LEWIS CARROLL'S METHOD OF TREES: ITS ORIGINS IN STUDIES IN LOGIC

FRANCINE F. ABELES
Math. & Comp. Sci. Dept., Kean College, SCNJ

Lewis Carroll's reputation as the author of the extraordinary "Alice" books is secure in the literary world. Respected but not as well known is his work in mathematical logic with which he was deeply involved throughout his life. Before taking up his post as Mathematical Lecturer at Christ Church, Oxford, Charles Lutwidge Dodgson, like his father, earned a first class honours degree in mathematics at Christ Church, heading the list of recipients on the examination in the Final Mathematics School, the last hurdle for the bachelor's degree. Dodgson formally began to work in logic in the early 1870's. about ten years after he began publishing mathematical texts, although his first book on logic did not appear until 1886. In logic as well as in mathematics, Dodgson was fascinated by problems in logical reasoning. In the Preface to the third edition of Curiosa Mathematica, Part I, A New Theory of Parallels, Dodgson wrote, "The validity of a Syllogism is quite independent of the truth of its Premisses. [and provided this example] 'I have sent for you, my dear Ducks,' said the worthy Mrs. Bond, 'to enquire with what sauce you would like to be eaten?' 'But we don't want to be killed!' cried the Ducks. 'You are wandering from the point' was Mrs. Bond's perfectly logical reply." [1]

In Curiosa Mathematica, Part 1 (1888), Dodgson presented an entire theory of valid inference in an axiomatic system in which he distinguished the axioms of euclidean geometry by their relative degrees of truth. In exploring valid logical inferences. Dodgson chose to stay within the domain of the classical forms of the syllogism and sorites. The syllogism, as the archtype of correct reasoning, appealed to him in the same way that Euclid's axiom system did as the paradigm of correct geometric reasoning. It did not concern him that Boole's algebraical logic was supplanting traditional logic. What interested Dodgson were the methods by which valid conclusions could be reached. In treating a sorites, which is a series of syllogisms, he had this to say, "Of all the strange things, that are to be met with in the ordinary text-books of Formal Logic, perhaps the strangest is the violent contrast one finds to exist between their ways of dealing with these two subjects....no less than nineteen different forms of Syllogisms [exist]...they have limited Sorites to two forms only, of childish simplicity; and these they have dignified with special names, apparently under the impression that no other possible forms existed!" [2]. Dodgson, the instructor and populariser of mathematics and logic; writer of extraordinarily imaginative literature; avuncular friend to children was compelled to set the matter right. In Part I of Symbolic Logic (1896). he outlined two methods to deal with soriteses. One he called the Method of Separate Syllogisms; the other, the Method of Underscoring. Methods, particularly mechanical methods as well as mechanical devices had a special appeal to Dodgson. He owned an electric pen and a chromograph, both predecessors of the typewriter, and he invented an aid to writing in the dark which he called a Nyctograph. He visited Charles Babbage in an attempt to acquire an analytical engine, unaware that it existed only in the mind of its originator. He experimented with different methods of constructing ciphers; devised labor-saving methods to perform arithmetical operations; experimented with memory retention techniques and framed an ingenious way to compute the determinant of a matrix. In July 1894 he devised the Method of Trees to deal with complex sorites problems. This work was unknown until 1977 when William Warren Bartley published the second part of Dodgson's Symbolic Logic, a book that had been thought lost. Bartley described it this way. "Carroll developed a 'Method of Trees' to determine the validity of what were, by the standards of his English contemporaries, highly complicated arguments. This provided, in effect, a mechanical test of validity through a reductio ad absurdum argument for a large part of the logic of terms....Carroll's procedure bears a striking resemblance to the trees employed with increasing popularity by contemporary logicians according to the method of 'Semantic Tableaux' published in 1955 by the Dutch logician E. W. Beth. The basic ideas are identical." [2].

Dodgson worked essentially alone; it was his nature to do so. But he was aware of the work of his contemporaries in both mathematics and in logic. His own writings, the contents of his personal library and entries in his diary all provide evidence of this. However, he did not adopt the approaches of others, preferring instead to develop his own often idiosyncratic methods which sometimes involved reinventing what already existed, as his published Alphabet Cipher, a reinvention of the well-known Vigenere cipher, illustrates. Nevertheless, he certainly was influenced by what he read in the work of others. Dodgson described his Method of Trees in this way. "The essential character of an ordinary Sorites-Problem may be described as follows. Our Data are certain Nullities, involving Attributes, some of which occur both in the positive and in the negative form, and are the Eliminands; while others occur in one form only, and are the Retinends. And our Quaesitum is to annul the aggregate of the Retinends (i.e. to prove it to be a Nullity). Hitherto we have done this by a direct Process:....In the Method of Trees this process is reversed. Its essential feature is that it involves a Reductio ad Absurdum. That is, we begin by assuming, argumenti gratia, that the aggregate of the Retinends (which we wish to prove to be a Nullity) is an Entity: from this assumption we deduce a certain result: this result we show to be absurd: and hence we infer that our original assumption was false, i.e. that the aggregate of the Retinends is a Nullity."[4]

For example, consider the set of propositions:

1. No a are b′
2. All b are c
3. All c are d
4. No e′ are a′
5. All h are e′

If we add the conclusion, all h are d to this set of Premisses, we have a Sorites. The terms a, b, c, e are its Eliminands; the terms d and h are its Retinends.

As an example of the Method of Trees, consider the following sorites problem [5]: Do

1 2 3 4 5 6 7

$hm_1k_0 \dagger d'e'c'_0 \dagger hk'a'_0 \dagger bl_1h'_0 \dagger ck_1m'_0 \dagger hc'e_0 \dagger ba_1k_0$

prove $bl_1d'_0$ (i.e. all bl are d)?

Here ′ indicates negation; † is the symbol for "and"; a is read as no a exists, while a means there exists some a. All a are b is written as a_1b_0'; not all a are b would be written as a b_1'.

The six eliminands and three retinends in these seven premises can easily be seen by setting up a "Register of Attributes" which Dodgson sets forth in the form,

a	b	c	d	e	h	k	l	m
7	4,7	5		6	1,3,6	1,5	4	1
3		2,6	2	2	4	3,7		5

The upper row refers to the premises where the attribute occurs in positive form; the lower row where it occurs in negative form. Every letter that has numbers under it in both rows is an eliminand; the remaining letters are retinends.

To construct the tree, the retinends bd′l are taken as the root.

Starting with the leftmost attribute b we see that it occurs in premises 4 and 7. Trying 4 first, we note that the retinend l also occurs in premise 4, as does eliminand h′. Two results are now available: we don't have to examine l further and we know that b and h′ are incompatible so that the "thing" assumed to have attributes b, d′ and l <u>cannot</u> have attribute h′. Therefore it must have attribute h. We signify this by drawing a branch of the tree with h under b and the 4 with the delimiter. . . placed to its left identifying the reference premise. Taking premise 7 next, we see that b is incompatible with the two eliminands a and k′ so it has <u>both</u> attributes a′ and k which would force us to divide the tree. This we avoid doing unless there is no other choice. Moving on to the next retinend d′ we observe that we would again have to divide the tree. So we go on to the next level and examine h. The attribute h occurs in premises 1, 3, 6. Using any of these forces a division of the tree so we <u>must</u> divide now. Returning to premise 1 (just to be orderly) to construct the two branches: since the "thing" cannot have the pair of attributes k and m, it must have the contradictory of the pair, i.e. k′or m′ (by Demorgan's Laws). Alternatively, the possible <u>pairs</u> that the "thing" may have are k′m, km′, k′m′, with the first two being sufficient. Examining the branch k′m, we see that m occurs only in the premise we are currently using so m leads nowhere. However, k also occurs in premise 5, so the tree should have two branches asserting that the supposedly existing "thing" that has the attributes bd′lh must also have either the attribute k′ (which it can follow with either m or m' as the left branch) or the <u>pair</u> of attributes km′ represented as the right branch with the reference premise 1 in the middle. Following the left branch first, k′occurs in premise 3. So k′ is incompatible with ha′. But since h is already in the tree, we must also have a. Attribute k′is also in premise 7, so k′ is incompatible with b. Since b is incompatible with a, both being in premise 7, we must have a′. Writing all this under k′ as 7,3.aa′, we see that this branch ends. So we draw a circle under it. Now going down the right branch, note that m′ occurs only in premise 5. Attribute c also occurs in premise 5 and we already have k. Therefore m′ is incompatible with the pair kc implying that of the possible pairs k′c, kc′ we must have the one with c′. We add this to the tree in the form 5.c′ Now c′ occurs in premises 2 and 6. Using premise 2, c′ being incompatible with e′d′ forces us to select the pair ed′ because we already have d′. Hence we have e too. Using premise 6, c′ is incompatible with eh, so we have the choice of the pairs eh′ or e′h. But not really because already having h, we must follow with e′. This ends the right branch and so we place a circle under 6,2.ee′. The tree is now complete.

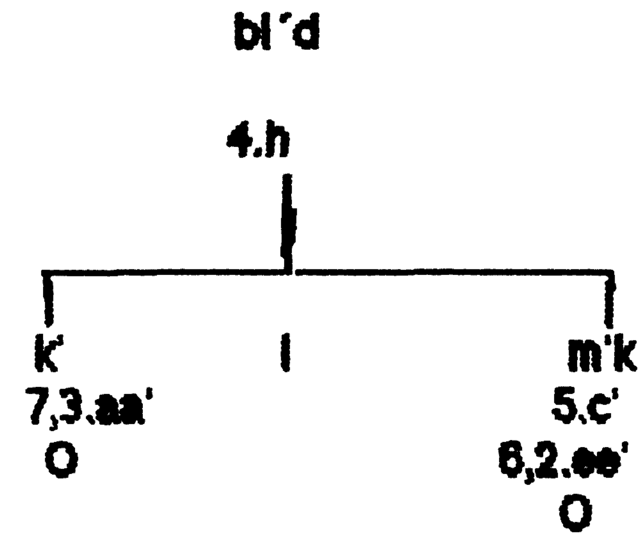
What we have proved is that if we assume that an existing "thing" has the attributes bd'l, it must also have either hk'aa' or hm'kc'ee'. Since each of these is impossible, bd'l cannot be an entity, i.e. cannot exist, so it must be the nullity, bld' because b and l being retinends gives us bl . The answer to the original question is, yes, as the figure below shows. [6]

Illustration to exhibit One of Carroll's Trees. The question: Do

1 2 3 4
hm,k. † d'e'c.' hk'a.' † bl, h.'
5 6 7
† ck. m.' † hc'e. ba, k.'

prove bl, d' ?

The method: Assume that the Premisse are true and the Conclusion false; i.e., assume that bld' is an Entity: bld'_1 , and reduce to absurdity.



∴ bld'_0 † (by premiss 4) bl_1

∴ $bl_1d'_0$

Figure 1

Dodgson modeled his logical tree on a genealogical tree, which always grows downwards, using two rules for its construction. The first establishes when two premises are incompatible: Assume the "thing" has attribute c. If a premise is the nullity ce_0 , c and e are incompatible, so we must have c and e'. The second rule determines branching in triliteral premises: Suppose a nullity contains attributes a, b, c and we know the "thing" has attribute a (because a is a retinend or an attribute already placed in the tree). Then the "thing" cannot have attributes b and c. so it must have (bc)', i.e. it must have either b' or c' by De Morgan's rule.

He began the work in July of 1894, recording on the sixteenth in his diary (unpublished),

"Today has proved to be an epoch in my Logical work. It occurred to me to try a complex Sorites by the method I have been using for ascertaining what cells, if any. survive for possible occupation when certain nullities are given. I took one of 40 premisses, with 'pairs within pairs,' & many bars, & worked it like a genealogy, each term proving all its descendants. It came out beautifully, & much shorter than the method I have used hitherto–I think of calling it the 'Genealogical Method.'" [7]

On August 4 he tentatively found a way to connect his Tree Method with his Method of Underscoring. His diary entry (unpublished) for that day was,

"I have just discovered how to turn a genealogy into a scored Sorites. The difficulty is to deal with forks. Say "all a is b or c " = "all A is b" & "all α is c" when the 2 sets A, α make up a – then prove each column separately [a doubtful success]."[8]
He was still actively at work on the Method of Trees in 1896. He wrote in his diary (unpublished) on November 12 and 13,

"Discovered method of combining 2 trees, wh [which] provide $abc0 \dagger abd0'$, into one proving $ab(cd)_0$, by using the Axiom $cd(cd)_0'$. [9]

Bartley has investigated Dodgson's awareness of the work of his contemporaries in logic and found it to be extensive. He notes particularly Dodgson's familiarity with Venn's version of Boole's logical algebra, Keynes' Studies and Exercises in Formal Logic, the work of R. H. Lotze in English translation, and with Studies in Logic, edited by Charles Sanders Peirce. This book appeared in 1883 and included chapters written by his students at Johns Hopkins: Allan Marquand, Christine Ladd, O. H. Mitchell, B.I. Gilman with the final chapter written by Peirce. Dodgson specifically cited this book in his Symbolic logic, Part II, along with books by W. Stanley Jevons, George Boole, and Augustus DeMorgan among others. In Studies in Logic the work of Ladd and Marquand must have appealed to him. Ladd's contribution, "On the Algebra of Logic" contains a novel treatment of the classical syllogism, the "inconsistent triad." The two premises of a syllogism together with the contradictory of its conclusion is an "inconsistent triad" because if we have three propositions, two of them being true, the third must be false. [If the two premises are true, the conclusion must be true, its contradictory being false.] Similarly, if the contradictory of the conclusion and one of the premises are true, the other premise must be false.] An "inconsistent triad" corresponds therefore to three valid syllogisms. Ladd provided, "a perfectly general rule, easy to remember and easy of application, for testing the validity of any syllogism, universal or particular, which is given in words. It is this:

Rule of Syllogism.- Take the contradictory of the conclusion, and see that universal propositions are expressed with a negative copula and particular propositions with an affirmative copula. If two of the propositions are universal and the other particular, and if that term only which is common to the two universal propositions has unlike signs, then, and only then, the syllogism is valid." [10] Ladd used the convention that particular propositions imply the existence of their subjects, while universal propositions do not. Arthur N. Prior wrote that Christine Ladd-Franklin used eight "copulae" to construct DeMorgan's eight categorical forms and exhibited syllogisms in different figures as derivable from "antilogisms". [11]

In Carroll's Method of Trees the essential feature is that it involves a reductio ad absurdum, a form of inverse reasoning that he enjoyed using in his mathematical writings. Seeing this same type of reasoning in logic in Ladd's work could have only been encouraging. The essential idea in the tree method is that when a conclusion following from a set of premises is assumed to be false, then if taking the premises together with the contradictory of the conclusion is an inconsistent triad, it will imply an inconsistent result. In the Method of Trees, if the tree does not continue to branch, sprouting leaves instead, then the original conclusion is derivable from the premises. Obviously, Dodgson extended Ladd's work beyond the syllogism to the sorites where the Method of Trees became an essential aid in threading through the complexity of a large number of premises. In Dodgson's final example in his chapter on the Method of Trees, he presented a problem with twenty-six premises which produces the tree below. [12]

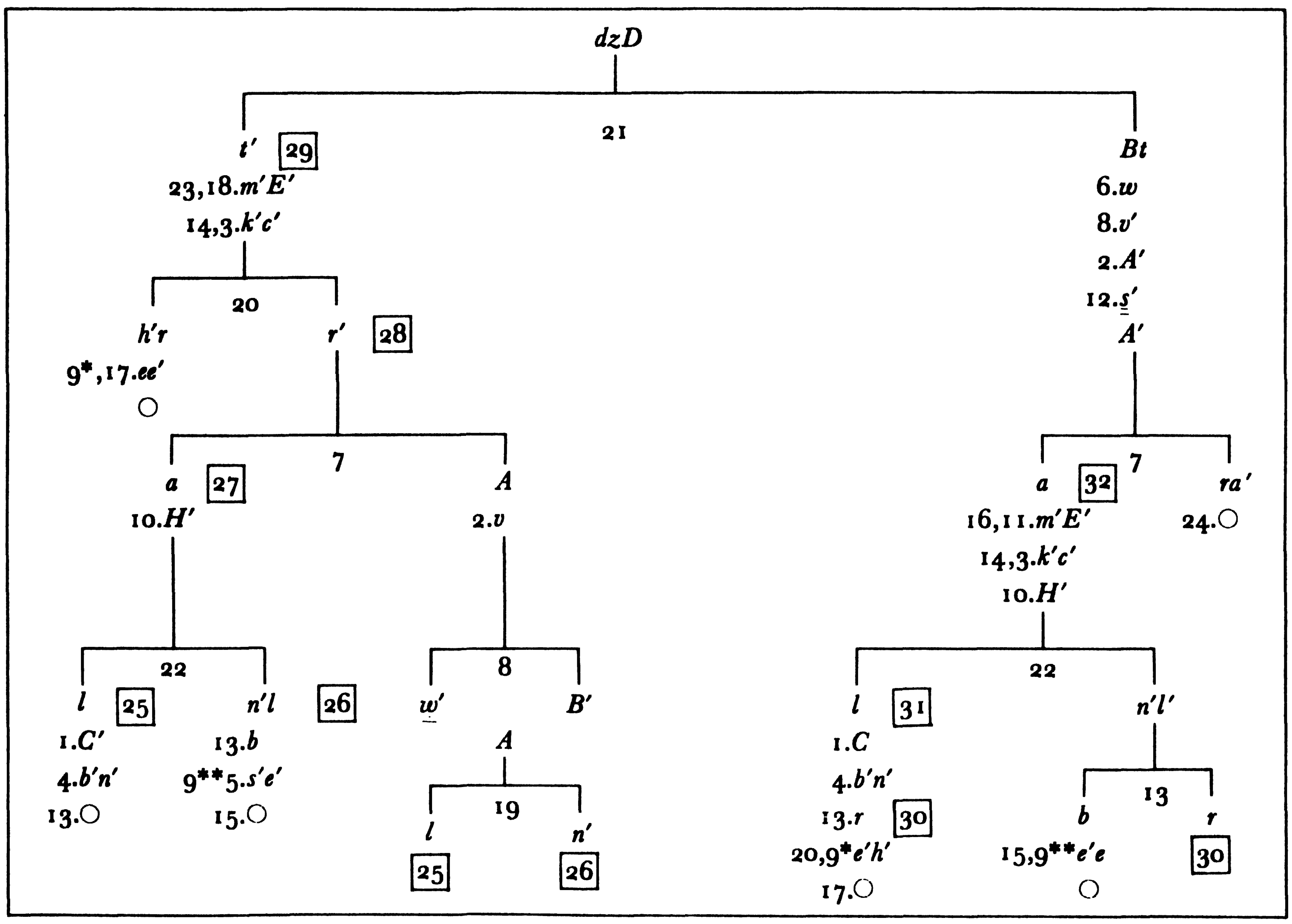


Figure 2

Ladd's work was not the only source of inspiration in Studies in Logic. Marquand's chapter would also have been especially attractive to Dodgson. Ten years before he published Symbolic Logic, Part I, Dodgson produced an elementary work, The Game of Logic, intended for his child audiences, in which he used a rectangular graph to diagram the premises of a syllogism of the sort Marquand had discussed in an article appearing in 1881. Apparently, Dodgson was not aware of Marquand's earlier work, as Martin Gardner remarked in his discussion of Carroll's diagrammatic method. [13] Marquand's short article and note, "A Machine for Producing Syllogistic Variations" and "Note on an Eight-Term Logical Machine" in Studies in Logic, contain the critical idea that when the premises of a syllogism are divided into excluded combinations (eliminands) and the non-excluded combinations constituting the conclusion, that these retinends can automatically be exhibited by a machine. Dodgson used the Register of Attributes to do precisely the same thing. Actually, Dodgson's earlier diagrammatic method, the one that appeared in 1886, in which he used colored counters to solve syllogisms had all the characteristics of a logic machine. Marquand's Eight-Term Logical Machine must have impressed Dodgson. It was capable of handling Jevon's "logical alphabet" of three terms, yet it was far simpler than what Jevons had devised. Marquand described his invention this way, "I have completed the design of an 8-term Logical Machine, of which a 4-term model is now nearly finished. If the premises be reduced to the form of the combinations to be excluded, as suggested by Boole and carried out by Venn, the operation of excluding these combinations may be performed mechanically by this machine. I have followed Jevons in making use of keys, but require for the 8-term machine only eight positive and eight negative letter keys and two operation keys." [14]. Marquand was able to reduce the number of keys because he did not use Jevon's equations, opting instead for the method devised by O. H. Mitchell, described in his article in Studies in Logic, "On a New Algebra of Logic."

Dodgson's plans for a third book on symbolic logic were thwarted by his death in 1898. The title for Part III would have been "Transcendental". Parts of the manuscript must have been written (and lost) because the book's projected contents appeared in an advertisement written by Dodgson for his set of books on symbolic logic. One of the major topics to have been included was the Theory of Inference. [15] Commenting on Carroll's "Logical Charts" in the second book, Bartley surmised, "Presumably it was Carroll's aim, either in a chapter that is now missing or in a part of the book that was never written, to teach his readers how to put disjunctions onto his diagrams and how to calculate with them." [16] The Method of Trees was one of the early steps in this direction.

Peirce's "existential graphs", the logic diagrams he invented in 1896, are certainly much more encompassing - a giant step - compared with Dodgson's rudimentary work. Nevertheless, they are both epistemologically related to the "Semantic

Tableaux" of E. W. Beth who characterised logic as a "theory of inference". Curiously, Dodgson and Peirce, both original thinkers of considerable ingenuity, displayed several characteristics that were similar. Both enjoyed creating unconventional terms and symbols for new ideas, "icons" to use Peirce's term. Both shared the belief that mathematical (deductive) reasoning was essential to the understanding of reality. Both recognized the importance of communicating the principles of logical analysis to young people. And both derived much from the work of Augustus DeMorgan. But Dodgson's scope was narrow. The syllogism and sorites, even in their most general forms belong to classical, traditional logic. Dodgson's interest was confined to the extensions of traditional logic and he did not communicate this interest well even to his contemporaries. One cannot fail to notice the literary flavor to Dodgson's serious writing, not a characteristic that would please the professional logician. As an ordained member of the Church of England and an essentially shy person, he remained throughout his life outside the mainstream of the professional logical - mathematical community and its increasingly secularized interests. His writings display an appreciation of history and appeal to humanistic sensibilities. These are the main reasons that his contributions to logic were overlooked even though they were very much on the right track.

References

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6. Marquand, A. (1883). "Note on an Eight - Term Logical Machine." In C. S. Peirce (ed.) Studies in Logic. Boston, Little, Brown, and Co.

7. Carroll, L. (1897 reprinted 1958). Symbolic Logic. New York, Dover.

NOTES

1. Bartley 1977:134. Bartley cites the third edition of Euclid and his Modern Rivals as the source rather than Curiosa. Only two editions of EMR were published.
2. Bartley 1977:250.
3. Bartley 1977:31-2.
4. Bartley 1977:279-80.
5. Bartley 1977:292-3.
6. Bartley 1977:295.
7. Dodgson 1894:16 July.
8. Dodgson 1894:4 August.
9. Dodgson 1896:12-13 November.
10. Ladd 1883:41.
11. Prior 1971:548.
12. Bartley 1977:312.
13. Gardner 1968:47-8.
14. Marquand 1883:16.
15. Carroll 1897 (Dover 1958:xi).
16. Bartley 1977:274.

ABSTRACT

In 1894, Charles L. Dodgson (Lewis Carroll) developed a mechanical method to test the validity of complicated multiliteral statements using a reductio ad absurdum argument. The basic ideas are similar to those in Beth's "Semantic Tableaux" and their seeds can be found in papers by Peirce's students, C. Ladd and A. Marquand, that appeared in Studies in Logic (1883), edited by Peirce. Dodgson named his approach, the Method of Trees. It was virtually unknown until 1977 when W. Bartley published the second part of Dodgson's Symbolic Logic, a book thought to have been lost. In this paper we examine the Method of Trees closely and establish the semantic connections between it, Ladd's "inconsistent triad" and Marquand's logic machines.