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Mark Balaguer
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THIN- AND FULL-BLOODED PLATONISM

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General Setting

Mathematical theories seem to be objectively true in the sense that they are true independently of us and of our mathematical theorizing. Several features of mathematical and scientific practice support such a view. Mathematical practice treats its discourse objectively. Gödel's first incompleteness theorem seems to show the independence of the mathematical realm – some mathematical theories are true in some sense that goes beyond the theory's means of deciding their truth. Moreover, scientific practice also seems to presuppose mathematical objectivity. The apparent indispensability of mathematics to scientific theorizing led philosophers like Putnam ([31, 32]) and Quine ([34, 35]) to mathematical objectivity: if we believe what our theories say about forces and fundamental particles, we ought also to believe what they say about functions and numbers, given that our best scientific theories involve an unavoidable appeal to an inextricable combination of physical and mathematical entities. Traditional Platonism provides the most straightforward way of upholding mathematical objectivity. This is the view that mathematical objects are abstract objects that exist independently of us and of our theorizing, and that our mathematical theories are true (false) to the extent that they correctly (incorrectly) characterize those objects.

However, mathematical objects of the kind required by Platonism appear to differ from their everyday and scientific cousins in at least two important respects. They are abstract. And their very existence and identity are intimately connected with what we say, think, and theorize about them. In his [4] and [3] respectively, Paul Benacerraf showed how these two features of mathematical objects – their abstractness and their discourse-relativity – lead to philosophical problems that are difficult to resolve. Much of philosophy of mathematics for the past twenty-five years has been driven by efforts to resolve the problems they brought to light. The problems are sufficiently intractable that many philosophers are driven to mathematical anti-Platonism. The most straightforward version of anti-Platonism is mathematical fictionalism: the view that mathematical objects such as numbers and sets are merely fictions and that all existentially generalized mathematical sentences are false.

The upshot is that considerations based on mathematical and scientific practice push us toward one extreme, Platonism, while considerations based on the peculiar nature of Platonist objects drive us toward another extreme, fictionalism. Much has been written, numerous intermediate positions have been explored, but surprisingly little agreement (even in philosophy where agreement is difficult to achieve) has been forthcoming.

Mark Balaguer's excellent book falls in this tradition. As I see it, the book accomplishes three tasks. First, it provides an exceptionally clear, insightful, and useful critical survey of the literature that has developed in response to these tensions. As such it would make a fine, comprehensive, sophisticated introduction to this quite large body of literature. But the book does far more. Second, it develops a (highly original) version of Platonism and a version of anti-Platonism and argues that the developed versions are the best of their kind. Finally, the book concludes, first, that there is no good reason to prefer one position to the other and more radically, second, that there is no fact of the matter as to whether Platonism or anti-Platonism is correct. This last point revitalizes the logical positivist doctrine, advocated originally by Carnap ([7, 8]) but since fallen into disrepute, that the Platonism/anti-Platonism dispute, like all metaphysical questions, has no factual basis. This is an attractive doctrine. If true, it explains why there has been such little agreement about the dispute. However, it is also a provocative doctrine. For well-known reasons – mainly their dependence on an inadequate theory of meaning – logical positivist critiques of metaphysics failed. Though Balaguer arrives at his conclusion by arguments that do not rely on positivistic tenets, the suspicion is likely to persist, among Platonists and anti-Platonists alike, that the conclusion cannot be correct.

Following a brief introduction that sets the stage and outlines what is to come, the book is divided into three parts: three chapters presenting and defending the best version of Platonism (Full-Blooded Platonism), three chapters presenting and defending the best version of anti-Platonism (Fictionalism), and a concluding chapter arguing (a) that Full-Blooded Platonism and Fictionalism are on a par, and (b) that there is no fact of the matter as to which is correct.

In this review I first outline the book and Balaguer's principal arguments, then provide critical discussion. In an important sense, I argue, Balaguer is correct – there is no fact of the matter whether Platonism or anti-Platonism (in their traditional manifestations or in Balaguer's best versions) is the correct metaphysics for mathematics. However, I also argue, this is so for reasons that are very different from those adduced by Balaguer.

1. Full-Blooded Platonism, Access, and Uniqueness

According to traditional Platonism (TP), mathematical theories are true (false) in virtue of correctly (incorrectly) characterizing determinate objects (like numbers, functions, and sets) that are abstract (non-mental, non-spatiotemporal, non-physical, and non-causal) and exist independently of us. Given that such objects lack any natural connection with human beings, how can Platonists even begin to explain how human beings could acquire any knowledge of them, have reliable beliefs about them, or even refer to them? This is the core of Benacerraf's challenge in [4]: assuming such objects exist, the challenge is to explain how we could come to have reliable beliefs or entertain thoughts about them, or even to refer to them. Henceforth I'll call it "the problem of access".

Balaguer provides a useful taxonomy of responses found in the literature and shows how each class of response fails. **Contact theories** – whereby we are in cognitive contact with the objects – comprise one class of response. Gödel ([13]) seems to have held a contact theory: we cognize mathematical objects by means of a faculty of mathematical intuition just as we cognize physical objects by means of a faculty of perception. Such a view, Balaguer argues, borders on the unintelligible since we have no idea how information could be transmitted from the inhabitants of Cantor's paradise to us inhabitants of this world. Maddy ([24, 25]) provides a naturalized version of a contact theory, replacing Gödel's mystical faculty with a neurophysiological mechanism that enables us to perceive the properties of (small) sets of concrete objects, on the basis of which we can piggy-back to knowledge of infinite sets of abstract objects. Balaguer rejects this kind of naturalized Platonism: perceptual encounters cannot provide the kind of abstract information required for basic mathematical beliefs, and even if they could, the move to the infinite would be problematic. Balaguer's rejection of contact theories seems correct: **contact theories** are precluded by the very nature of the problem.

In order to respond to the challenge, mathematical Platonists must therefore use **no-contact theories** that attempt to explain how we could refer to and have reliable beliefs about mathematical objects while conceding that we can have no contact with such objects given their abstract nature. Balaguer discusses and criticizes several classes of theories of this type. I postpone discussion of one class that relies on confirmational holism ([35, 37, 41]) and the idea that there's nothing special about mathematical knowledge until §2. Another class of no-contact theory comes in a variety of sub-classes, common to each of which is the idea that mathematical knowledge has some special (typically *a priori*) feature that differentiates it from *a posteriori* empirical knowledge in such a way that we can acquire the former, but not the latter, without contact with the objects of that knowledge. For Wright ([43]) and Hale ([16]), the special feature is conceptual: we can justifiably come to have beliefs about mathematical objects, for example, by analysis of mathematical concepts. For Katz ([19]) and Lewis ([22]), the special feature is necessity: once we construct a sufficiently detailed concept of 4, we see that it must be the successor of 3. For the structuralists, Resnick ([37]) and Shapiro ([38]), the special feature is axiomatic characterization: we can acquire knowledge of mathematical structures (and thus of the mathematical objects which are the positions in those structures) by constructing axiom systems that implicitly define the structures. In each case, the special feature allows us to know mathematical truths by *a priori* means without contact.

Suppose we grant Platonism's ontological presuppositions – that abstract mathematical objects exist independently of us. It will still be the case, Balaguer argues, that all of these special-feature versions of the no-contact strategy share a common problem: without contact, they do not help us to understand how we can refer to, and have reliable beliefs about, the objects presupposed. Even if we grant that abstract mathematical objects exist, Platonists must still explain how the definitions, concepts, or axioms that we happen to formulate or construct can actually pick them out. Given a concept, definition, or axiomatic characterization, how does the no-contact Platonist know that there exist objects that satisfy it? How does he know, for example, that our definition of 4 as the successor of 3 actually picks out an independently existing number 4 whose nature involves being the successor of 3? Even if we grant a Platonist ontology, each of the special-feature versions of the no-contact strategy falls short of guaranteeing that the appropriate mathematical objects exist. If 4 exists, then it is (conceptually or necessarily) the successor of 3. If an ω-sequence of abstract objects exists, then the Peano axioms implicitly characterize its structure. But these are big *IF*s.

This is the core of Benacerraf's problem of access for Platonists. Given the existence and nature of mathematical objects, we can have no contact with them. So contact theories are ruled out. But no-contact theories lack the resources needed to explain how what we say, think, and theorize when we engage in mathematical practice is actually about the independently existing objects. If contact theories are precluded by the very nature of the problem, the only way to uphold Platonism is somehow to augment the resources available to no-contact theories.

This is precisely what Balaguer proposes. But instead of concentrating on augmenting our epistemic wherewithal to bridge the gap, Balaguer proposes to augment the platonic universe. This is his thesis of full-blooded Platonism (FBP). According to FBP, every mathematical object that logically could exist actually does exist, or (as he alternatively formulates it) every consistent mathematical theory truly describes part of the mathematical realm. What makes our number-beliefs be *about* numbers? What, for example, makes our belief that 4 is the successor of 3 be *about* 4 (or 3)? Balaguer distinguishes two senses of *about*: to have a belief that is *thickly about* an object, one needs to be connected to the object in some non-trivial information generating way; to have a belief that is *thinly about* an object, no such requirement needs to be met. Merely by serious epistemic engagement with a fiction, a child can have beliefs – *e.g.*, that Santa Claus is fat – that are thinly about an object – Santa Claus – without any contact with the object and even if the object does not exist. Provided number theory is consistent, we can have beliefs (that 4 is the successor of 3, *e.g.*) that are thinly about numbers (about 4, *e.g.*) without any contact with the numbers and even if they do not exist. Moreover, given FBP, it automatically follows that the objects do exist, because FBP guarantees that every mathematical object that could exist does exist. The answer to the question "What makes mathematicians' beliefs about numbers reliable (= likely to be true)?" is equally simple. If mathematicians are reliable in arriving at consistency judgments (and presumably they are), then if most mathematicians believe some mathematical sentence, p, then p is probably consistent. But if p is consistent, then (by FBP) p truly describes part of the mathematical realm. Acceptance is a reliable indicator of consistency, which in turn guarantees (by FBP) true description of part of the mathematical realm. If FBP is true, we have quite simple answers to the problem of access posed in [4].

We now turn to the problem of non-uniqueness posed in [3]. Traditional Platonism presupposes that numbers and sets are objects with determinate conditions of identity, that mathematical theories truly describe *unique* collections of mathematical objects. But there are numerous ways to identify the collection of mathematical objects that a given theory describes. The standard natural number sequence, for example, can be reduced to set theory following the method of Zermelo, who identified a natural number with the set of its immediate predecessor, or the method of von Neumann, who identified a natural number with the set of all its predecessors, or various other methods. However, the methods cannot all be correct. According to the Zermelo method, $2 = \{\{\emptyset\}\}$; according to the von Neumann method, $2 = \{\emptyset, \{\emptyset\}\}$; but $\{\{\emptyset\}\} \neq \{\emptyset, \{\emptyset\}\}$. The methods give different identity conditions for the number 2. Any set theoretic isomorphic copy of the Zermelo numbers will work as well as any other; yet there seems to be nothing about the numbers, about the objects we identify them with, or about our mathematical practice that can determine a unique set theoretic progression as the natural numbers. Furthermore, Benacerraf argues, any ω-sequence at all will do, because the only constraint on being a natural number sequence is that it satisfy the Peano axioms and account for cardinality, and any ω-sequence will perform these tasks. It follows that the traditional Platonist assumption that arithmetic truly describes a unique collection of mathematical objects is false. The numeral '2' does not refer to a unique object, because there are no unique objects that are the numbers. Even worse, the argument generalizes ([20]): there seems to be no fact of the matter whether real numbers should be identified with Dedekind cuts or with equivalences classes of Cauchy sequences, whether ordered pairs should be identified with Wiener or with Kuratowski pairs, *etc.*

Balaguer's response to the problem of uniqueness is a mixture of the good news-bad news variety. The good news is that informal considerations, captured by what Balaguer calls "our full conception of natural numbers" (FCNN), can be used to augment the purely mathematical constraints captured by the Peano axioms so that we can cut down the range of ω-sequences that are admissible candidates for the natural numbers. FCNN precludes sequences that identify a number with concrete objects like the Eiffel Tower, for example. Such objects are corruptible, but numbers are not. The bad news is that FCNN lacks the resources to single out a unique natural number sequence: lots of sequences of objects with properties we have never thought of are likely to qualify as natural number sequences. Balaguer's positive proposal is that we should accept non-uniqueness. Given any mathematical theory supplemented with all the relevant informal considerations to form our full conception of the theory's domain, no unique collection of objects is thereby characterized. Given FBP, non-uniqueness is attractive: since every mathematical object that could exist does exist, it is natural to think there are numerous ω-sequences that satisfy FCNN (everything we think characterizes the natural numbers) but differ in ways that we have never thought about. Again, if FBP is true, we have a simple answer to the problem of non-uniqueness: there isn't one.

2. Fictionalism, Applications, and Indispensability

Many think that access to a Platonic realm of mathematical reality is sufficiently mysterious to undermine Platonism altogether. Instead they opt for anti-Platonism: either mathematical theories are not true (in any straightforward sense) or, if they are true, their truth is not to be understood in terms of their characterizing abstract objects that exist independently of us. Balaguer puts anti-Platonists into two camps. **Realist anti-Platonists** solve the problem of access by constructing the mathematical universe out of materials that permit epistemic and semantic access. Mathematical theories are true (false) to the extent that they correctly characterize mental objects (19th century psychologism), or physical objects ([27]), or the collecting and matching operations of an ideal agent ([21]). Since we have unproblematic access to our own ideas, to ordinary physical objects, or to our own ideal constructions, the problem of access is moot. However, pressures build up elsewhere. If mathematics is about ideas, then there's no hope of accounting for (a) mathematical objectivity (your "2 + 3" is different from my "2 + 3"), (b) the ability to talk meaningfully about mathematical objects we have never mentally constructed (very large numbers) or cannot mentally construct (the class of all real numbers). If mathematics is about physical objects, then we cannot make sense of (a) the difference between {Bill Gates} and {{Bill Gates}} (both sets are associated with the same physical stuff), (b) infinity (there is no reason to suppose that there are sufficiently many physical objects), (c) mathematical practice (where proof, not empirical investigation, is central). If mathematics is about the operations of a non-existent ideal agent, then mathematics is vacuous – its universal generalizations (over these operations) are vacuously true, its existential generalizations false. Kitcher's account thus collapses into antirealism.

Antirealist anti-Platonists solve the problem of access by denying that there is a mathematical universe: either mathematical theories are not true (fictionalism), or their truth is not to be understood in terms of their characterizing collections of objects of any kind (reinterpretationism). Conventionalists ([1], [18], [7] and [8]) maintain that our mathematical theories are true by conventional stipulation; the Peano axioms, for example, are true in the sense that they are stipulated to be so. Deductivists ([30]) maintain that standard mathematical sentences, T, should be reinterpreted as conditionals of form (AX $\rightarrow$ T), where 'AX' is the conjunction of the axioms appropriate for a T-assertion. Formalists ([11]) maintain that standard mathematical statements, T, should be reinterpreted as metamathematical statements about syntax: T is a theorem of some formal system AX. All versions of antirealist anti-Platonism, Balaguer argues, encounter two problems. First, they require non-standard, non-face-value interpretations of mathematical theory that run counter to standard, face-value interpretations of what mathematicians say and do. When mathematicians assert the existence of prime numbers between 0 and 10, they mean to say categorically that there are such numbers; they do not mean to say that there are such numbers conditional on the Peano axioms being stipulatively given or assumed, or that the assertion is a theorem of Peano arithmetic. As far as possible, we should take what mathematicians say at face-value; if so, then all of these reinterpretationist accounts are unsatisfactory. Second, there is a better antirealist approach – mathematical fictionalism – that provides a standard, face-value interpretation of what mathematicians say and do.

According to mathematical fictionalism, there are no mathematical objects, mathematical terms do not refer, and mathematical theories are simply useful fictions whose existence claims are all false. However, FBP and realism in general seem to have one advantage that fictionalism and antirealism in general lack. The former, but not the latter, can account for the apparently indispensable use that is made of mathematics in scientific application.

In §1, we postponed discussion of a class of no-contact Platonist theory that relies on confirmational holism ([35]) and the idea that there's nothing special about mathematical knowledge. The basic idea is that mathematical theories are inextricably bound up with, and central to, our scientific worldview; confirmation is holistic. When an experiment confirms or disconfirms a part of a theory, it confirms or disconfirms all those parts of the theory, including its mathematical parts, that are used to deduce the confirming or disconfirming instance; since our scientific worldview is repeatedly confirmed by standard empirical procedures using standard scientific methodology, its mathematical parts are also repeatedly confirmed by the same empirical procedures. So, we do not need contact with mathematical objects to know that our mathematical theories are true: the reliability of our beliefs about sets and numbers is no different from the reliability of our beliefs about molecules and physical fields; all are useful theoretical posits that have withstood the test of experience. This is the core of the indispensability argument for Platonism and against fictionalism. Balaguer denies confirmational holism. He correctly points out that mathematicians often know their theories are true prior to any applications of those theories; they do not wait upon empirical application for after-the-fact empirical justification. More generally, confirmational holism does not fit mathematical practice very well. As we said in connection with Millian mathematics, proof, not empirical investigation, is central to what mathematicians do when they justify their theories.

Nevertheless, Platonists can marshal the fact that mathematical theories are applicable and apparently indispensable to scientific theorizing without holism. Granted, the evidence for mathematics is distinct from the evidence for science, they can argue, but we are justified in drawing scientific conclusions only if we are also justified in accepting as true the mathematics indispensably used in drawing those conclusions. We cannot both accept the deliverances of empirical science and at the same time deny (with fictionalists) the truth of the mathematical premises that were among the hypotheses used to deliver them ([37]).

Balaguer explores two strategies for undermining this argument. The tougher strategy argues that mathematics is dispensable to empirical science by showing that any standard scientific theory, T, can be nominalized: it can be rewritten as a theory, N, so that (a) no sentence of N requires quantification over abstracta, (b) T is a conservative extension of N with respect to nominalistic consequences, (c) N will do the same scientific work as T. Field ([12]) produced such a nominalization, N, of classical gravitational field theory, T. The axioms of N are formulated without using ordinary mathematics. Assignments of particles to quadruples of reals in T are replaced by the primitive notion of a particle occupying a space-time point in N. Functors in T are replaced in N by primitive comparative predicates that apply only to nominalistically acceptable entities (space-time points and regions) and are axiomatized so that the structure of $\mathbb{R}^4$ and of physical magnitude functions from $\mathbb{R}^4$ onto $\mathbb{R}$ is preserved. Field proves a representation theorem showing the existence of the structure-preserving mapping.

Thus, Field claims to show that in the case of T mathematics can be dispensed with.

Malament ([26]) objects that, even if Field's nominalization of T is successful, the example cannot be generalized to all scientific theories. In particular, it cannot be generalized to quantum mechanics. The only obviously similar representation theorem for quantum mechanics would need to show that the lattice of propositions (of form 'a measurement of observable A yields a value in Borel set Δ with probability r') is isomorphic to the lattice of subspaces of some Hilbert space. The problem is that propositions are not nominalistically acceptable: measurement events associated with an observable do not generally occur and so are abstract.

Balaguer tackles this problem head-on. The closed subspaces of Hilbert space, he argues, need not be taken as representing abstract propositions; instead they can be taken as representing physically real propensity properties of quantum systems. Instead of taking them to represent propositions of form 'a measurement of observable A yields a value in Borel set Δ with probability r', we may take them as representing properties of form 'the r-strengthened propensity of a state-Ψ system to yield a value in Δ for a measurement of A'. Balaguer sketches how to generate the tools to prove a representation theorem showing that the structure of propensities of a system can be embedded in the ordinary Hilbert-space structure.

Unlike the space of possible events, the space of a quantum system's propensities is formed from nominalistically acceptable notions. A quantum system's propensity, Balaguer argues, is a real physical property. In a given state, a quantum system has associated with it a set of <u>actual, causally efficacious</u> propensities that are <u>located in space-time</u> by being instantiated by the system. Propensities, in other words, are not abstract. One may object that propensities are not nominalistically acceptable because their specification in terms of Δ-values and r-strengthened probabilities require reference to real numbers. Balaguer proposes (following Field) to eliminate references to values in the Borel set by introducing comparative predicates between quantum systems: a state-Ψ electron's r-propensity to have a momentum-value in the closed interval $[m_1, m_2]$ is for it to be <u>momentum-between</u> a state-Ψ_1 electron and a state-Ψ_2 electron, where Ψ_i is the state of having a momentum-value, m_i. Similarly, we eliminate 'r-strengthened propensity' references: 'a state-Ψ electron has an r-strengthened propensity to yield a value in Δ for a measurement of A' will be replaced by sentences like 'a state-Ψ electron is $\underline{(A, \Delta) - \text{propensity-between}}$ a state-Ψ_1 electron and a state-Ψ_2 electron'. Field eliminated reference to real numbers in favor of comparative properties defined on space-time points and regions; similarly Balaguer proposes to eliminate reference to propositions in favor of comparative propensity properties defined on quantum systems.

The easier strategy, and the strategy clearly preferred by Balaguer, for avoiding the indispensability objection to fictionalism denies that mathematical indispensability entails mathematical truth. Life would be easier for the fictionalist if, instead of having to construct a nominalistic variant of each scientific theory, he only had to show that any given theory has a nominalistic content that captures its complete picture of the physical world. Balaguer presents a philosophical argument for this conclusion, relying only on very general principles about the nature of mathematical and physical objects. Take any true mathematized scientific theory; *e.g.*, 'The temperature of physical system $S = 40°C$' ['$\mathrm{T}(S) = 40$']. It appears to express a mixed fact relating S's physical state to a number. This mixed fact cannot be what Balaguer calls a "bottom-level" physical fact, because genuinely physical objects have a causal nature, and the number 40 is acausal. The truth of '$\mathrm{T}(S) = 40$' must then depend on two sets of causally independent facts – purely nominalistic bottom-level physical facts involving causal determinants (connected with S's energy) of S's behavior and purely Platonic facts involving acausal numbers. The root intuition is that Platonic objects like numbers cannot really have anything to do with determining the behavior of empirical systems; that behavior and its physical determinants are what empirical science is trying to capture; it is successful insofar as the physical world "holds up its end of the empirical science bargain". Given the natures of physical and mathematical objects, we know that some purely physical fact (with no mathematical trappings) holds up its end of the '$\mathrm{T}(S) = 40$' bargain. We may not be able to represent that fact without using mathematical tools. But we know such a fact – the nominalistic content of '$\mathrm{T}(S) = 40$' – exists. Thus, <u>any</u> mathematized scientific theory has a nominalistic content that captures its complete picture of the physical world. Moreover, because mathematical objects are completely independent of this nominalistic content, it could still obtain even if they did not exist. Thus, even if mathematics is indispensable, fictionalism is reasonable: it is reasonable to hold that the nominalistic content of a theory obtains yet deny its Platonistic content.

According to Balaguer, empirical theories use mathematical-object talk only to construct descriptive frameworks in which to make assertions about the physical world. Empirical theories are not primarily concerned with mathematical facts or with mixed mathematical-physical facts. They are concerned only with the underlying nominalistic facts. Quantum mechanics is primarily concerned with the nature and behavior of quantum systems and their states. It may need to employ a background descriptive framework referring to vectors, operations, and Hilbert spaces to represent quantum phenomena, but it employs this framework because it provides us with a convenient way of saying what we want to say about the phenomena, not because we think they are in any way responsible for the phenomena. Since mere descriptive aids need not be true to be good, this account of mathematical applications dovetails nicely with fictionalism. It also dovetails nicely with FBP because, given FBP's plenitude, it's not surprising that for most physical situations, there will be an applicable mathematical theory that will represent it. However, it does not dovetail with TP, because it is a mysterious how a non-plenitudinous realm of causally inaccessible objects should be useful in scientific applications to the physical world.

3. Saving Mathematical Objectivity

Objectivity minimally requires that there is a difference between getting things right and getting them wrong. If fictionalism is correct, no mathematical assertion is (non-vacuously) true. If FBP is correct, all consistent mathematical assertions are true in the sense that they truly describe part of the mathematical realm. This "make-them-all-true-or-all-false" aspect of both FBP and fictionalism seems to fly in the face of mathematical practice, where some sentences are true and some are false. This does not look very propitious for theories that are supposed to provide a standard, face-value interpretation of what mathematicians say and do. How are we to square the promiscuity of FBP and fictionalism with the selectiveness of mathematical practice?

Balaguer proposes that there are numerous consistent theories (or stories) mathematicians could invent. Suppose ([39]) a mathematician develops a consistent theory of quasi-numbers, where the sequence of objects runs out after a googol and 3 is the smallest prime, and asserts that there's a greatest prime and 2 is not prime. FBP entails that this theory truly describes part of the mathematical realm. According to FBP, '2 is prime and there's no greatest prime' truly describes part of the mathematical realm (numbers), while '2 is not prime and there is a greatest prime' truly describes a different part of the mathematical realm (quasi-numbers). According to fictionalism, '2 is prime and there's no greatest prime' is true in one story, while '2 is not prime and there is a greatest prime' is true in another story. All parts of the FBP universe are ontologically on a par, coexisting independently of us. Similarly all mathematical stories enjoy equal opportunity. It is purely a sociological phenomenon that mathematicians become interested in, study, and develop some theories/stories rather than others. It is just a fact about us with no ontological significance that we prefer number theory to quasi-number theory. Call the story we prefer "the standard story" (if you are a fictionalist) and the part of the universe accurately described by the standard story "the standard model" (if you are an FBPist). The standard story is just the story provided by FCNN. When we assert '2 is prime' or say it is true, we are saying it is true in the standard story (according to the fictionalist) or in the standard model (according to the FBPist).

Non-uniqueness presents a complication. If FCNN lacks the resources to single out a unique model of the natural numbers, it will presumably also lack the resources to single out a unique future elaboration of our standard story. Just as lots of sequences of FBP objects differing only in ways we have never thought of can qualify as natural numbers, so lots of elaborations of FCNN differing only in ways we have never thought of will conform to our standard story. We ordinarily think of the truth conditions for a sentence like '2 is prime' as given by: '2 is prime' is true if and only if the referent of the singular term '2' satisfies the predicate 'x is prime'. This kind of treatment presupposes that numerals refer uniquely. If non-uniqueness is unavoidable, how do we handle the presupposition of uniqueness? Balaguer simply builds non-uniqueness into the truth conditions and explains away the presupposition of uniqueness. When we say '2 is prime' is true we are saying that it is true in the class of standard models (FBP) or in the class of elaborations of the standard story (fictionalism), where in each case the class is that compatible with everything contained in FCNN. For FBP, '2 is prime' is true if and only if the class of standard models is non-empty and '2 is prime' is true in all standard models. Put another way, '2 is prime' is true if and only if there's at least one object that satisfies all FCNN constraints for being 2 and any object that satisfies all FCNN constraints for being 2 also satisfies all FCNN constraints for being prime. For fictionalism, '2 is prime' is true if and only '2 is prime' is true in all elaborations of the standard story. The presupposition of uniqueness that we often make is a false but harmless convenience. It is convenient to think of mathematical reference as being unique just like non-mathematical reference; it is harmless because mathematicians are interested only in the structural features of the objects they study.

Thus, according to Balaguer, both FBP and fictionalism can save all the appearances of mathematical objectivity. Saying a sentence is objectively true (false) amounts to saying it is true (false) in all standard models (FBP) or true (false) in all elaborations of the standard story (fictionalism). Given the dependence of the notion of a standard model or story on the notion of a full conception of the objects, a sentence is true (false) if it (its negation) follows from our full conception. Furthermore, Balaguer argues, this way of looking at truth allows for a neglected but important possibility: that both a sentence and its negation are consistent with our full conception of the objects yet neither of them follows from that conception. Mathematicians treat any arithmetical sentence as either true or false (= true in all standard models or false in all standard models), because they are convinced that FCNN is categorical. The situation is more complicated with set theoretical sentences. We know, *e.g.*, that CH is independent of ZFC, but we do not know whether it is independent of our current full conception of sets. It may turn out that our current full conception, no matter how much we refine it with principles already implicit in it, is not strong enough to settle CH. In that case CH is now strongly open: nothing in our current full conception decides it; it is objectively open (neither true nor false). Or it may turn out that some future theoretical refinement of our current conception (by adding to ZFC new independent axioms that are implicit but currently unnoticed in our conception) will decide CH; in that case it is weakly open (objectively true or false but currently undecided). Balaguer argues convincingly that we want our philosophy to leave open the answer to the question, "Is CH undecidable or merely undecided?" This attitude fits well with current mathematical practice, where some set theorists argue that there's no fact of the matter about CH (it's objectively open) and others argue that it's currently undecided but could be decided down the road (it's weakly open). Mathematicians, not philosophers, should decide such questions.

4. FBP versus Fictionalism: No Fact of the Matter

So far, we have simply helped ourselves to the assumptions of FBP and Fictionalism, exploring how Balaguer develops them and how they respond to certain problems. But how does he support them? In each case by an inference-to-the-best-explanation style of argument. Balaguer argues for FBP and for fictionalism on two fronts. First, FBP is the only version of Platonism that solves the Benacerraf problems, and fictionalism is the only version of anti-Platonism that provides a face-value interpretation for mathematics and can save the appearances of objectivity. Second, in accounting for mathematical practice, each has independent advantages over traditional versions of Platonism and anti-Platonism. Take the issue of undecidable propositions, for example. Full-blooded, but not traditional, Platonists cannot account for strongly open questions. For traditional Platonists all open questions must have correct answers determined in the mathematical universe. On the other side of the coin, traditional anti-Platonists cannot account for weakly open questions. For traditional anti-Platonists, if a question is not settled by our conventions, axioms, or rules, then it is not settled at all. According to Balaguer, this is a plus for FBP and fictionalism and a minus for traditional Platonism and anti-Platonism, because in practice mathematicians observe a distinction between strong and weak openness. Thus Balaguer concludes: FBP is the best version of Platonism (if one is going to be a Platonist, one should be a full-blooded Platonist); fictionalism is the best version of anti-Platonism (if one is going to be an anti-Platonist, one should be a fictionalist).

According to Balaguer, FBP and fictionalism are on a par. Each avoids the problem of access – FBP by guaranteeing no-contact knowledge of abstract mathematical objects, fictionalism by denying that any such objects exist. Each accepts non-uniqueness – there are many FBP models/standard stories that answer to our full conception underlying a given theory. Each preserves a standard semantics for mathematese and the appearances of mathematical objectivity – mathematical theories are objectively correct if they are true (true in all standard FBP-models or true in all refinements of the standard story). For both FBP and fictionalism, mathematical knowledge is just logical knowledge in the final analysis. We get mathematical knowledge from the logical knowledge that certain sentences are logical consequences of theories or stories we recognize as logically consistent. The relevant notions of consistency and consequence must be primitive intuitively understood notions. Given that the project requires us to piggyback from non-mathematical knowledge of consistency and consequence to mathematical knowledge, Balaguer cannot employ standard mathematical construals of these notions in terms of syntactic or semantic consistency/consequence.

In the final section of the book, Balaguer argues that (a) FBP and Fictionalism are on a par *qua* philosophies of mathematics, (b) there is no fact of the matter as to whether abstract objects exist. By (a), each is as correct as a philosophical account of mathematics can be;

by (b) neither can be correct. The argument for (a) is this. The only significant difference between FBP and fictionalism is that the former asserts what the latter denies: the existence of abstract objects. Given that mathematical objects, if they exist, are non-causal, we could never settle the FBP/Fictionalism issue directly on the basis of information received from the objects. But, neither can we settle the question indirectly on the basis of the consequences of the two theses. Only mathematical practice – what mathematicians say and do – can be the final arbiter with respect to any comparison of consequences, and FBP and fictionalism have the very same set of consequences for mathematical practice. FBP maintains that all consistent purely mathematical theories truly describe part of the mathematical realm; fictionalism maintains that no consistent purely mathematical theory truly describes anything. So, on either doctrine, what makes any mathematical theory "good", or "objectively true" is a function of its fitting well with our conceptions, interests, and aesthetic and pragmatic valuations. As a result, each doctrine can give substantially the same account of strongly and weakly open questions, of mathematical knowledge (as following logically from consistent stories), of mathematical applications (in terms of useful descriptive frameworks), of the thinness of mathematical reference, of the relative contributions of invention and discovery of mathematical truths, *etc.* It looks like, Balaguer argues, nothing could settle the dispute between FBP and fictionalism.

Moreover, (b) nothing could settle the dispute because there is no factual basis for the dispute. Consider the English sentence:

(*) There exist abstract objects that do not exist in spacetime.

Given two possible situations that are identical except that (*) is true in one and false in the other. Balaguer argues that we have no idea that distinguishes them. If we have no idea of what a world would have to be like in order for (*) to be true or false, then our usage does not determine possible world truth conditions for (*). In turn, if our usage does not determine truth conditions for (*), nothing does. There is no fact of the matter as to whether (*) is true. It follows that both FBP and fictionalism are incorrect in their metaphysical claims and equally correct in their vision of mathematical practice.

5. Critical Remarks

In this final section I raise some questions about (1) fictionalism, (2) full-blooded Platonism, and I suggest (3) an alternative account – thin-blooded Platonism – that promises to preserve the positive features of

Balaguer's positive findings, avoid its problematic features, and explain why there is no fact of the matter whether fictionalism or Platonism is correct.

5.1. **Not Fictionalism.** There is room for doubt whether Balaguer's arguments successfully undermine either indispensability or the link to mathematical realism. I begin with Balaguer's proposals for nominalizing quantum mechanics. Many nominalists, I think, would baulk at accepting primitive propensities because of their modal entanglements. If so, they will have to be mimicked by nominalistic notions. But if events that are generally unrealized are not nominalistically acceptable, it's hard to see how the mimicking comparative relations Balaguer proposes will be nominalistically acceptable. The relations used by Field were defined on points and regions, and to the extent that a case could be made for the nominalistic acceptability of those objects, the construction works. But the betweenness relations proposed by Balaguer are defined to hold between an actual quantum system (like a state-Ψ electron) and other quantum systems (state-Ψ_1 and state-Ψ_2 electrons). It is difficult to see how these other quantum systems can be actual concrete systems. There must be enough of them (at least continuum many), for example, to enable us to define all the possible probability- and observable-values needed for completeness. So, the resulting theory will have the same formal power (and arguably the same existential commitments) as the ordinary theory. If they are not actual, then they are in the same position as events or propositions – they are abstract. We seem to have switched from a theory that posits an infinite number of abstract unrealized events associated with the state of a particular quantum system to a theory that posits an infinite number of unrealized quantum systems associated with the state of the particular quantum system. We may only have substituted one set of abstracta for another.

The easy strategy for undermining the indispensability argument rests on denying that indispensability entails truth. Any mathematized scientific theory has a nominalistic content that captures its complete picture of the physical world and is completely independent of its Platonistic content, so we can accept that its nominalistic content obtains and deny its Platonistic content. I am not convinced. First, assuming – as our best evidence currently indicates – the mathematics is ineliminable, we have no specification of what it is we are accepting. That there obtain purely physical facts of the kind that would have to obtain in order for theory T to be true seems far too weak and non-committal as a notion of theory acceptance.

Second, mathematics provides a descriptive framework for representing empirical facts, but it also provides an inferential framework for empirical reasoning. A major virtue of mathematics is its ability to characterize and support long chains of precise, detailed reasoning and calculations that guide our practical and scientific inferences. If the mathematical premises we employ are false, we have no justification for supposing that our reasoning is sound. And if we have no justification for supposing our reasoning sound, we have no reliable guidance in our practical and scientific inferences. For example, solutions to numerous important physical problems require the determination of a function satisfying a differential equation. The theory of differential equations provides important information about these solutions. Sometimes (*e.g.*, if the differential equation is linear) the existence of a solution for initial value problems can be established by directly solving the equations. In the general case, where direct methods fail, the existence of a solution must be established indirectly, generally by constructing a sequence of functions that converges to a limit function that satisfies the initial value problem. Moreover, the solution to an initial value problem very often cannot be evaluated by analytic methods, and scientists must rely on discrete variable or finite element numerical methods to approximate the solution. Mathematical analysis of local and accumulated errors arising from different methods of approximation provides further useful information governing the choice of approximation method, the step size of the element, and the number of elements needed for the approximation to reach a desired level of precision. Both the analytical proofs and the numerical approximation methods do important work. Proofs of the existence and uniqueness of solutions (a) support the practice of searching for them, (b) unearth the initial and boundary value conditions that must obtain for the solution's existence, (c) provide us with other valuable qualitative information about the solution (*e.g.*, whether it varies continuously or jumps with small changes in initial conditions). The numerical methods often provide our only way of extracting an actual solution. Mathematical physicists rely on the theories presupposed in proving the existence of the solutions and approximating them. It is difficult to see how they could do this while adding the fictionalist disclaimer, "But, you know, I don't believe any of the mathematics I'm using". It is difficult to see how a fictionalist pursuing the easy strategy can account for the soundness of mathematical reasoning in mathematical physics and elsewhere in the sciences.

Balaguer discusses this problem in a footnote (fn 13, pp. 201-2). Take any sound scientific argument with some mathematical premises, $\{P_1, \ldots, P_n\} \models C$. Since the nominalistic content of C, $NC(C)$, is part of C, it will follow that $\{P_1, \ldots, P_n\} \models NC(C)$. Moreover, since the Platonistic content of $\{P_1, \ldots, P_n\}$ is causally irrelevant to $NC(C)$, there is nothing that is in $\{P_1, \ldots, P_n\}$ and not in $\{NC(P_1), \ldots, NC(P_n)\}$ that is relevant to the truth of $NC(C)$. It follows that $\{NC(P_1), \ldots, NC(P_n)\} \models NC(C)$, given $\{P_1, \ldots, P_n\} \models C$. More briefly, any mixed mathematical-empirical sound inference is underwritten by a nominalistic sound inference, given the causal isolation of mathematical objects. According to Balaguer, that suffices to explain the usefulness of mathematics in supporting inferences to the physical world.

Perhaps it does, but it sidesteps the issue. First, let us remind ourselves what needs to be accounted for. We come to believe C – a physical initial value problem has a solution or the approximate solution is such-and-such – by a proof or calculation from premises $\{P_1, \ldots, P_n\}$ that include mathematical statements. What we are asking the fictionalist to explain compatibly with his fictionalist stance is how we can have justification for believing C without believing both $\{P_1, \ldots, P_n\}$ and $\{P_1, \ldots, P_n\} \models C$. Even if true, it is beside the point to learn that the premises and conclusion each have <u>some</u> nominalistic contents that we are not able to express but are such that if the nominalistic content of the premises obtains then the nominalistic content of the conclusion also obtains. It is beside the point because our reasons for believing $\{NC(P_1), \ldots, NC(P_n)\} \models NC(C)$ and $\{NC(P_1), \ldots, NC(P_n)\}$ derive, respectively, from our reasons for believing $\{P_1, \ldots, P_n\} \models C$ and $\{P_1, \ldots, P_n\}$. But these latter reasons will presuppose standard mathematics and will not be available to the fictionalist.

Neither the tough nor the easy route toward undermining our need to take mathematical theories as true is promising. Mathematical theories are unlike fictions in several respects, one of which is central to indispensability, as I am understanding it. Unlike fictions, when we treat mathematical theories as true, we take them as really true in the sense that we rely on what they say to guide us in serious epistemic and practical activities. Mathematical fictionalism is an inadequate philosophy of mathematics because it glosses over this important difference between mathematical and fictional discourse.

The fact that we have little choice but to treat mathematical theories as true need not in itself commit us to Platonism. Realist anti-Platonist programs often reinterpret mathematical discourse in a background that makes the appropriate sentences true yet avoids the problematic commitment of mathematical discourse to abstract objects. The strategy behind these reinterpretations is to translate mathematical sentences into sentences that involve no references to abstract objects, thereby reaping the advantages of Platonism without taking on its problems. Balaguer correctly criticizes several such programs (conventionalism, *etc.*) on grounds that they do not secure the right kind of truth conditions for mathematical sentences. In order to secure truth conditions for mathematical discourse that make standard mathematical assertions non-vacuously and objectively true, any successful reinterpretation will likely have to appeal to some modal notions. Although Balaguer does not discuss these modal reinterpretation programs, there are problems with them, as [38, chap. 7] and [37, chap. 4] convincingly show. Take the standard sentence that asserts the existence of a natural number structure. Hellman's ([17]) modal structuralist reinterpretation asserts that it is logically possible for there to be such a structure. Chihara's ([10]) constructibility reinterpretation asserts that an open formula satisfying a condition of a certain type is constructible. But, given the nature of the project of reinterpretation, the modalities appealed to cannot be understood in the standard way in terms of the existence of models with domains of abstract objects. And it is not clear that we have a better way of understanding them. Reinterpretation strategies often appeal to an unanalyzed, primitive notion of possibility or consistency and support this appeal by claiming that we employ such a pre-theoretic notion in our everyday practices. Undeniably we do have pre-theoretic notions of possibility and consistency. However, it is disputable that these notions are sufficiently refined to secure the possibilities the reinterpretationist needs to play the role of mathematical existence claims. And it is disputable whether we have any sufficiently refined cognitive purchase on those notions absent the mediation of standard mathematics. The problem with these reinterpretationist strategies is that to the extent that they are successful in recovering standard mathematics, their theories will be equivalent to standard theories in a sufficiently strong way that they will also inherit problems that are as intractable as are the traditional problems of Platonism. Both fictionalism and modal reinterpretationism face a common problem. If we accept a mathematical theory of a certain formal strength or of a certain kind of structure, we cannot get rid of its standard ontology simply by rewriting it in a new syntactic guise ([42]). So, we seem to be stuck with Platonism.

5.2. **Not Full-Blooded Platonism.** Balaguer's full account combines three theses that appear to be independent of each other. First, there's the thesis of FBP proper: every consistent purely mathematical theory truly describes part of the plenitudinous mathematical realm. Second, there's the thesis of thin-aboutness: what our full conception of a mathematical domain says about the objects in that domain is about those objects, if they exist. Third, there's the thesis of full conceptions: our full conception of a mathematical domain includes not only the axioms that characterize it but also the body of truths that are commonly accepted to hold of it. These theses work in tandem to provide a Platonistic account of truth conditions for mathematical sentences that both preserves a face-value, uniform semantic treatment and avoids the problem of access. But they are distinct theses that appear to be independent of each other and to perform separate tasks.

FBP proper says that every consistent purely mathematical theory truly describes part of the mathematical realm. This thesis has liberal and strict readings that depend on how we answer two questions that it naturally invites. First, what is the class of consistent purely mathematical theories? If we think of mathematical theories as developed in formal languages with a clearly specified mathematical vocabulary, then it is easy to identify the purely mathematical theories by their syntax. But purely mathematical theories do not wear their syntax on their sleeves. They are generally presented informally at least in part, and we cannot identify their syntax without looking at the language employed by mathematical practitioners. On a liberal reading, a theory will count as purely mathematical merely in virtue of its being expressed in a language that uses only predicates that are orthographically similar to predicates employed by mathematicians. On this reading, semantic features of standard mathematical usage play no rôle in determining the class of purely mathematical theories. It is not clear that this liberal reading is plausible. A sufficiently large deviation from a predicate's or theory's standard usage may produce a more drastic result than a mere change in subject matter or interpretation; it may entirely discount its claim to be a mathematical predicate or theory. Suppose a mathematician develops a consistent theory of the quasi-numbers mentioned earlier (no number greater than a googol, 2 not prime) and asserts that there's a greatest prime. I suspect that mathematicians might not include this theory among the purely mathematical theories. The problem is not that the theorist has a false belief that there's a greatest prime number. Rather, the problem is that the theorist is diverging so radically from existing usage of the predicates 'x is prime', 'x is a number', *etc.* and is misusing language to such an extent that practicing mathematicians are likely to refuse to count the proposed theory as mathematical.

According to Balaguer, mathematicians are free to characterize any collection of objects they like. As long as the new theory is consistent, the quasi-number theorist cannot be criticized on grounds that the objects do not exist. FBP guarantees that the objects exist and that the novel theory truly describes some part of the mathematical realm. Yes, mathematicians are free to characterize new collections of objects; without creative freedom, mathematics would have never progressed. But their freedom is not as unbridled as Balaguer suggests. Sometimes a new theory is introduced to organize a pre-existing practice or conception. In such cases, the theory had better deliver the organization it promises. Even brand new theories are rarely introduced in a vacuum. Typically, their introduction is constrained to fit the historically developing practice and past usage. Apart from questions about their consistency, questions that are internal to mathematics can be raised about them. Apart from consistency constraints, other mathematical constraints generally apply. New theories need to be not only consistent internally; generally they also need to preserve a certain amount of prior practice, to be fruitful in accomplishing the goals set for them, to illuminate established theories, *etc.* ([21]). The history of mathematics contains ample evidence of the operation of these constraints. The development of generalized notions of infinite summability had to preserve Cauchy-summability (for convergent series), satisfy fundamental operations appropriate to numbers, and agree when alternative methods assigned a sum to the same series. The development of complex numbers had to satisfy operations appropriate to numbers and provide solutions for all polynomial equations of arbitrary degree ([23]). These constraints should not be counted as mere expressions of psychological or sociological preferences, as Balaguer sometimes seems to suggest. They appear to be constraints of a mathematical character.

On stricter readings, which theories count as purely mathematical will be a function of standards (over and above consistency) employed by mathematicians. The problem now is that it is not clear that stricter readings will do for Balaguer's purposes. Take the class of all consistent theories that appear to be purely mathematical. Now cut down the class to include only those that mathematicians are willing to count as purely mathematical. If that reduced class includes only theories that characterize standard mathematical objects (the kinds of domains and structures that pass mathematical muster and satisfy semantic constraints imposed by standard usage), then FBP threatens to collapse into traditional non-plenitudinous Platonism: every consistent purely mathematical theory that comports with the standards of correctness of mathematical practice truly describes part of the mathematical realm.

Second, what are we attributing to the consistent purely mathematical theories when we say they truly describe part of the mathematical realm? If "truly describes part of the mathematical realm" merely means "has a model in the mathematical realm", few would disagree: if a theory is consistent we expect it to have a model. Quasi-number theory is consistent; it has a finite model in an initial segment of the natural numbers. This very liberal reading cannot be what Balaguer intends, since it is as available to traditional as it is to full-blooded Platonists. On the other hand, strict readings will again not suffice for Balaguer's purposes. If true description of part of the mathematical realm requires (in addition to consistency) comporting with the standards of correctness of mathematical practice, FBP will again threaten to collapse into traditional non-plenitudinous Platonism. Balaguer makes it clear that he intends neither the very liberal nor the strict readings. "Truly describes part of the mathematical realm" means "is true in a language that interprets the theory to be about the objects it intuitively is about" (p. 60). So, quasi-number theory is true in quasi-English where the extension of 'x is a number' is a finite set and the extension of 'x is prime' does not include 2, *etc.* This will support plenitudinous Platonism. However, it is now radically divorced from mathematical practice. Mathematical practice is transacted in languages where the appropriate truth predicates are 'true-in-English', 'true-in-French', 'true-in-German', 'true-in-Chinese', 'true-in-Japanese', *etc.* On the current reading, FBP entails that quasi-number theory (expressed in quasi-English) is true in quasi-English. Quasi-number theory truly describes quasi-numbers, quasi-primes, *etc.* But this bears no relation to what mathematicians who are English speakers say, think, or do. If white objects were called 'green' in quasi-English, then 'Snow is green' would be true in quasi-English, but this tells us nothing about the world described in English and snow would still be white. Similarly, if quasi-primes were called 'prime' in quasi-English, then 'There's a greatest prime' would be true in quasi-English. but this would tell us nothing about mathematical reality as described in English and there would still be no greatest prime.

The upshot is that readings of FBP proper that support a plenitudinous mathematical universe fail to connect with mathematical practice, and readings that connect with mathematical practice do not support a plenitudinous universe. Indeed, the only work FBP proper really does is to resolve the problem of access. When a traditional no-contact Platonist constructs a consistent theory or description, even if we grant him that mathematical objects exist, he cannot account for how he knows that one of them satisfies that theory or description. He is like our artist trying to paint a realistic representation of an object he has never seen and has no information about. FBP resolves the problem by ensuring that "any consistent representation of a mathematical object that someone could construct will be an *accurate* representation of an actually existing mathematical object" (p. 43). So, what FBP does is provide a metaphysical postulate guaranteeing the existence of the intended objects: if FBP is true, every consistent theory will be true of its intended objects. The notions of thin-aboutness and full conception provide only a conditional guarantee: a consistent theory is true of its intended objects, if those objects actually exist. FBP asserts their existence. But notice what a weak guarantee it provides. The only argument for FBP itself, a style of inference to the best Platonist theory, will be undermined to the extent we can advocate an equally good or better theory (like TBP below).

FBP proper only provides a metaphysical existence postulate. What is plausible in Balaguer's package and what does all the work of securing objectivity, truth, and agreement with mathematical practice is not FBP proper. What is plausible and does the work is what he sees as common to both FBP and fictionalism – the combination of the theses of thin-aboutness and full conceptions: what our full conception of a mathematical domain says about the objects in that domain is about those objects, if they exist. Since these theses are independent of, and severable from, FBP proper, they can be used to support an alternative account of mathematical reality.

5.3. **Thin-Blooded Platonism.** Perhaps the most striking feature of Balaguer's overall argument is the unstable position he arrives at. We are told that FBP is the best version of Platonism (and, I think, of mathematical realism), yet there is no fact of the matter whether it is true. One wonders how we were led to this position. And one suspects that something went wrong rather early in the argument. According to Platonism, mathematical discourse has truth conditions only if its face-value interpretation in terms of Platonistic objects is the correct interpretation. This assumption has three parts: (P1) the interpretation is face-value; (P2) it appeals to abstract objects; (P3) it appeals to objects that exist independently of us. The conjunction of these assumptions is responsible for the problem of access, and proposed solutions to the problem typically consist in denying one or more of them. But instead of denying them, perhaps a better idea would be to see what they come to. In particular, what does (P3), the independent existence assumption amount to?

In discussing the failures of fictionalism, we saw that we have little choice but to endorse mathematical practice by accepting its deliverances to guide our practical and scientific reasoning. The practice is central to our cognitive and practical activities, but we can temporarily step back from it to see what its features are. Endorsing the practice (in the sense of transacting serious business within it) involves asserting some statements ('there is no greatest prime') and denying others ('there is a greatest prime'). It involves having a conception of an objective world that is distinct from what we say about it and accepting that there is a conceptual gap between our ability to prove statements and their holding true. This gap between word and object, between language and world, is marked by our applying semantic concepts to the language. It involves giving the discourse in which the practice is transacted a face-value interpretation in terms of an apparatus of reference, predication, and quantification. Some of the sentences of this discourse – "There are prime numbers" – are correct by the standards of the practice; on the face-value interpretation, they are true and tell us that numbers and other mathematical objects exist. Other sentences of the discourse – "There is a greatest prime" – are incorrect by the standards of the practice; on the face-value interpretation, they are false. So, endorsing the practice commits one to (P1).

Reflection on the size of the universe presupposed in (P1) readily leads to commitment to (P2). Since there are not enough concrete objects to serve the needs of the conception, since concrete objects are corruptible but mathematical objects are not, we recognize that the objects presupposed by the practice are abstract objects. Such a move requires no philosophical interpretation external to the practice itself. The internal standards of the practice guarantee that 2 is prime. So they guarantee the existence of prime numbers. Since we recognize that any prime number (indeed any mathematical object) is an abstract object, we conclude that abstract objects exist by simple existential generalization and universal instantiation.

Next, we consider the commitment to a conceptual gap between our ability to prove statements and their holding true, the difference between our thinking we have a proof of a statement and our having a proof of it. The possibility of everyone being wrong for a time leads us to try to characterize the gap between truth and proof. We are led to (P3): the objects' existence and nature are mind- and language-independent. We search for ways to give content to this idea and picture the objects as denizens of Platonic heaven just like concrete objects are denizens of the physical world. This is the step at which trouble begins. Until we began to consider (P3), we did not have to engage in any philosophical interpretation of the practice. Everything produced by our exploration of the practice's commitments followed by applying the internal standards of the practice. But now we are in trouble, because the problem of access looms immediately. The mathematician is like an artist attempting to paint a realistic portrait of an object he has never seen, lacking any template, photograph, or sketch to guide him. Once we start along this path, we are pushed to try to restore the artist's contact with the world he depicts. FBP is one such attempt that ends with the realization that there is no fact of the matter. The picture we began with has no real content that we can grasp clearly.

However, it is not clear why a thin-blooded version of Platonism could not do the same job without taking the step to independence that causes the trouble in the first place. [The name "Thin-Blooded Platonism" is my own, but similar or related views can be found in [2], [6], [14], and [39, 40].] Thin-blooded Platonism (TBP) accepts (P1) and (P2): there is no compelling reason to deny a mathematical theory its face-value interpretation in terms of abstract objects. These theses, we argued, follow from mathematical practice itself without external philosophical intervention. But TBP refuses to make the final move that involved understanding mind- and language-independence in terms of a model of concrete objects. This thesis provided the bad picture that led to our problems. Instead, TBP holds that mathematical objects have the kind of independent existence that one who endorses mathematical practice is committed to, the kind of existence that prime numbers have and that a greatest prime number lacks. Their independent existence "is constituted by the fact that there is a legitimate practice involving discourse with a certain structure, and that certain of the products of this discourse meet the standards of correctness that it sets" ([39]). TBP declines deeper philosophical interpretation.

TBP can co-opt the plausible core of Balaguer's account we get when we omit FBP proper. What our full conception of a mathematical domain says about the objects in that domain is about those objects. Mathematical existence and mathematical truth amount to no more than what exists and what is true, according to our full conception of the particular subject matter. In particular, TBP questions the intelligibility of understanding them in terms of an independently existing Platonic universe, full-blooded or traditional. For TBP, there is no problem of access. The mathematician is not like an artist trying to paint a realistic rendering of something he never saw and lacks a template for. His conception of the objects he intends to represent provides him with the template he needs. So long as it is consistent, and he follows the standards set by that conception, his existence assertions will be true. Moreover, the proponent of TBP can also claim the independent advantages claimed by Balaguer on behalf of FBP. For TBP, truth is just correctness according to the standards set by the conception and the practice. A sentence is true (false) if it is true (false) in all models that satisfy our full conception of the objects the theory characterizes; it is neither true nor false if it is true in some of those models and false in others. A question may be strongly open in the sense that neither our full conception nor any future refinement of it with principles implicit in it helps to decide it. It may be weakly open in the sense that some future theoretical refinement of our current conception (by adding to ZFC new axioms that are implicit but currently unnoticed in our conception) will decide it. The problem of accounting for how extrinsic, pragmatic justification works in mathematical practice can be handled in a natural way by TBP. When mathematicians add an axiom A to a theory solely on grounds that it will help solve an open question or better organize a body of results or smooth out the conception of T's objects, they are modifying or extending the conception of those objects. If their study of T + A receives endorsement from the community, their modification or extension will have been successful. The FBP gloss that T + A is guaranteed to describe part of the mathematical universe seems hardly relevant to the practice. What mathematicians will want to know is whether the modification or extension is plausible given the current conception associated with T not whether it truly describes some part of the mathematical realm. Finally, TBP reconciles mathematical objectivity with mathematicians' freedom in what appears to be a more natural way than the reconciliation provided by FBP. Mathematical invention is not completely free; novel theories must comport with standards of mathematical practice over and above consistency.

FBP proper, as we saw, simply adds a metaphysical existence postulate. TBP rejects the need for such a postulate. The need for an existence postulate becomes pressing only if we understand the mind- and language-independence of mathematical objects in terms of a model of concrete objects – only if we think of mathematical objects as residing in some Platonic heaven as concrete objects reside in the physical universe. But TBP refuses to understand the mind- and language-independence of mathematical objects in these terms, opting instead to understand them in the way they are understood in mathematical practice. In practice, we consider mathematical objects to be independent of particular minds and particular languages. The number 2 is distinct from any particular individual's ideas of it; arithmetical operations apply to the former but not the latter. In practice, we consider the arithmetical fact that there are infinitely many primes to be independent of any particular linguistic expression or proof of that fact whether presented in classical Greek or in modern English.

Moreover, TBP rejects the possibility of a metaphysical support of existence (whether grounded in FBP or another metaphysical thesis) that is likely to improve on the existence guarantees provided by mathematics itself. The proof that there are infinitely many primes is not helped by additional metaphysical premises, which are likely to be less trustworthy than the mathematical premises used in the standard proof. We cannot ground mathematical existence "in any domain or theory that is more secure than mathematics itself" ([38, p.135]).

TBP adopts a similar stance with respect to the problem of uniqueness. If the full conception of a domain has the formal or informal resources to single out a unique intended model for the theory, then uniqueness is secured mathematically without metaphysical help. For categorical theories, we have all the uniqueness we need. For non-categorical theories, uniqueness would require appeal to informal considerations drawn from our conception of the subject matter. Perhaps uniqueness arguments can be constructed in some cases; perhaps not in other cases. But in any case uniqueness is a mathematical problem. It is useful to compare this response with Balaguer's reaction to a common structuralist answer to the problem of non-uniqueness ([17, 29, 37, 38]). Granted, the structuralist says, our mathematical theories do not describe unique collections of objects, but they describe unique structures. The (second order) Peano axioms, for example, single out a unique structure that is shared by all ω-sequences. This structure is implicitly characterized by the Peano axioms; there is nothing more to being a natural number than being a position in that structure; there is nothing more to being 2 than being the 3rd position in that structure. Balaguer rejects these structuralist responses to the problem of non-uniqueness. First, the thesis that mathematical objects have no properties other than their relations to other objects in a structure, he points out, is central to such responses. If this thesis is false, then there could be two structures that were isomorphic to each other (sharing all the same positions interrelated in the same way), yet were not identical, since their corresponding positions might possess different non-structural properties. If this were so, then even a categorical theory, like second order arithmetic, would not describe a unique structure; it would describe a collection of isomorphic structures. Balaguer argues that the thesis is (a) false and (b) incoherent. It is false because mathematical objects (even *qua* positions in structures) have lots of non-structural properties; FCNN tells us that no number is red, a set, or identical with Julius Caesar, *etc.* It is incoherent because the property of having only structural properties is itself not a structural property. The upshot is that mathematical objects cannot have only structural properties. Moreover, the closely related thesis that mathematical objects are to be identified with positions in structures is also central to structuralism. But this numbers-are-positions thesis is no more built into FCNN than is the numbers-are-Zermelo-numbers thesis. Any FCNN-admissible progression has as much right to be identified with the natural numbers as have the positions (understood as objects) in the structure. So, non-uniqueness remains with us.

I doubt that the structuralist insight is either false or incoherent. Concrete objects seem to have the kind of nature where for any given property it is well defined whether the object has the property. Traditional and full-blooded Platonists, because they think of mathematical objects on the model of concrete objects, tend to think of mathematical objects as having the same kind of completeness. Thus, Balaguer's version of FCNN tells us that numbers have lots of non-structural properties, so that two isomorphic structures could each have what is required to realize the natural numbers yet be different because their elements might have different, non-structural properties that no one has ever thought of. Thin-blooded Platonists, however, think of mathematical objects as having no distinguishing properties other than the mathematical properties their construction and employment in reasoning bestows upon them. For TBP, only structural properties distinguish mathematical objects; only arithmetical properties – those that are bestowed upon numbers by arithmetical practice – distinguish numbers from each other and from other mathematical objects. As a result, thin-blooded Platonists will endorse a less robust version of FCNN. It takes little reflection on our conception of numbers to recognize that Julius Caesar's death did not impugn the existence of any number. But this has little to do specifically with numbers; it applies to any mathematical object, indeed to any abstract object. Our conception of mathematical objects entails that they are abstract. Consequently, no mathematical object, and *a fortiori* no number, is identical to Julius Caesar or is red. Any non-structural properties (being non-identical to Julius Caesar, being non-red) that any mathematical object might have, it would seem to share with all mathematical objects. For TBP, FCNN can (though it need not) accept that mathematical objects have those non-structural properties that all mathematical objects possess simply because they are abstract. This will not compromise a categorical theory's ability to characterize a unique structure, because the non-structural properties cannot distinguish mathematical objects in the way Balaguer envisions. And it is neither false nor incoherent to hold that only structural properties distinguish mathematical objects. This is less a defence of structuralism than of its fundamental insights: mathematical objects are radically incomplete with respect to distinguishing properties; the only distinguishing properties they have are the structural properties bestowed upon them by the conception of them underlying the mathematical practice; to the extent that they have any non-structural properties, these play no role in their identification.

It seems then that TBP does at least as well as, if not better than, FBP. All the important features of mathematical practice – reliability, correctness, objectivity, truth – are captured in FBP by the thin-aboutness of full conceptions – by the idea that our consistent mathematical theories truly characterize the objects intended in our full conception of those objects. TBP incorporates these notions in a natural fashion. FBP proper is neither necessary nor desirable from the viewpoint of TBP.

Moreover, TBP enables us to understand why there is no fact of the matter whether FBP or fictionalism is correct. TBP amounts to little more than the doctrine of the thin-aboutness of our full conception of a domain of mathematical objects – a doctrine that is shared by FBP and fictionalism alike. This is what secures all the important features of mathematical practice. In order to engage in mathematical and scientific reasoning, we have little choice but to talk as if numbers exist. In so talking, we are committed to their existence. All the existence of infinitely many primes amounts to is that this fact follows from our full conception of natural numbers. FBP adds "and they really exist"; fictionalism adds "but they don't really exist". From the point of view of mathematical practice, the former is trivially true and the latter trivially false. There is no external (metaphysical) point of view that we can seriously endorse, clearly grasp, or entertain without being led to unsolvable conundrums.

I close with a challenge TBP must face. Because TBP eschews independently existing objects in Platonic heaven, it is likely to be accused of being just a form of fictionalism. If the objects are just a product of a face-value interpretation of the assertoric discourse in which our practices are transacted, then no matter how constrained those practices are, the objects are our creations. We have a practice, in it we create normatively constrained stories about objects, but why should any of this lead us to accept that those objects exist? Because accepting the normatively constrained stories is all there is to accepting that the objects exist, according to TBP. Moreover, the fact that we create the objects supports their existence: they exist because we construct them, just as houses exist because we construct them. But then, the objector says, the objects are not independent of us in the right kind of way – *i.e.*, in the way needed to secure mathematical objectivity. Moreover, continues the objector, language- and mind-independence are intimately bound up with completeness in the case of physical objects. But if commitments to completeness should be avoided in the TBP understanding of mathematical objects, how is TBP going to secure objectivity and differentiate mathematical from fictional objects, if it admits that mathematical objects share an important characteristic – incompleteness – with fictional objects. For TBP, the answer must again lie in our different attitudes to the practices of writing fiction and doing mathematics. Writing fiction does not engage with our inferential and truth practices in the way that doing mathematics does. Typically the practice of fiction lacks any associated conception of getting it right that is uniformly agreed upon. The practice of mathematics has such a conception. Theories must be consistent; results must be proved; but most importantly the practice operates with a truth-like conception of a distinction between our words and the mathematical world they are taken to describe. Endorsing the practice includes having a conception of an objective world that is distinct from what we say about it, and accepting that there is a conceptual gap between our ability to prove statements and their holding true. Although there are no mathematical facts in Platonic heaven, there are mathematical facts – that there are infinitely many primes is one of them – that we think of as transcending our theories about them. Most mathematical sentences we think are either true or false. Even when we lack a proof in either case, we can imagine ourselves in an improved situation in which either our full conception of the domain or an improved elaboration of our full conception delivers a proof one way or the other. Clearly much more detailed work would be needed to provide an adequate presentation and defence of TBP. Nevertheless, TBP seems to me the most promising current approach to philosophy of mathematics[1].

References

[1] Ayer, A.J., *Language, Truth, and Logic*, New York: Dover, 1946.

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[2] Azzouni, J., *Metaphysical Myths, Mathematical Practice*, Cambridge: Cambridge University Press, 1994.

[3] Benacerraf, P., "What Numbers Could Not Be", *Philosophical Review* **74** (1965), pp.47–73.

[4] ———, "Mathematical Truth", *Journal of Philosophy* **70** (1973), pp.661–79.

[5] Benacerraf, P. and Putnam, H. (eds.), *Philosophy of Mathematics*, Cambridge: Cambridge University, 1983.

[6] Blackburn, S., *Essays in Quasi-Realism*, Oxford: Oxford University Press, 1993.

[7] Carnap, R., *The Logical Syntax of Language*, trans. by A. Smeaton, New York: Harcourt Brace, 1937.

[8] ———, "Empiricism, Semantics, and Ontology", 1956, in [9, pp.205–21].

[9] ———, *Meaning and Necessity*, Chicago: University of Chicago Press, 1956.

[10] Chihara, C., *Constructibility and Mathematical Existence*, Oxford: Oxford University Press, 1990.

[11] Curry, H.B., *Outlines of a Formalist Philosophy of Mathematics*, Amsterdam: North-Holland, 1951.

[12] Field, H., *Science Without Numbers*, Princeton: Princeton University Press, 1980.

[13] Gödel, K., "What is Cantor's Continuum Problem?", 1964, in [5, pp.470–85].

[14] Grandy, R., "Shadows of Remembered Ancestors: Mathematics as the Epitome of Story-telling", 1996, in [28, pp.167–89].

[15] E. Grosholz and H. Breger (eds.), *The Growth of Mathematical Knowledge*, Dordrecht: Kluwer, 2000.

[16] Hale, R., *Abstract Objects*, Oxford: Basil Blackwell, 1987.

[17] Hellman, G., *Mathematics Without Numbers*, Oxford: Clarendon Press, 1989.

[18] Hempel, C., "On the Nature of Mathematical Truth", *American Mathematical Monthly* **52** (1945), pp.543–56.

[19] Katz, J., *Language and Other Abstract Objects*, Totowa, New Jersey: Rowman and Littlefield, 1981.

[20] Kitcher, P., "The Plight of the Platonist", *Nous* **12** (1978), pp.119–36.

[21] ———, *The Nature of Mathematical Knowledge*, Oxford: Oxford University Press, 1984.

[22] Lewis, D., *On the Plurality of Worlds*, Oxford: Basil Blackwell, 1986.

[23] Liston, M., "Mathematical Progress: Ariadne's Thread", 2000, in [15, pp.257–68].

[24] Maddy, P., "Perception and Mathematical Intuition", *Philosophical Review* **89** (1980), pp.163–96.

[25] ———, *Realism in Mathematics*, Oxford: Oxford University Press, 1990.

[26] Malament, D., Review of Field's *Science Without Numbers*, *Journal of Philosophy* **79** (1982), pp.523–34.

[27] Mill, J.S., *A System of Logic*, London: Longmans, Green, and Co., 1843.

[28] Morton, A. and Stich, S., *Benacerraf and his Critics*, Oxford: Basil Blackwell, 1996.

[29] Parsons, C., "The Structuralist View of Mathematical Objects", *Synthese* **84** (1990), pp.303–46.

[30] Putnam, H., "Mathematics Without Foundations", *Journal of Philosophy* **64** (1967), pp.5–22.

[31] ———, "Philosophy of Logic", 1971, in [33, pp.323–57].

[32] ———, "What is Mathematical Truth?", 1975, in [33, pp.60–78].
[33] ———, *Mathematics, Matter and Method*, Cambridge: Cambridge University Press, 1979.
[34] Quine, W.V.O., "On What There Is", 1948, in [36, pp.1–19].
[35] ———, "Two Dogmas of Empriricism", 1951, in [36, pp.20–46].
[36] ———, *From a Logical Point of View*, New York: Harper and Row, 1961.
[37] Resnick, M., *Mathematics as a Science of Patterns*, Oxford: Oxford University Press, 1997.
[38] Shapiro, S., *Philosophy of Mathematics*, Oxford: Oxford University Press, 1997.
[39] Stalnaker, R., Critical Notice of Lewis's *On the Plurality of Worlds*, *Mind* **97** (1988), pp.117–28.
[40] ———, "On What Possible Worlds Could Not Be", 1996, in [28, pp.103–18].
[41] Steiner, M., *Mathematical Knowledge*, Ithaca, New York: Cornell University Press, 1975.
[42] Wilson, M., "The Double Standard in Ontology", *Philosophical Studies* **39** (1981), pp.409–27.
[43] Wright, C., *Frege's Conception of Numbers as Objects*, Aberdeen: Aberdeen University Press, 1983.

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