The Review of Modern Logic
Volume 9 Numbers 3 & 4 (December 2003–August 2004) [Issue 30], pp. 203–207.

Michael Dummett
Elements of Intuitionism. Second Edition.
New York: Oxford University Press, 2000
xii + 331 pp. ISBN 0198505248

REVIEW

JAN WOLEŃSKI

The first edition of this book appeared as the second volume in the series, Oxford Logic Guides, in 1977 and became the standard introduction to the intuitionistic philosophy of mathematics very soon. The present edition satisfies the highest standards of publishing quality, compared to the rather poor features of the former printing. The second edition preserves the original structure of the content adopted by the author for the first edition. Chapter 1 outlines preliminaries concerning constructive proofs, the meaning of logical constants, a sample of logical laws typical for intuitionism, and functional completeness. Elementary intuitionistic mathematics (arithmetic, real numbers, order, the axiom of choice) is discussed in Chapter 2. More advanced topics (infinity, the fan theorem, bar induction, the continuity principle, the Bar Theorem and its proof by Brouwer, continuous functionals and their representation, the uniform continuity theorem) are presented in Chapter 3. The next chapter is devoted to logical matters (natural deduction, the sequent calculus, cut-elimination, decidability of intuitionistic sentential logic, normalization). Chapter 5 contains an exposition of semantics for intuitionistic logic (valuation systems, lattices and finite model property, topological spaces, Beth trees, the semantics for intuitionistic predicate logic, the completeness of intuitionistic predicate logic, generalized Beth tree, compactness). Intuitionistic formal systems, realizability, and the Creative Subject (an idealized mathematician used in thought-experiments for motivating axioms and basic principles of mathematical reasoning) are discussed in Chapter 6 (under the title "Some further topics"). Concluding philosophical remarks (philosophical foundations of constructive mathematics, the notion of proof, partial functions, logical constants as represented on Beth trees, the notion of a choice sequence). A revised (relative to the first edition) bibliography closes the book.

Although the structure of the book is preserved, the second edition contains changes and additions. They concern: the presentation of Brouwer's proof of the Bar Theorem, the new and simplified account of valuation systems, the new and revised treatment of generalized Beth trees and the completeness proof of intuitionistic first-order predicate logic, and a sketch of Friedman's proof of the completeness of negation-free first order logic (it is added). As Dummett informs us in the Preface to the second edition, he lost his initial enthusiasm for Kreisel's theory of constructions for providing the semantics for intuitionistic logic, because this semantic theory "did not prove fruitful" (p. v).

In general, the book considers intuitionism as a complex proposal in the foundations of mathematics without specially entering into the more general philosophical problems suggested by this way of thinking. Although Michael Dummett is known as a consequent proponent of verificationism in the philosophy of language and epistemology, *i. e.*, the view based on constructive logic, the reviewed book is written from an external, though sympathetic point of view. In particular, Dummett sees various points of intuitionism as open and difficult and thereby requiring further studies. It concerns, for instance the limitations of intuitionistic mathematics, the rôle of Church's thesis in intuitionism or the completeness of intuitionistic first-order logic (see below for the last problem).

The general aim of the book is stated in the following words: "The purpose of this book is to provide, in a form readily intelligible to one having no prior knowledge of the subject, basic information about the fundamental ideas of intuitionistic mathematics. It is particularly aimed at those who are concerned to acquire an explicit knowledge of intuitionistic logic and the established results concerning it." (p. vii) Accordingly, the book develops in three directions. First, we have the mathematical part (Chapters 2-3) in which selected elementary as well as more advanced fragments of intutionistic mathematics are summarized. Secondly, there is the logical dimension (Chapters 4-6). Logical matters are considered more extensively than other problems. This feature of the book documents a considerable change in looking at intuitionism compared with the way of thinking characteristic of the Fathers (Brouwer and Heyting) who did not pay much attention to logic (Heyting more than Brouwer). Contemporary intuitionism is strongly concerned with logic. In fact, the reviewed book contains a fairly formalized version of intutionistic logic. Thirdly and finally, Dummett conducts the philosophical discussion of intuitionistic assumptions and consequences of intuitionism as the philosophy of mathematics (Chapter 1, Chapter 6). These three dimensions are unified in a coherent whole. Thus, Dummett certainly succeeded in producing an elementary text, accessible to a wide audience, philosophical as well as mathematical. However, if the book is evaluated from the point of view of its completeness, one should point out that the history of intuitionism is almost entirely neglected. Historical notes are very restricted and omit many important facts: for example, that Kripke semantics was anticipated by Grzegorczyk or that Wajsberg proved the separation theorem. The same applies to the variety of constructivism. Although Dummett refers to other books (for example, [1], [16], [17]) and sends his readers to them, one could expect more comprehensive historical information to be included in a book of this character. Moreover, the bibliography is not complete. In particular, the following sequence should be added: [2], [3], [4], [5], [6], [7], [8], [9] (it is important to know that Markov's fundamental book is available in English), [12], [13] and [14] appeared before 2000. One should add [10] and [15] were published simultaneously with Dummett's book. I also mention [11], which is an excellent treatment of constructive logic from the classical point of view; I think that this book should be translated into English. Of course, I do not claim that this supplementary bibliographical information exhausts the books that could be added.

Let me stress some points more closely. The book presents well-known motives for intuitionism, in particular why, according to intuitionism, mathematics needs a justification and how this view differs from the Platonism of classical mathematics. Perhaps the problem of truth is critical in this respect. Dummett discusses the relation between truth and provability, focusing on the meaning of such phrases as "provable", "we can prove" or "has been proved" as possible explicata for the concept of constructive truth (pp. 10-12). This is followed by remarks on constructive proofs with important notes about the Markov principle (MP), that is, the formula $\forall x(A(x) \vee \neg A(x))\&\neg\forall x\neg A(x) \rightarrow \exists x A(x)$ which is not valid in intuitionistic logic. On p. 23 one can find an explanation of why the principle of the least number does not hold in intuitionistic mathematics, showing why the principle of mathematical induction has another meaning in intuitionism than in classical mathematics. In the latter both principles are equivalent. The author shows further (p. 37) that the axiom of choice holds in intuitionistic mathematics, provided that the quantifiers are constructively understood. These two examples document that intuitionistic mathematics is not as simple as it is thought sometimes. The simplified view is that all principles of classical arithmetic are automatically available in intuitionism, but stronger set theoretical tools must be dropped. Dummett develops some parts of metamathematics for intuitionistic formal systems. Perhaps the most fascinating problem concerns the completeness of first-order logic. This property can be proved easily intuitionistically (that is, by methods acceptable to the intuitionists) for propositional calculus. However, it is still a controversial matter whether the completeness of intutionistic predicate logic is intuitionistically provable. Dummett is rather sceptical as far as the matter concerns known proofs. For example, he thinks that the proofs using Beth's generalized trees raise some doubts from the intuitionistic point of view. This evaluation is to be contrasted with that of van Dalen and Troelstra (see [17]). The reviewer regards this question as the most critical for intuitionism, because its solution determines the range of applicability of intuitionistic methods.

I mentioned above some inaccuracies in the historical information. There are other omissions as well. The lack of a treatment of the Dialectica interpretation is mentioned in the Preface and it would unfair to make an objection at this point. Although the selection of missing topics depends on my personal choices and expectations, I think that they are justified from a general perspective. Nothing is said about the relation between intutionism and modal logic. The status of the incompleteness theorems in intuitionism is almost not discussed. More attention should be paid to a comparison of the Hilbert program and intuitionism, in particular, in the context of reverse mathematics. Both questions are serious. It is known that if Heyting arithmetic is consistent, Peano arithmetic is consistent too. This result holds intuitionistically. Classically we can prove that since Peano arithmetic is interpretable in Heyting arithmetic, the consistency proof for the latter is not available in it. However, consistency is a necessary, although insufficient condition of existence under the intuitionistic requirements. Thus, one of the most important points in intuitionism is intuitionistically unclear. Thus, it seems that the intuitionists must believe in consistency, because they cannot prove it. Does theology enter into intutionism at this point?

References

[1] M. J. Beeson, *Foundations of Constructive Mathematics*, Berlin: Springer Verlag, 1995.

[2] D. van Dalen, *Mystic, Geometer, and Intuitionist. The Life of L. E. J. Brouwer, v. I: The Dawning Revolution*, Oxford: Clarendon Press, 1999.

[3] A. G. Dragalin, *Mathematical Intuitionism. Introduction to Proof Theory*, Providence: American Mathematical Society, 1987 (Russian original, 1987).

[4] M. Fitting, *Proof Methods for Modal and Intuitionistic Logic*, Dordrecht: D. Reidel, Publishing Company, 1983.
[5] P. Fletcher, *Truth, Proof and Infinity. A Theory of Constructions and Constructive Reasoning*, Dordrecht: Kluwer Academic Publishers, 1998.
[6] D. M. Gabbay, *Semantical Investigations in Heyting's Intuitionistic Logic*, Dordrecht: D. Reidel, Publishing Company, 1981.
[7] D. E. Hesseling, *Gnomes in the Fog. The Reception of Brouwer's Intuitionism in the 1920's*, Utrecht: Universiteit Utrecht, 1991.
[8] F. von Kutschera, *Der Satz von ausgeschlossenem Dritten*, Berlin: de Gruyter, 1985.
[9] A. A. Markov, *Theory of Algorithms*, Jerusalem: The Israel Program for Scientific Translations, 1962 (Russian original, 1954).
[10] G. Mints, *A Short Introduction to Intuitionistic Logic*, Dordrecht: Kluwer Academic, 2000.
[11] P. S. Novikov, *Constructive Mathematical Logic from the point of View of Classical Logic* (in Russian), Moskva: Nauka, 1977.
[12] T. Placek, *Mathematical Intuitionism and Intersubjectivity*, Dordrecht: Kluwer Academic Publishers, 1999.
[13] G. Sambin, J. Smith (eds.), *Twenty-Five Years of Constructive Type Theory*, Oxford: Oxford University Press, 1998.
[14] K. Schütte, *Vollständige Systeme modaler und intuitionistischen Logik*, Berlin: Springer Verlag, 1968.
[15] G. Sommaruga (ed.), *History and Philosophy of Constructive Type Theory*, Dordrecht: Kluwer Academic Publishers, 2000.
[16] W. P. van Stigt, *Brouwer's Intuitionism*, Amsterdam: North-Holland Publishing Company, 1990.
[17] A. S. Troelstra, D. van Dalen, *Constructivism in Mathematics: An Introduction*, Amsterdam: Elsevier, 1988.

Jagiellonian University, Institute of Philosophy, Kraków, Poland
E-mail address: `wolenski@theta.uoks.uj.edu.pl`