

Since *FPC* is faithful, Theorem 3.8 is equivalent to

$$FPC \vdash A[\top] \leftrightarrow A[A[\top]].$$

We consider the relation between Theorem 3.8 and the Explicit Fixed Point Theorem of *PrL* below, when we discuss translations into modal logic.

3.9 PROPOSITION.

- (1) $\vdash \sigma \vee \tau$ holds in *FPC* if and only if $\vdash \sigma$ or $\vdash \tau$ hold.
- (2) *FPC* is faithful.

PROOF: The model constructions in the proof of Proposition 3.7 turn models of *FPC* into models of *FPC*. \dashv

Provability is usually considered in the context of modal logic. Translations of *IPC* into the modal logic date back to [Gödel 1933]. Now we form a translation from the language of *BPC* into the language of *K4* ([Boolos 1979]; [Smoryński 1985] calls it *BML*). The translation $A \mapsto A'$ is inductively defined by

$$\begin{aligned} \top' &= \top; \\ \perp' &= \perp; \\ p' &= p \wedge \Box p, \quad p \text{ an atom}; \\ (A \wedge B)' &= A' \wedge B'; \\ (A \vee B)' &= A' \vee B'; \quad \text{and} \\ (A \rightarrow B)' &= \Box(A' \rightarrow B'). \end{aligned}$$

The system *K4* is axiomatized by *CPC*, the axioms

$$\begin{aligned} \Box(A \rightarrow B) \vdash \Box A \rightarrow \Box B, \quad \text{and} \\ \Box A \vdash \Box \Box A, \end{aligned}$$

and the rule

$$\frac{\vdash A}{\vdash \Box A}.$$

The extension *S4* satisfies the additional

$$\Box A \vdash A;$$

and *PrL* equals *K4* plus the extra schema

$$\Box(\Box A \rightarrow A) \vdash \Box A.$$

Note that the additional sequents defining *IPC* and *FPC* as extensions of *BPC* are translated into the additional sequents defining *S4* and *PrL* as extensions of *K4*.

The connection between *IPC* and *S4* can be generalized to

3.10 PROPOSITION.

$$\begin{aligned} BPC, \varphi \vdash \psi \text{ if and only if } K4, \varphi' \vdash \psi' \\ IPC, \varphi \vdash \psi \text{ if and only if } S4, \varphi' \vdash \psi' \\ FPC, \varphi \vdash \psi \text{ if and only if } PrL, \varphi' \vdash \psi' \end{aligned}$$

PROOF: Exercise. \dashv

Proposition 3.10 for *FPC* versus *PrL* and a special case of the Explicit Fixed Point Theorem [Smoryński 1985, p. 78] imply the Explicit Fixed Point Theorem 3.8 for *FPC*.

§4. BASIC PREDICATE CALCULUS

The extension of *BPC* to first-order Basic Predicate Calculus *BQC* presents us with the same challenges as *BPC*, plus one: universal quantification. The extra problem shows similarities to the use of partial elements in the original [Heyting 1930], [Heyting 1930a]. In the discussion below we use the related additional existence predicate *E* of [Scott 1979] as a starting point.

In [Scott 1979] variables are allowed to range over partial elements like $1/x$ over the reals when we don't know whether x is invertible or equals 0 or is somewhere in between. The expression Ex stands for 'x exists.' For intuitionistic predicate logic *IQC*—with full *modus ponens*—this leads to the following rules defining \exists and \forall :

$$\frac{A \vdash Ex \rightarrow C}{A \vdash \forall x C} \qquad \frac{Ex \wedge B \vdash A}{\exists x B \vdash A}$$

where x is not free in A .

The existence predicate *E* allows for a different interpretation. For *IQC* with total elements only, Ex equals \top . This implies that we may eliminate the subexpressions Ex from the rules altogether. For *BQC* with total elements only, this is true for the rules concerning existential quantification but not for universal quantification.

Although x is 'total', its existence is 'partial' in the sense that it depends on its context. A proof of $\varphi(x) \vdash \psi(x)$ consists of having a proof of $\psi(x)$ assuming the existence of an element x and assuming $\varphi(x)$. Therefore the usual rule for existential quantification is acceptable. The sequent $\varphi x \vdash \exists x \varphi x$, for example, is acceptable since the assumption φx and the assumption that x is an element immediately imply $\exists x \varphi(x)$. We tacitly assume that there exists at least one element in the domain.

Universal quantifier elimination is not acceptable: the formula $\forall x \varphi(x)$ expresses $\vdash \varphi(x)$, that is, it expresses that from the assumption that x is an element we derive $\varphi(x)$. So the sequent $\forall x \varphi(x) \vdash \varphi(x)$ is unacceptable: from an *assumed* proof of $\varphi(x)$ from assuming that x is an element, and the assumption that x is an element, we conclude the existence of an *actual* proof of $\varphi(x)$. Even the assumption $\vdash \forall x \varphi(x)$ is not sufficient to get $\vdash \varphi(x)$. The most we can derive from $\forall x \varphi(x)$ is $\top \rightarrow \varphi(x)$. More generally, let $\forall x: \varphi(x). \psi(x)$ be the formula expressing $\varphi(x) \vdash \psi(x)$. Then $\forall x: \varphi(x). \psi(x)$ entails $\varphi(x) \rightarrow \psi(x)$.

For the language of *BQC* we use $\forall x: B.C$ as standard notation; $\forall x C$ is an abbreviation for $\forall x: \top.C$. Similarly, the expression $\exists x: B.C$ is equivalent to $\exists x(B \wedge C)$.

The extension *BQC*₀ of *BPC* is formed by the following axioms and rules; the interpretation of existential quantification is considered straightforward.

$$\frac{Ax \vdash Bx}{At \vdash Bt} \dagger$$

$$\frac{A \vdash B \rightarrow C}{A \vdash \forall x: B.C} \dagger \quad \frac{B \vdash A}{\exists x B \vdash A} \ddagger$$

$$A \wedge \exists x B \vdash \exists x(A \wedge B) \ddagger$$

$$\top \vdash x = x$$

$$(x = y) \wedge Ax \vdash Ay *$$

The special form of the substitution rule and the extra axiom for the existential quantifier are essential because we don't have *modus ponens*. As with BPC_0 , we need additional axioms for the new rules. The following two suffice to complete the axiomatization of BQC :

$$A \rightarrow (B \rightarrow C) \vdash A \rightarrow \forall x: B.C \ddagger$$

$$(B \rightarrow A) \vdash \exists x B \rightarrow A \ddagger$$

We allow the substitution of new variables for bound variables. In case \dagger , the term t does not contain a variable bound by a quantifier of A or B ; in cases \ddagger , the variable x is not free in A ; and in case $*$, the variables x, y are not bound by a quantifier of A .

The presence of variables makes associating sequents with rules slightly more complicated. For example, the sequent associated with the substitution rule reads

$$\forall x: Ax.Bx \vdash At \rightarrow Bt$$

and immediately follows from universal quantifier elimination and the substitution rule.

Because of the relative weakness of implication \rightarrow , nested universal quantifications $\forall x \forall y \dots$ are weaker than a single quantification over strings $\forall(x, y, \dots)$. Therefore we allow strings $\mathbf{x} = (x_1, x_2, \dots, x_n)$ as replacement for x in the rule and sequent axioms for universal implication:

$$\frac{A \vdash B \rightarrow C}{A \vdash \forall \mathbf{x}: B.C} \text{ and}$$

$$A \rightarrow (B \rightarrow C) \vdash A \rightarrow \forall \mathbf{x}: B.C,$$

where none of the variables in the string \mathbf{x} is free in A . So when we write $\forall x, y \varphi$ we mean something essentially different from $\forall x \forall y \varphi$.

BQC satisfies the equivalent of Proposition 3.1. However, a proper formulation requires us to use universal closures if any of the sequents $\sigma_i \vdash \tau_i$ share free variables. Suppose all free variables that occur in the formula $\varphi \rightarrow \psi$ are among the variables $\mathbf{x} = (x_1, \dots, x_n)$. Then the *universal closure* of $\varphi \rightarrow \psi$ is the sentence $\forall \mathbf{x}: \varphi.\psi$. Now the equivalent of Proposition 3.1 reads: If T is a theory of sequents such that

$$T \cup \{\sigma_1 \vdash \tau_1, \dots, \sigma_n \vdash \tau_n\} \vdash (\sigma \vdash \tau),$$

then

$$T, (\forall \mathbf{x}: \sigma_1. \tau_1) \wedge \cdots \wedge (\forall \mathbf{x}: \sigma_n. \tau_n) \vdash \forall \mathbf{x}: \sigma. \tau.$$

A *faithful* theory satisfies the reverse implication.

BQC satisfies the equivalent of Proposition 3.2 if φ and $\sigma \vdash \tau$ don't share any free variables.

A (*generalized*) *Kripke model* of *BQC* consists of a triple $\mathbf{K} = \langle \mathbf{P}^{\mathbf{K}}, D^{\mathbf{K}}, I^{\mathbf{K}} \rangle$ where

- (1) $\mathbf{P}^{\mathbf{K}} = \mathbf{P}$ is a transitive structure as in propositional Kripke models,
- (2) $D^{\mathbf{K}} = D$ assigns to each node α a nonempty set $D\alpha$, and to each ordered pair $\alpha < \beta$ a function $\sigma_\beta^\alpha: D\alpha \rightarrow D\beta$ such that if $\alpha < \beta < \gamma$, then $\sigma_\gamma^\beta \sigma_\beta^\alpha = \sigma_\gamma^\alpha$, and
- (3) $I^{\mathbf{K}} = I$ assigns to each n -ary predicate p a function I_p on domain P such that $I_p\alpha$ is a subset of $(D\alpha)^n$, and $(\sigma_\beta^\alpha)^n(I_p\alpha) \subseteq I_p\beta$ whenever $\alpha < \beta$. To each n -ary function symbol g , I assigns a function I_g on domain P such that $I_g\alpha: (D\alpha)^n \rightarrow D\alpha$ is a function satisfying $\sigma_\beta^\alpha(I_g\alpha) = (I_g\beta)(\sigma_\beta^\alpha)^n$ whenever $\alpha < \beta$. Constant symbols are treated as 0-ary functions.

To each term t we assign a function I_t on P that, for each α , is the composition of functions $I_g\alpha$ where the g are the function symbols that make up term t . For atomic sentences $p(t)$ we write $\alpha \Vdash p(t)$ exactly when $I_t\alpha \in I_p\alpha$. Augment the language of *BQC* with new constant symbols for all the elements of all the nodes $D\alpha$. Then we write φ_α if all constants in φ are from elements in $D\alpha$; given $\alpha < \beta$ and φ_α , then φ_β is the sentence constructed from φ_α by replacing all constant symbols c_α by $\sigma_\beta^\alpha(c_\alpha)$. The canonical extension of \Vdash to all sentences of *BQC* is inductively defined in the familiar way for \top , \wedge , and \vee , and by the additional

$$\alpha \Vdash c = d \iff c, d \in D\alpha \text{ and } c = d;$$

$$\alpha \Vdash (\varphi \rightarrow \psi)_\alpha \iff \beta \Vdash \varphi_\beta \text{ implies } \beta \Vdash \psi_\beta, \text{ for all } \beta \succ \alpha;$$

$$\alpha \Vdash (\forall \mathbf{x}: \varphi(\mathbf{x}). \psi(\mathbf{x}))_\alpha \iff \beta \Vdash \varphi(\mathbf{c})_\beta \text{ implies } \beta \Vdash \psi(\mathbf{c})_\beta \text{ for all } \beta \succ \alpha \text{ and } \mathbf{c} \in (D\beta)^n;$$

and

$$\alpha \Vdash (\exists x \varphi(x))_\alpha \iff \alpha \Vdash \varphi(c)_\alpha \text{ for some } c \in D\alpha.$$

For formulas $\varphi(\mathbf{x})_\alpha$ and $\psi(\mathbf{x})_\alpha$, we write $\alpha \Vdash (\varphi(\mathbf{x}) \vdash \psi(\mathbf{x}))_\alpha$ if $\beta \Vdash \varphi(\mathbf{c})_\beta$ implies $\beta \Vdash \psi(\mathbf{c})_\beta$, for all $\beta \succeq \alpha$ and $\mathbf{c} \in (D\beta)^n$.

4.1 PROPOSITION. If $\beta \succ \alpha$ and $\alpha \Vdash \varphi_\alpha$, then $\beta \Vdash \varphi_\beta$. If $\beta \succ \alpha$ and $\alpha \Vdash (\varphi \vdash \psi)_\alpha$, then $\beta \Vdash (\varphi \vdash \psi)_\beta$.

PROOF: By induction on the complexity of formulas. \dashv

Borrowing the notation and definitions for \models from *BPC* we get

4.2 THEOREM (COMPLETENESS THEOREM). Let $\Gamma \cup \{\varphi \vdash \psi\}$ be a set of sequents. Then $\Gamma \vdash (\varphi \vdash \psi)$ if and only if $\Gamma \models (\varphi \vdash \psi)$.

PROOF: We present only a hint of a proof. Soundness follows from a straightforward induction on the complexity of proofs. Conversely, suppose $\Gamma \not\vdash (\varphi \vdash \psi)$. To construct a model \mathbf{K} of Γ such that $\mathbf{K} \not\models (\varphi \vdash \psi)$, we take as the set of nodes the collection of consistent and closed theories extending $PBQC + \Gamma$, where $PBQC$, *prime BQC*, is the extension of $PBPC$ and BQC with the rule

$$\frac{\vdash \exists x Ax}{\vdash Ac \text{ for some constant } c}$$

The language has to be repeatedly augmented with additional constants. These technicalities are beyond the scope of this paper; rather, cf. [Smoryński 1973, pp. 330ff]. We set $\Delta < \Delta'$ if $\lambda \wedge \mu \vdash \nu \in \Delta'$ whenever $\lambda \vdash \mu \rightarrow \nu \in \Delta$. Define

$$D\Delta = \{c \mid c \text{ is a constant existing at node } \Delta\} / \sim,$$

where $c \sim d$ is the equivalence relation $\Delta \vdash c = d$. The resulting Kripke model satisfies Γ but fails to satisfy $\varphi \vdash \psi$. \dashv

The extension IQC of Intuitionistic Predicate Calculus equals $IPC + BQC$; the extension FQC of Formal Predicate Calculus equals $FPC + BQC$; CQC equals $IQC + (\vdash A \vee \neg A)$. The systems $PIQC$ and $PFQC$ are obtained by replacing BQC by $PBQC$ in the two definitions above. $PCQC = PIQC + CQC$ is the theory of complete theories.

4.3 PROPOSITION (EXPLICIT DEFINABILITY). Let $\exists x\varphi(x)$ be a sentence. Then

- (1) $BQC \vdash \exists x\varphi(x)$ if and only if $BQC \vdash \varphi(t)$ for some term t without variables bound by a quantifier of φ .
- (2) $IQC \vdash \exists x\varphi(x)$ if and only if $IQC \vdash \varphi(t)$ for some term t without variables bound by a quantifier of φ .
- (3) $FQC \vdash \exists x\varphi(x)$ if and only if $FQC \vdash \varphi(t)$ for some term t without variables bound by a quantifier of φ .

PROOF: The implications from right to left immediately follow from the derivability of $\varphi(t) \vdash \exists x\varphi(x)$. Conversely, suppose that for all suitable terms t there is a model \mathbf{K}_t such that $\mathbf{K}_t \not\models \varphi(t)$. We may assume the \mathbf{K}_t 's to have smallest nodes α_t such that $\alpha_t \not\vdash \varphi(t(c))$ for some substitution of constants c from $D\alpha_t$ for the variables of t . Construct a new model \mathbf{L} by adding a new node α to the disjoint union of the models \mathbf{K}_t , with $\alpha < \beta$ for all nodes β from the \mathbf{K}_t . $D\alpha$ consists of the terms of the language, augmented with sufficiently many constant symbols to cover the new constants in all the $t(c)$ (countably many will do); the σ_β^α are defined accordingly. We make sure that each term $t(c)$ of $D\alpha$ is mapped to some $t(c) \in D\alpha_t$ with $\alpha_t \not\vdash \varphi(t(c))$. Set $\alpha \not\vdash p(t(c))$ for all predicates p and terms $t(c)$ over $D\alpha$. Then $\alpha \not\vdash \exists x\varphi(x)$. Thus $BPC \not\vdash \exists x\varphi(x)$.

The case for IQC is well-known. The proof for FQC is identical to the proof for BQC since if the \mathbf{K}_t are models for FQC , then so is \mathbf{L} . \dashv

There is a translation between BQC and the first-order extension $QK4$ of $K4$ satisfying the equivalent of Proposition 3.10. The system $QK4$ is axiomatized by $K4 + CQC$. The translation $A \mapsto A'$ from the language of BPC into the language of $K4$ is extended to the quantifiers by

$$\begin{aligned} (\forall x: A.B)' &= \Box \forall x(A' \rightarrow B') \quad \text{and} \\ (\exists x A)' &= \exists x A'. \end{aligned}$$

One easily shows that

$$BQC, \varphi \vdash \psi \text{ if and only if } QK4, \varphi' \vdash \psi'.$$

Weakening some of the logical connectives of IQC to get BQC could have made the first-order Basic Calculus too weak to be useful. We show that it isn't. On one hand, most intuitionistic first-order mathematics (and classical first-order mathematics) extends to basic first-order mathematics; on the other hand, Basic Arithmetic BA , the equivalent of Heyting Arithmetic HA and Peano Arithmetic PA , is still a powerful theory.

A formula sequent is *geometric* if it does not contain any occurrences of \rightarrow or \forall . A set of sequents is called a *geometric theory* if it contains geometric sequents only.

4.4 PROPOSITION. Let T be a geometric theory, and $\varphi \vdash \psi$ a geometric sequent. Then $T \vdash (\varphi \vdash \psi)$ if and only if $CQC, T \vdash (\varphi \vdash \psi)$.

PROOF: Each node $D\alpha$ of a Kripke model \mathbf{K} , with its structure borrowed from \mathbf{K} , is a classical CQC model for the language of BQC . A trivial induction on the complexity of formula sequents shows that if $\sigma \vdash \tau$ is a geometric sequent, then $\mathbf{K} \models (\sigma \vdash \tau)$ if and only if $D\alpha \models (\sigma \vdash \tau)$ for all nodes α of \mathbf{K} . \dashv

A glance through [Mines et al. 1988] convinces that a substantial portion of its contents can be generalized from IQC to BQC without major revisions. A significant part is geometric, and another significant part assumes equality to be decidable.

For BQC , decidability of $=$ means that there is a relation \neq such that both $\vdash (x = y) \vee (x \neq y)$ and $(x = y) \wedge (x \neq y) \rightarrow \perp$ hold. Many interesting structures in constructive mathematics don't have a decidable equality. The expression $x = y \rightarrow \perp$ is considerably weaker than the equivalent 'denial inequality' of IQC and therefore less useful. Instead, we take the relation \neq to be primitive, with an appropriate axiomatization. What axiomatization for \neq is right for BQC ? We propose a generalized theory of inequality along the lines of [Ruitenburg 1988]; the idea behind the theory of difference relations in that paper is to include a geometric theory for inequality that is as strong as possible.

Basic Arithmetic BA is the basic calculus equivalent of Heyting Arithmetic HA and Peano Arithmetic PA . It has axioms

$$\begin{aligned} Sx = 0 &\vdash \perp \\ Sx = Sy &\vdash x = y \\ \vdash x + 0 &= x \\ \vdash x \cdot 0 &= 0 \\ \vdash x + Sy &= S(x + y) \\ \vdash x \cdot Sy &= (x \cdot y) + x \end{aligned}$$

and rule

$$\frac{A(x) \vdash A(Sx)}{A(0) \vdash A(x)}$$

where x is not free in $A(0)$. The sequent schema $\forall x: A(x). A(Sx) \vdash A(0) \rightarrow A(x)$ that is associated with the induction rule follows from BA : apply the induction rule to the formula $A(0) \rightarrow A(x)$. So the equivalent of Propositions 3.1 and 3.2 applies to BA .

We employ the usual abbreviations 1 for $S0$, 2 for $S1$, etc. We easily verify that BA satisfies the schema $1 = 0 \vdash A$, so we can replace \perp by $1 = 0$. Note that HA equals $BA + ((\top \rightarrow A) \vdash A)$.

4.5 PROPOSITION.

- (1) (Explicit Definability) $BA \vdash \exists x\varphi(x)$ if and only if $BA \vdash \varphi(n)$ for some numeral n .
- (2) $BA \vdash \varphi \vee \psi$ if and only if $BA \vdash \varphi$ or $BA \vdash \psi$.
- (3) BA is faithful.

PROOF: (1). Let $\mathbf{K}_0, \mathbf{K}_1, \dots$ be a sequence of models of BA such that $\mathbf{K}_n \not\models \varphi(n)$. Form a new model \mathbf{L} by adding a new bottom node α to the disjoint union of the \mathbf{K}_n 's, and set $D\alpha = \mathbf{N} = \{0, 1, 2, \dots\}$. Then $\mathbf{L} \models BA$ and $\alpha \not\models \exists x\varphi(x)$.

Explicit Definability implies the disjunction property: replace $\varphi \vee \psi$ by $\exists x((x = 0 \wedge \varphi) \vee (x = 1 \wedge \psi))$ and apply (1).

(3). Suppose $BA, (\forall \mathbf{x}: \sigma_1. \tau_1) \wedge \dots \wedge (\forall \mathbf{x}: \sigma_n. \tau_n) \vdash \forall \mathbf{x}: \sigma. \tau$, and let $\mathbf{K} \models BA$ be such that $\mathbf{K} \models (\sigma_i \vdash \tau_i)$ for all i . Form a new model \mathbf{L} by adding a new bottom node α with $D\alpha = \mathbf{N}$. Then $\mathbf{L} \models BA$, and $\alpha \models \sigma_i \rightarrow \tau_i$ for all i ; hence $\alpha \models \sigma \rightarrow \tau$, and thus $\mathbf{K} \models (\sigma \vdash \tau)$. \dashv

Let FA be the system of Formal Arithmetic $BA + FQC$. One easily verifies that the model constructions in the proof of Proposition 4.5 preserve FA . So FA satisfies the corresponding properties of Explicit Definability and faithfulness. Similar statements for Intuitionistic Arithmetic (Heyting Arithmetic) HA are well-known.

Beside the fact that BA is a 'nice' theory, we need it to be a 'strong' theory. We show that the equality relation is decidable and that a substantial part of standard arithmetic is derivable.

Formulas φ and ψ are called *complements* over a theory T if $T \vdash \varphi \vee \psi$ and $T, \varphi \wedge \psi \vdash \perp$. Define $x < y$ to be an abbreviation for $\exists z(x + Sz = y)$, and $x \neq y$, inequality, to be an abbreviation for $x < y \vee y < x$.

4.6 PROPOSITION. BA satisfies the Trichotomy Law $x < y \vee x = y \vee y < x$. So $=$ and \neq are complements over BA .

PROOF: The proof consists of a few trivial applications of the induction rule. We mention some intermediate stages, leaving the details as exercises. Prove by induction the associativity, and then the commutativity, of addition. A straightforward calculation proves $BA, x < y \vdash Sx < Sy$. By induction one shows $BA \vdash 0 = x \vee 0 < x$. Then derive $(Sx < y \vee Sx = y \vee y < Sx)$ from each of the three assumptions $x < y$, $x = y$, and $y < x$, and apply induction. \dashv

Formulas of PA can be embedded into the language of BA by first writing them in prenex normal form where the quantifier-free part is a combination of equalities $=$, inequalities \neq , disjunctions \vee , and conjunctions \wedge . Then translate this prenex form by replacing the occurrences of \neq with \neq defined in the language of BA . A strengthening of Proposition 4.4 shows that a large portion of BA immediately follows from $PA = BA + CQC$ using this translation.

Let β be a node of a Kripke model \mathbf{K} . Then \mathbf{K}_β denotes the submodel of \mathbf{K} with set of underlying nodes all $\alpha \geq \beta$. We write $\beta, (\sigma_1 \vdash \sigma_2) \Vdash (\tau_1 \vdash \tau_2)$ if $\beta \Vdash (\sigma_1 \vdash \sigma_2)$ implies $\beta \Vdash (\tau_1 \vdash \tau_2)$. Recall that we also write $D\alpha$ for the classical structure above a node α of a Kripke model \mathbf{K} .

4.7 PROPOSITION. Let β be a node of a Kripke model \mathbf{K} , and let $\sigma \vdash \tau$ be a geometric sequent whose free variables are among the ones in the sequence $\mathbf{x} = (x_1, \dots, x_n)$. Then $\beta \Vdash (\sigma \vdash \tau)$ if and only if $D\alpha \models \forall \mathbf{x}: \sigma, \tau$ for all nodes α of \mathbf{K}_β .

PROOF: $\beta \Vdash (\sigma \vdash \tau)$ if and only if $\alpha \Vdash (\sigma(\mathbf{c})_\alpha \vdash \tau(\mathbf{c})_\alpha)$ for all nodes α of \mathbf{K}_β and $\mathbf{c} \in (D\alpha)^n$. Apply the proof of Proposition 4.4. \dashv

4.8 COROLLARY. Let β be a node of a Kripke model \mathbf{K} , and let $\sigma_1 \vdash \sigma_2$ and $\tau_1 \vdash \tau_2$ be geometric sequents whose free variables are among the ones in the sequence $\mathbf{x} = (x_1, \dots, x_n)$. Then $\beta, (\sigma_1 \vdash \sigma_2) \Vdash (\tau_1 \vdash \tau_2)$ if and only if $D\alpha \models \forall \mathbf{x}: \sigma_1, \sigma_2$ for all nodes α of \mathbf{K}_β implies $D\alpha \models \forall \mathbf{x}: \tau_1, \tau_2$ for all nodes α of \mathbf{K}_β . \dashv

4.9 THEOREM. Let T be a faithful theory, and let $\sigma_1 \vdash \sigma_2$ and $\tau_1 \vdash \tau_2$ be geometric sequents whose free variables are among the ones in the sequence $\mathbf{x} = (x_1, \dots, x_n)$. Then $T, \forall \mathbf{x}: \sigma_1, \sigma_2 \vdash \forall \mathbf{x}: \tau_1, \tau_2$ if and only if for all $\mathbf{K} \models T$ and nodes β of \mathbf{K} , if $D\alpha \models \forall \mathbf{x}: \sigma_1, \sigma_2$ for all nodes α of \mathbf{K}_β , then $D\alpha \models \forall \mathbf{x}: \tau_1, \tau_2$ for all nodes α of \mathbf{K}_β .

PROOF: Since T is faithful, $T, \forall \mathbf{x}: \sigma_1, \sigma_2 \vdash \forall \mathbf{x}: \tau_1, \tau_2$ if and only if $T, (\sigma_1 \vdash \sigma_2) \vdash (\tau_1 \vdash \tau_2)$. Apply Corollary 4.8. \dashv

BA is faithful, so by Theorem 4.9, the theory of all nodes of all Kripke models of BA satisfies the Π_2^0 induction rule of PA . So again by Theorem 4.9, BA satisfies the Π_2^0 fragment of the theory of the nodes.

§5. RUSSELL'S PARADOX REVISITED

Basic Calculus was not designed with the intent of offering a solution to Russell's Paradox or to any other set-theoretic paradox. Therefore it came as a little bit of a surprise that over Basic Calculus the proof of Russell's Paradox turns into a valuable theorem. The results below are partially influenced by the indexed implications $\rightarrow_1, \rightarrow_2, \dots$ of [Myhill 1975]. In fact, if we define $A \rightarrow_1 B = A \rightarrow B$, and $A \rightarrow_{n+1} B = \top \rightarrow (A \rightarrow_n B)$ for all n , then Myhill's demonstration of how to circumvent Russell's Paradox gets close to the proofs below.

We introduce an incomplete set theory F with full comprehension. The only axiom schemas and rules that are included in this theory are those that at the least should be valid and that allow us to derive our main theorem. So the goals of this section are modest compared to the ones of a paper like [Flagg 1987]. At this moment we don't have a consistency proof for F .

The language of F is a BQC -style language with binary relation \in , and such that for each formula φ we have a term $\{x \mid \varphi\}$. The construction of these new terms is iterated countably many times so that the φ themselves may contain constructions of the form $\{y \mid \psi\}$. The only logical axioms and rules of F are the ones of BQC . Following Frege, the terms $\{x \mid \varphi\}$ are such that F satisfies

$$\frac{A \vdash y \in \{x \mid Bx\}}{A \vdash By},$$

where x and y are not bound by a quantifier of B . The sets are completely determined by their elements, so F satisfies the rule of extensionality

$$\frac{A \wedge x \in y \vdash x \in z \quad A \wedge x \in z \vdash x \in y}{A \vdash y = z},$$

where x is not free in A .

The symbol \perp is redundant if we have a constant symbol \emptyset satisfying the schema $x \in \emptyset \vdash x \in y$. Let $V = \{x \mid \top\}$. Then we can replace \perp by the sentence $\emptyset = V$.

Rather than constructing ways to circumvent Russell's Paradox, the following procedure converts its traditional proof into a useful theorem.

Let φ be a formula in which x does not occur. Define $[\varphi] = \{x \mid x \in x \rightarrow \varphi\}$.

5.1 LEMMA. F satisfies the schema $[A] \in [A] \vdash \top \rightarrow A$.

PROOF: Use

$$[\varphi] \in [\varphi] \vdash (\top \rightarrow [\varphi] \in [\varphi]) \wedge ([\varphi] \in [\varphi] \rightarrow \varphi)$$

and transitivity of \rightarrow . \dashv

5.2 COROLLARY. F satisfies the schema $\vdash [A] \in [\top \rightarrow A]$. \dashv

5.3 THEOREM. F satisfies Löb's Axiom Schema $(\top \rightarrow A) \rightarrow A \vdash \top \rightarrow A$.

PROOF: Let φ be a formula and let x be a variable that does not occur in φ . Then $(x \in x \rightarrow (\top \rightarrow \varphi)) \wedge ((\top \rightarrow \varphi) \rightarrow \varphi) \vdash x \in x \rightarrow \varphi$. So $x \in [\top \rightarrow \varphi] \wedge ((\top \rightarrow \varphi) \rightarrow \varphi) \vdash x \in [\varphi]$. Obviously, $x \in [\varphi] \vdash x \in [\top \rightarrow \varphi]$, hence $(\top \rightarrow \varphi) \rightarrow \varphi \vdash [\varphi] = [\top \rightarrow \varphi]$. So by Corollary 5.2, $(\top \rightarrow \varphi) \rightarrow \varphi \vdash [\varphi] \in [\varphi]$. Thus by Lemma 5.1, $(\top \rightarrow \varphi) \rightarrow \varphi \vdash \top \rightarrow \varphi$. \dashv

The systems FQC and IQC are relatively inconsistent, so Theorem 5.3 implies that $F + IQC$ is inconsistent, which is Russell's Paradox. Thus Theorem 5.3 presents new evidence that BQC is a better foundation than IQC .

Many studies of Frege-style systems are less concerned with a proper foundation than with consistency. In light of Theorem 5.3 it is, therefore, an important question whether there are faithful, consistent, and preferably maximal, extensions of FPC and FQC that may be added to F . A natural candidate is the addition of the axiom schema of linearity

$$\vdash (A \rightarrow B) \vee (B \rightarrow A).$$

FPC plus linearity is a maximal faithful consistent theory.

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