Modern Logic
Volume 8, Number 1/2 (January 1998–April 2000), pp. 28–46.

ALGEBRAIC EQUIVALENTS OF KUREPA'S HYPOTHESES

R. M. DIMITRIĆ

Abstract. Kurepa trees have proved to be a very useful concept with ever growing applications in diverse mathematical areas. We give a brief survey of equivalent statements in algebra, particularly in valuated vector spaces, abelian p-groups and non-abelian periodic groups. The survey is prefaced by an outline of the illustrious history of Kurepa's Hypothesis. An interesting aspect of the work in this area is the equivalence (via Kurepa's Hypotheses) of some statements in abelian group theory with statements in non-abelian group theory. This kind of relationship would be hard to establish, without Kurepa trees. The goal of the paper is to alert as well as familiarize the readers with this active research amalgam of set theory and algebra, but also to entice at least some to take part in the work.

1. Kurepa's trees and hypotheses

We first set the terminology in its modern form:

A strict partially ordered set $(T, \leq)$ is a *tree* if for every $x \in T$, the set $\{y \in T : y < x\}$ is well ordered in the induced ordering. The *height of* $x \in T$, denoted by $ht(x, T)$, is the ordinal that is order equivalent to the well ordered set $\{y \in T : y < x\}$. If α is an ordinal then $Lev_\alpha(T) = \{x \in T : h(x, T) = \alpha\}$ is the *α-th level* of T. *The height of* $(T, \leq)$, denoted $ht(T)$, is the least ordinal τ such that $Lev_\tau(T) = \emptyset$. A *branch of* T is a maximal linearly ordered subset of T; it is well ordered by the ordering of T. If b is a branch and $b \cap Lev_\alpha(T) \neq \emptyset$ then the intersection is a singleton and $b \cap Lev_\beta(T) \neq \emptyset$ for all $\beta < \alpha$; the *length of* b is the least ordinal λ such that $b \cap Lev_\lambda(T) = \emptyset$. Note that this ordinal is order equivalent to b. If the length of a branch b is λ we shall

1991 *Mathematics Subject Classification.* Primary: 03E65, 03E35, 04A20, 20K10, 20F24; Secondary: 20K40, 20K45, 20F50, 20D25.

Key words and phrases. Valuated vector space, Kurepa's Hypothesis, abelian p-group, C_{ω_1}-group, disco group, discy group, the Tor functor, balanced projective dimension, extraspecial groups, FC-groups, weak Kurepa Tree, classes $\mathcal{Z}_\kappa$ and $\mathcal{Y}_\kappa$, Easton forcing.

refer to b as a λ-*branch*. In this paper, κ will denote a regular cardinal and $\mathcal{B}$ will denote the set of all κ-branches.

A tree $(T, \leq)$ is a κ-*Kurepa tree* if:

(1) $ht(T) = \kappa$;
(2) for every $\alpha < \kappa$, $|Lev_\alpha(T)| < \kappa$;
(3) T has at least κ^+ cofinal (κ-)branches.

The κ-*Kurepa Hypothesis* (κ-KH) is the statement "there is a κ-Kurepa tree." When $\kappa = \omega_1$ we have Kurepa's Hypothesis (KH).

2. History of proofs of the consistency and independence of KH[1]

The outline in this section has the following goals: to provide a framework for more thorough historical investigations, to set additional terminology as well as state consistency and independence results, and to pose some questions of interest, whose answers are not at present known to the author.

[Kurepa, 1935] (his dissertation reprinted) is among the first to consider tree-like structures, called "ramification systems" (*ensemble ramifié*, or *tableau ramifié*). In [Kurepa, 1942, p143], a discussion of Souslin's problem led the author to consideration of trees T of height ω_1, with at most countable levels (and nodes). It was stated (among other hypotheses) that he did not know what the cardinality of the set of maximal (linearly) ordered subsets of T of cardinality $\aleph_1$ may be, and it was proved that this cardinality is at least $\aleph_1$ (but it obviously cannot exceed $2^{\aleph_1}$). Unable to prove many of his conjectures, Kurepa in [Kurepa, 1935] speaks of working in the field of "many undecidable postulates and principles", just as in [Kurepa, 1952] there is a mention of " . . . immensité inconcevable d'hypothèses montrant des possibilités logiques incroyables au sein du transfini".

KH (or KC – Kurepa's Conjecture as originally known) was initially (c. 1943) the statement about Kurepa families: Let κ be a (regular) cardinal (hence, the initial ordinal of that cardinality); a κ-*Kurepa family* is a family $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ of cardinality at least κ^+ such that, for each $\alpha < \kappa$, $|\{\alpha \cap Y : Y \in \mathcal{F}\}| < \kappa$ [Vaught, 1965, p85]. By a *Kurepa tree* or a *Kurepa family* we shall mean an ω_1-Kurepa tree or ω_1-Kurepa family respectively. The existence of a κ-Kurepa family is equivalent to the existence of a κ-Kurepa tree (this was shown by Kurepa for $\kappa = \omega_1$; see also [Ricabarra, 1958, spec. p. 344], where a great deal of Kurepa's pre 1955 work had been explored).

[1] Dj. Kurepa died in 1993. A brief sketch of Kurepa's life and work may be found in [Dimitrić, 1993].

[Kurepa, 1935, 1951] and [Ricabarra, 1958] (see also [denjoy, 1946, vol. III]) are the best sources on the theory of trees, before the emergence of the method of forcing, Kurepa's monograph not having lost its vitality even nowadays. Kurepa initially conjectured that the negation of KH was valid (but later changed his mind, not being perhaps fully aware of what was to come).

New light was shed on these problems with the emergence of powerful forcing methods in the 1960's, as set out in [Cohen, 1963/4, 1966] and their first applications as in [Feferman, 1965]. While looking into the Löwenheim-Skolem theorems for pairs of cardinals, [Vaught, 1963, p. 309] realized that KH implies that there exists an (ω_2, ω_1) algebraic structure of at most denumerable type without (ω_1, ω_0) elementary substructures. Moreover κ^+-KH is stated [Vaught, 1965, p. 85] to be equivalent to the existence of a sentence in a first order language (with equality) that has a relational structure model (A, U, R_β) of type (κ^{++}, κ) (i.e. $|A| = \kappa^{++}, U \subseteq A, |U| = \kappa$). If $\kappa \geq \lambda$, the author (*ibid.*) was interested in the following gap-two, two-cardinal conjecture (under the GCH): For any countable set Σ of a first order language sentences, if Σ has a model of type (κ^{++}, κ), then Σ has a model of type (λ^{++}, λ). The affirmation of this conjecture would imply that κ^+-KH $\rightarrow \lambda^+$- KH.

Vaught's observations from 1963 were strengthened in Rowbottom's 1964 Dissertation (see addendum in [Rowbottom, 1971], pp. 41–43), where it was claimed that (if I represents the statement "there exists a strongly inaccessible cardinal", or the cardinal itself, when no confusion arises) consistency of ZF + I implies consistency of ZFC + CH + KH. The author stated that A. Lévy had discovered independently the same results about three months earlier using forcing methods. No detailed proofs or references were given. These observations can also be described as follows: KH follows from the assumptions that ω_1 is inaccessible in L and that no ordinal between ω_1 and ω_2 is a cardinal in L, for in this case, the family $\mathcal{F} = \{S \subseteq \omega_1 : S \in \mathrm{L}\}$ is a Kurepa family. This is also the idea [Bukovský, 1966] uses to show that if ω_α is a (regular) strongly inaccessible cardinal and there are no cardinals between ω_α and $\omega_{\alpha+1}$, then ω_α-KH holds in $\mathcal{P}(\omega_\alpha) \cap \mathrm{L}$.

It turned out that the assumption of the existence of a strongly inaccessible cardinal was not needed as shown by [Stewart, 1966], (a student of Rowbottom), who proves in his Masters Thesis (taking advantage of the forcing ideas developed by [Lévy, 1965]) that consistency of ZF implies consistency of ZFC + CH + KH. Solovay (unpublished; see also [Přikrý, 1968]) shows that ZF + V = L $\vdash$ KH, (after [Jensen's, 1968] construction of a Souslin tree in L — accomplished by Drake too); in other words, KH is true in L and if $X \subseteq \omega_1$, then KH is true in L$[X]$ (see a sketch of proofs in [Jech, 1971]). Jensen has abstracted Solovay's proof into $\diamondsuit^+ \vdash$ KH (see e.g. [Kunen, 1980]; note that ZF + V = L $\longrightarrow \diamondsuit^+$). Jensen also introduced ineffable cardinals in order to generalize Kurepa trees: If κ is ineffable, then κ-KH does not hold. KH (and $\neg$SH) was shown to be a consequence of the following statement satisfied in L [Silver, 1973, p. 164]: There exists a function Q on ω_1, which assigns to each ordinal $\alpha < \omega_1$ a countable collection $Q(\alpha)$ of subsets of α, such that the following holds: if X is any subset of ω_1, there is a club C of ω_1, such that $X \cap \alpha$ and $C \cap \alpha$ are both members of $Q(\alpha)$, for all $\alpha \in C$. Thus were the consistency questions regarding KH settled.

The question of independence of KH is also of interest, thus models satisfying the negation of Kurepa's Hypothesis are needed. [Silver, 1971a] was able to construct a model of ZF + GCH+$\neg$KH+ω_2-KH as follows: Start with a countable transitive model M of ZF + V = L + I (V = L may be replaced by GCH); the desired model $M[G]$ is obtained from M by adjoining a generic sequence $\{f_\alpha : \alpha < \mathrm{I}\}$ of collapsing functions $f_\alpha : \omega_1^M \xrightarrow{\text{onto}} \omega_{\underline{\alpha}}^M$, so that $M[G] = \{f_\alpha : \alpha < \mathrm{I}\}$. One of the consequences is the independence of the above gap-two, two-cardinal conjecture (modulo I and GCH). Recast in the relative consistency language: ZFC + I is equiconsistent with ZFC + $\neg$KH and with ZFC + GCH + $\neg$KH. The negation of KH implies that ω_2 is inaccessible in L; thus, unlike the case of consistency of KH, for the consistency of $\neg$KH, inaccessible cardinals are required, as was pointed out by Solovay. [Silver, 1971a] claims that even a more general result holds: If κ is a cardinal in some countable transitive model M of ZFC and less than an inaccessible $\underline{I}$ in M, then a Cohen extension N of M can be found satisfying: $\underline{\kappa}^N = \underline{\kappa}^M, \underline{\kappa}^{+N} = \underline{\kappa}^{+M}$, and $\neg\, \underline{\kappa}^+$-KH, $\underline{\kappa}^{++}$-KH hold in N.

Chang's conjecture is the following statement: Any relational structure $(\omega_2, \omega_1, R_\beta)$ has an elementary substructure $(C, C \cap \omega_1, {}^2C \cap R_\beta)$, where $|C| = \aleph_1$ and $C \cap \omega_1$ is countable. Chang's conjecture implies the negation of KH, for it implies the negation of the following weaker statement: There is a cardinality $\aleph_2$ family $\mathcal{F}$ of functions from ω_1 to ω_0 such that, for every $f, g \in \mathcal{F}$ with $f \neq g$, there is an $\alpha \in \omega_1$, such that, $f(\beta) \neq f(\alpha)$ for all $\beta > \alpha$ [Silver, 1971b]. For related considerations see [Chang, 1972].

Note that if ZF is consistent, then so is GCH + $\diamondsuit$ + $\neg$KH.

Some still open questions?

Q1: Can Bukovský's proof be improved by not assuming I? The affirmative answer would give consistency of κ-KH, for $\kappa > \omega_1$.

Q2: In Silver's 1971a construction of the model, how does one get rid of the assumption of GCH?

Q3:[2] For a (regular) cardinal κ, can one get "decent" models of ZFC (without assuming I or GCH) that have κ-Kurepa trees with a prescribed number $\aleph$ of κ-branches ($\kappa^+ \leq \aleph \leq 2^\kappa$)? The emphasis is on the *prescribed* number of branches, thus κ-KH may be assumed. If GCH is assumed, then would Silver's above claim suffice? A note in [Jech, 1971] affirms the question for $\kappa = \omega_1$. [Jin, 1991] and [Jin and Shelah, 1993] may be related to this question.

3. Valuated vector spaces

We now want to transfer the discussion about trees into the algebraic language. We start with a tree $(T, \leq)$ and an arbitrary field (or a ring) F. At each level of the tree we may generate the direct sum $\bigoplus_{x \in Lev_\alpha} Fx$, and form the product

$$P = \prod_{\alpha < ht(T)} \bigoplus_{x \in Lev_\alpha} Fx.$$

This two-dimensional product (an F-vector space in our case) reflects the "ramified table" nature of a tree. We want to get even a closer translation by introducing valuation into P. First a few general words on valuation: A vector space V over a field F, is a *valuated vector space* with a (logarithmic) valuation $v : V \to \text{Ord} \cup \{\infty\}$, if the following axioms hold: $v(a) = \infty$ iff $a = 0$, $v(ta) = v(a)$ for all scalars $t \neq 0$, and $v(a + b) \geq \min\{v(a), v(b)\}$. By $V(\alpha)$ we mean the subspace $V(\alpha) = \{x \in V : v(x) \geq \alpha\}$. If λ is a limit ordinal then by the λ-*topology* on V we mean the linear topology having as a base for the neighborhoods of 0 the set $\{V(\alpha) : \alpha < \lambda\}$. All the topologies in this paper are of this kind. It is easy to see that if $a, b \in V$ with $v(a) \neq v(b)$ then $v(a + b) = \min\{v(a), v(b)\}$. If U and W are subspaces of V then by $V = U \breve{\oplus} W$ we mean the valuated direct sum, i.e., $V = U \oplus W$ and, for all $a \in U$ and $b \in W$, $v(a + b) = \min\{v(a), v(b)\}$. All valuated vector spaces in this paper are subject to some cardinality restrictions in relation to the cardinality of the ground field F; these conditions will always be ensured for countable F. For more details on valuated vector spaces see [Fuchs, 1975].

Given an abelian p-group A, its p-socle $A[p] = \{a \in A : pa = 0\}$ is a vector space over the finite field $Z(p)$ of the integer remainders mod p. The dimension $\dim_{Z(p)} A[p] = r_p(A)$ is called the *p-rank of*

[2] There is an answer to this question in the Addendum at the end of this paper.

A. Another dimension of extreme importance is the following p-rank: $f_\alpha(A) = \dim_{Z(p)}(p^\alpha A)[p]/(p^{\alpha+1}A)[p]$, called the *$\alpha$-th Ulm invariant of A*. $p^\beta A$ is defined inductively: $p^0 A = A$, $p^{\beta+1}A = p(p^\beta A)$, and for limit β, $p^\beta A = \cap_{\alpha<\beta} p^\alpha A$. The smallest ordinal λ such that $p^{\lambda+1}A = p^\lambda A$ is the *p-length of A*, denoted by $l_p(A) = l(A)$. A valuation (the *height valuation*) may be defined by the p-height of an element:

$$v(a) \;=\; h_p(a) \;=\; \begin{cases} \alpha, & \text{where } a \in p^\alpha A \setminus p^{\alpha+1}A; \\ \lambda = l(A), & \text{if } a \in p^\lambda A; \\ \infty, & \text{if } a = 0. \end{cases}$$

Thus, an α-neighborhood $A(\alpha)$ of zero consists of elements whose height is $\geq \alpha$, and we have $A(\alpha) = p^\alpha A$. Likewise the length of A is the smallest ordinal α for which $A(\alpha) = A(\alpha+1)$. A subgroup of A is *α-high*, if it is maximal with respect to trivial intersection with $A(\alpha)$.

Returning to the ramified product, we define a valuation $v : P \longrightarrow \mathrm{Ord} \cup \{\infty\}$ by $v(g) = \min\{\alpha \in \kappa : g(\alpha) \neq 0\}$. Each κ-branch of T may be seen as an element of P — its α-th coordinate is simply its unique intersection with the α-th level. If $b \in \mathcal{B}$ is a branch, then we may form its α-th section b_α to be the element of P coinciding with b on indexes $< \alpha$ and 0 from α and upward. If a (valuated) vector space V is defined to be generated by all the α sections ($\alpha < ht(T) = \kappa$) of all the κ-branches of T, then the completion $\widetilde{V}$ in the κ-topology is of cardinality at least $|\mathcal{B}|$; this is because, for every branch b, $\{b_i\}_{i<\kappa}$ is a Cauchy net in V, converging to b in the κ-topology of V. A stronger statement is in place if we concentrate on Kurepa trees:

Theorem 1. [Cutler and Dimitrić, 1993] *Let κ be an uncountable regular cardinal and $\aleph$ a cardinal greater than κ. Then there is a κ-Kurepa tree with at least $\aleph$ κ-branches if and only if for every field F of cardinality $< \kappa$, there exists a valuated F-vector space V with the following properties:*

(a) $|V| = \kappa$,
(b) $V(\kappa) = 0$,
(c) *for every* $i < \kappa$, $|V/V(i)| < \kappa$,
(d) *the completion* $\widetilde{V}$ *of* V *in the κ-topology has cardinality* $\geq \aleph$.

Remark: If in the theorem above $|\widetilde{V}| = \aleph$, then the constructed κ-Kurepa tree T has exactly $\aleph$ κ-branches. Apparently an answer to Q3 in section 2 would help us fix the cardinality of the completion, rather than have only an inequality.

The reverse implication, starting from a valuated vector space with the prescribed properties, uses the fact that the completion $\widetilde{V}$ may be viewed as the inverse limit $\varprojlim V/V(i)$, and the tree $T = \cup_{i<\kappa} V/V(i)$ is ordered via the inverse system maps.

Still, the last equivalence can be improved even further (alas only one implication benefits); to this end we need the following natural property: For a limit ordinal λ and an uncountable cardinal κ, we will say that the valuated vector space V has the (λ, κ)-*closure property* if it is Hausdorff in the λ-topology and if, for every subspace W of V of cardinality $< \kappa$, its closure $\overline{W}$ in the λ-topology is also of cardinality $< \kappa$. If V is Hausdorff in the λ-topology and is of cardinality $< \kappa$, then V has the (λ, κ)-closure property; also free valuated vector spaces have the (λ, κ)-closure property.

The following is a strengthened version of Theorem 1 (from the existence of a certain valuated vector space to the existence of a Kurepa family); it is proved by building up the family using (two-dimensional) κ-filtrations of V, and the technicalities of the proof can be found at the source.

Theorem 2. [Cutler and Dimitrić, 1993] *Let κ be an uncountable regular cardinal and $\aleph$ a cardinal greater than κ. Then there is a κ-Kurepa family of cardinality $\geq \aleph$ if and only if there exists a valuated vector space V of cardinality κ, over a field of cardinality $< \kappa$, with the following properties:*

(a) $V(\kappa) = 0$,
(b) *for every (limit) $i < \kappa$, $V/V(i)$ has the (i, κ)-closure property.*
(c) *the completion $\widetilde{V}$ of V in the κ-topology has cardinality $\geq \aleph$.*

It is our hope to extend elsewhere the work on valuated vector spaces and their relationship with (not necessarily Kurepa) trees. The rest of the paper shows further applications and equivalencies. We think however that the methods used in arriving at these applications are still too complicated to be called perfectly beautiful and the aim is to further simplify them. This simplification would be realized exactly through still better understanding of the relatinships between valuations and trees.

4. Abelian p-groups

A well known and a relatively recently established relationship between axiomatic set theory and (abelian) group theory is that found in the problem of J. H. C. Whitehead concerning torsion-free abelian groups. The problem asks whether it is true that, if every extension of the integers $\mathbb{Z}$ by an abelian group A is splitting, (*i.e.* $\mathrm{Ext}(A, \mathbb{Z}) = 0$), then A has to be a free abelian group. Here $\mathrm{Ext}(A, \mathbb{Z})$ stands for the group of all congruence classes of exact sequences $0 \longrightarrow \mathbb{Z} \longrightarrow X \longrightarrow A \longrightarrow 0$ (extensions of $\mathbb{Z}$ by A) — the congruence roughly defined by an isomorphism of the middle terms (this may be postulated to be a small group — *i.e.* the underlying structure is a set, not a class). Of course, if A is free, then every such extension splits. After a series of incomplete and restricted answers to Whitehead's problem (for instance, if A is countable, then it has to be free), a somewhat surprising answer was given in the 1970's by Shelah, who showed that the answer to Whitehead's problem depended on the underlying axioms of set theory; thus, in a model of $\mathrm{ZFC} + \mathrm{V} = \mathrm{L}$, the group A has to be free, while in a model of $\mathrm{MA} + \neg\mathrm{CH}$, A does not have to be free. This result was perhaps rather more surprising at the time to abelian group theorists than to set- and model-theorists, for long is the history of intertwining of (abelian) group theory with set theory and logic; it goes back at least as far as works of Tarski and Mal'cev in the 1950's. In fact, one may broadly say that uncountable cardinalities in algebraic structures inevitably involve, often implicitly, model theoretical questions.

The theory of abelian groups (a referential monograph: [Fuchs, 1970]) nowadays seems to be divided into mainly non-intersecting areas (this is, we think, because of the lack of better, unifying methods) of torsion-free, torsion and (genuinely) mixed groups. While Whitehead's problem involves torsion-free groups, we are in the realm of torsion groups in this paper.

For the latter class of groups, another functor plays a prominent role. It is the *torsion product* $\mathrm{Tor}(A, B)$ of groups A and B (see for instance [Mac Lane, 1963]). This is the abelian group generated by $\{(a, n, b) : a \in A, b \in B, n \in \mathbb{Z}, na = 0, nb = 0\}$, subject to the following relations: $(a_1 + a_2, n, b) = (a_1, n, b) + (a_2, n, b)$, $(a, n, b_1 + b_2) = (a, n, b_1) + (a, n, b_2)$, $(a, nm, b) = (na, m, b) = (a, n, mb)$. The name apparently comes from the fact that only the torsion elements in A and B are used for generating the product. Moreover the isomorphism $\mathrm{Tor}(A, B) = \oplus_p \mathrm{Tor}(A_p, B_p)$, where the summation is over all prime p and A_p is the p-primary component of A, gives us freedom to concentrate on the p-groups only when considering torsion products. Every p-group is representable as a torsion product: $A = \mathrm{Tor}(Z(p^\infty), A)$. Note that by the symmetry of the definition, the torsion product is commutative and it can be shown ([Mac Lane, 1960]) that it is also associative, if the iterated torsion product is defined inductively by $\mathrm{Tor}(A_1, \ldots, A_n) = \mathrm{Tor}(\mathrm{Tor}(A_1, \ldots, A_{n-1}), A_n)$, with $\mathrm{Tor}(A) = A$. The latter two properties of the torsion product would persuade us to adopt a new notation $A \textcircled{T} B$ for the tensor product of A and B, proposed in [Keef, 1996] and suggested by C. Metelli.

One of the first and in some sense the most important results in abelian group theory comes from Frobenius and Stickelberger in 1878; it establishes that every finite abelian group is a direct sum of a finite number of cyclic groups each of order equal to a power of a prime number. This can be extended to say that every finitely generated group is a direct sum of a finite number of cyclic groups. The simplicity of the representation here stems from two aspects: the representation is by a direct sum and the summands are rather easy groups to understand. Divisible groups have this nice appearance too; they are of the form $\{\oplus_{r_0} Q\} \oplus \{\oplus_p (\oplus_{r_p} Z(p^\infty))\}$ (where r_0 and r_1 are the torsion-free rank and p-rank, respectively, of the given divisible group) . The group A is *divisible* if, for every $a \in A$ and every $n \in \mathbb{Z}$, the linear equation $nx = a$ has a solution in A. At the other extreme, A is *reduced*, if it does not contain a divisible subgroup. We will assume here that our groups are reduced, for the maximum divisible subgroup is always a direct summand.

Although finite abelian groups have numerous applications in diverse mathematical subjects, they are not so interesting to abelian group theorists, perhaps because their structure is fully known. Thus the first class of abelian groups to look at for excitement is that of (reduced) countable groups and their direct sums — acronymed *disco* groups, as well as direct sums of cyclic groups — acronymed *discy* groups (direct sums are in a sense the simplest among the most important constructions). For a p-group A, it is a discy group if and only if it is a disco group without elements of infinite height. By a result of Kolettis from 1960, two reduced disco p-groups are isomorphic iff their Ulm invariants coincide; thus the group isomorphism is reduced to comparing the cardinal invariants.

A subgroup of a free group is again a free group. The analogous result need not be true for subgroups of disco groups. Thus it would be useful and interesting to know what conditions are needed that subgroups of disco or discy groups are again of the same type. A theorem by Kulikov states that if a p-group is of length ω_0, then subgroups of disco are again disco. Every subgroup of a countable disco is again disco. By a result of Kaplansky, a direct summand of disco groups is again a disco group. We say that a subgroup B is *pure* in A, when, for every $n \in \mathbb{Z}$, if the equation $nx = b \in B$ has a solution $x \in A$, then it has a solution in B too. This is a familiar condition, known under different names in set theory. Yet a more general notion for p-groups is that of p^α-purity ($\alpha = \omega$ describes the above purity): a subgroup

A of G is *p^α-pure* (or *α-pure*) in G if the short exact sequence $0 \longrightarrow A \longrightarrow G \longrightarrow G/A \longrightarrow 0$ represents an element of $p^\alpha \operatorname{Ext}(G/A, A)$. It was shown by Richman, Walker and Hill that even under stronger conditions than purity, a subgroup of a disco group need not be of the same kind: there are balanced subgroups of disco groups that are not disco groups. An exact sequence $0 \longrightarrow B \longrightarrow A \longrightarrow C \longrightarrow 0$ is called *balanced* (and B is a *balanced subgroup* of A) if the sequence $0 \longrightarrow p^\alpha B \longrightarrow p^\alpha A \longrightarrow p^\alpha C \longrightarrow 0$ is exact for every ordinal α. Balanced projectives are called *totally projective* p-groups. There are enough balanced projectives, *i.e.* for every group A there is a balanced projective B with an exact sequence $B \longrightarrow A \longrightarrow 0$. The *balanced projective dimension* of a group A is the smallest n (or ∞ if there is no such n) with the property that there exists an exact sequence $0 \longrightarrow B_n \longrightarrow \ldots \longrightarrow B_1 \longrightarrow B \longrightarrow A \longrightarrow 0$, where all the B's are balanced projective.

[Nunke, 1967b] gives conditions under which disco groups contain p^{ω_1}-pure subgroups which are not disco groups. Among the subgroups of disco groups, the following class is singled out: A group is called *ω_1-disco*, if it can be embedded as an p^{ω_1}-pure subgroup in a disco group. The mentioned Nunke's conditions are in terms of non-completeness; a question arises whether an ω_1-disco group is a disco group, if it is complete in the ω_1-topology. Translating this question by considering the corresponding quotients we ask an equivalent question: Is there a C_{ω_1}-group of length ω_1 with balanced projective dimension at most 1? An abelian p-group G is a *C_{ω_1}-group* if $G/p^\alpha G$ (or $G/G(\alpha)$ — height valuation!) is a disco group for every $\alpha < \omega_1$. In fact the answer is 'no', for any group of length ω_1 has balanced projective dimension at least 2. A new question is therefore: Does there exist a C_{ω_1}-group of length ω_1 with balanced projective dimension exactly 2? Answers will follow in the sequel.

Some results concerning the Tor functor are dependent on the underlying model of set theory. For instance, it may be shown (see [Keef, 1996]) that for certain classes of groups A, $\operatorname{Tor}(A, A)$ is a discy group if and only if the continuum hypothesis holds.

[Nunke, 1967a] tries to solve a question as to when $\operatorname{Tor}(A, B)$ is a disco group, for reduced A, B. Subsequently Keef treats the remaining unsolved (more difficult) case of this question, namely when A and B have the same length (not exceeding ω_1). For the case of length ω_1, [Keef, 1988] shows that a necessary condition is for A, B to be C_{ω_1} groups and if their balanced projective dimensions are at most 1, then the $\operatorname{Tor}(A, B)$ is a disco group. Consequently this problem of Nunke can be resolved by treating the above mentioned question of the existence of C_{ω_1} groups of length ω_1 and the balanced projective dimension 2.

The results on valuated vector spaces come in handy in special situations of abelian p-groups: the above Theorems 1 and 2 are instrumental in proving two results of P. Keef, utilizing some of the techniques similar to those he used. The approach in [Cutler and Dimitrić, 1993] is such that it only deals with the socles of the groups in question whenever possible.

Proposition 3. [Keef, 1989b; Cutler and Dimitrić, 1993] *Kurepa's Hypothesis is equivalent to the existence of a C_{ω_1}-group G of length ω_1 and cardinality $> \aleph_1$ with a p^{ω_1}-pure subgroup A of cardinality $\aleph_1$ such that the closure of A in G in the ω_1-topology has cardinality $\kappa > \aleph_1$.*

A group G with the properties as in the last proposition, is called a *κ-Kurepa extension* of A, whereas A is a *κ-Kurepa subgroup* of G (all for $\kappa \geq \aleph_2$). It may be shown that if such an extension exists, then there is one satisfying $|G| = \kappa$. Thus, there exists a κ-Kurepa extension if and only if there exists a Kurepa family of cardinality κ. Note that a Kurepa family can have cardinality at most $2^{\aleph_1}$; hence there is no κ-Kurepa extension for $\kappa > 2^{\aleph_1}$. The existence of G is proved with the aid of Theorem 1 and a result of F. Richman and E. Walker on valuated groups. A is constructed as the countable union of groups A_i that are in turn arrived at with the aid of a number of tools. Proposition 3 is the backbone for the proof of the result that follows. In addition, a number of definitions and results are needed. For example: Suppose that G is a C_{ω_1}-group, λ is a limit ordinal, and $\{A_i\}_{i<\lambda}$ is a filtration of p^{ω_1}-pure subgroups of G. Then the closure of the union of the A_i's is the union of their closures in the ω_1-topology on G. These are some of the ingredients used to prove the following:

Theorem 4. [Keef, 1989b; Cutler and Dimitric, 1993] *Kurepa's Hypothesis is equivalent to the existence of a C_{ω_1}-group of length ω_1 and balanced projective dimension 2.*

If A is an ω_1-pure subgroup of a disco group G, then it is called *pseudo-disco*, if for $K = G/A$, the group $K/K(\omega_1)$ is likewise a disco group.

Theorem 5. [Keef, 1989b] *The following are equivalent*

(1) $\neg$ KH.
(2) *Every C_{ω_1}-group of length ω_1 has balanced projective dimension at most 1.*
(3) *For any C_{ω_1}-groups A, B of length ω_1, Tor(A, B) is a disco group.*

(4) *For every C_{ω_1}-group A of length ω_1, and every group B, the group Ext(A,B) / Ext$(A,B)(\omega_1)$ is complete in the ω_1-topology.*
(5) *The class of ω_1-disco's coincides with the class of pseudo-disco's.*
(6) *An ω_1-disco is a disco iff it is complete in its ω_1-topology.*

One of the consequences of Proposition 3 and Theorem 5 is as follows: Existence of a Kurepa tree is equivalent to the existence of a Kurepa extension, or to the existence of C_{ω_1}-groups A, B of length ω_1 such that $\mathrm{Tor}(A,B)$ is not a disco group.

A slight extension of the notion of a κ-Kurepa subgroup is as follows: Given a C_{ω_1}-group G of length ω_1, we say that an ω_1-pure subgroup A of cardinality $\aleph_1$ is an $\aleph_1$-*Kurepa subgroup* if it is either an $\aleph_2$-Kurepa subgroup or else, a closed subgroup which is not a disco group. The notion is apparently related to the iterated Tor functor. The following chain of results is mainly proved by induction on n and by expanding groups G_i into continuous filtrations of subgroups. One can say that the possible size of the family of sets satisfying KH is to a great extent determined by the Tor functor.

Theorem 6. [Keef, 1989a] *Let $G_1, \dots, G_n$ be C_{ω_1}-groups of length ω_1, of cardinality at most $\aleph_n$. Then Tor $(G_1, \dots, G_n)$ is not a disco group iff there is a permutation ϕ of the set $\{1, \dots, n\}$ such that $G_{\phi(i)}$ has an $\aleph_i$-Kurepa subgroup, for every $i = 1, \dots, n$.*

Theorem 7. [Keef, 1989a] *For a natural number n, there is no $\aleph_n$-Kurepa extension if and only if, for every family of C_{ω_1}-groups $G_1, \dots, G_n$, of length ω_1, the group Tor $(G_1, \dots, G_n)$ is a disco group.*

Theorem 8. [Keef, 1989a] *For a natural number n, the following are equivalent:*
(1) *There are no $\aleph_{n+1}$-Kurepa extensions.*
(2) *The balanced projective dimension of Tor$(G_1, \dots, G_n) \leq 1$, for any set $G_1, \dots, G_n$ of C_{ω_1}-groups of length ω_1.*
(3) *For any set $G_1, \dots, G_n$ of C_{ω_1}-groups of length ω_1, the group Tor $(G_1, \dots, G_n)$ is a disco group, and, if $C_1, \dots, C_n$ are closed ω_1-pure subgroups of $G_1, \dots, G_n$ respectively, then Tor $(C_1, \dots, C_n)$, is likewise a disco group.*

5. Non-abelian groups

Kurepa's Hypotheses appear, perhaps not surprisingly, in the field of non-commutative groups, in extension of various characterizations of center-by-finite and finite-by-abelian groups due to [B.H. Neumann, 1955].

A group G is a *center-by-finite group* if $|G/Z(G)| < \omega$ (each subgroup has only finitely many conjugates); G is a *finite-by-abelian group*, if $|G'| < \omega$ (here the *commutator subgroup* $G' = \langle [G, G] \rangle$, where $[S, T] = \{g^{-1}h^{-1}gh : g \in S\, h \in T\}$).

Recall that, for $U \leq G$ the *normalizer* is $N_G(U) = \{h \in G : h^{-1}Uh = U\}$; the *centralizer* is defined by $C_G(U) = \{h \in G : \forall u \in U \quad h^{-1}uh = u\}$ and *the center* of G is $Z(G) = C_G(G)$. A group is called a κC*-group*, if $\forall g \in G$, $[G : C_G(g)] < \kappa$ (each $g \in G$ has at most κ conjugates). ωC-groups are also called FC-*groups*; in other words, in such groups, every element has finitely many conjugates. A subclass of κC, denoted by $\mathcal{Y}_\kappa$, is the class of groups for which $[G : N_G(U)] < \kappa$, whenever $U \leq G$ is generated by fewer than κ elements. Denote by $\mathcal{Z}_\kappa$ the class of groups G for which $[G : C_G(U)] < \kappa$, whenever $U \leq G$ is generated by fewer than κ elements (for $\kappa > \omega$, this is equivalent to saying that $[G : C_G(U)] < \kappa$, for $U \leq G$ with $|U| < \kappa$). Note that $\mathcal{Z}_\kappa \subseteq \mathcal{Y}_\kappa \subseteq \kappa C$ and $\mathcal{Y}_\omega = \mathcal{Z}_\omega = \omega C$.

The following are equivalent for any group G:

(1) G is center-by-finite.
(2) $\forall U \subseteq G \quad [G : N_G(U)] < \omega$.
(3) If in addition, G is an FC group, then the conditions are equivalent to U/U_G being finite, for all $U \leq G$ (U_G denotes the largest normal subgroup contained in U).

[Faber and Tomkinson, 1983] generalize this result to the following:

Theorem 9. *For an infinite cardinal κ and any group G in $\mathcal{Z}_\kappa$, the following are equivalent:*

(1) $|G/Z(G)| < \kappa$
(2) $[G : N_G(U)] < \kappa$*, for all* $U \leq G$
(3) $[G : N_G(A)] < \kappa$*, for all abelian* $A \leq G$
(4) $|U/U_G| < \kappa$*, for all* $U \leq G$
(5) $|A/A_G| < \kappa$*, for all abelian* $A \leq G$.

For an uncountable cardinal κ, [Tomkinson, 1984, p. 149] asks a question as to whether there is an FC-group G with $|G/Z(G)| = \kappa$, but $[G : N_G(U)] < \kappa$, for all (abelian) subgroups U of G. The question is whether it is necessary to have $G \in \mathcal{Z}_\kappa$ in the above theorem and to investigate some alternative conditions. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4) $\Rightarrow$ (5) are always true; (3) $\Rightarrow$ (1) is a part of Neumann's original result, and the example of a group that is a semidirect extension of the Prüfer group $Z(p^\infty)$, by the automorphism that inverts each element, is an example that (4) $\not\Rightarrow$ (1), at least for $\kappa = \omega$.

In all the considerations here, it suffices only to look into periodic FC-groups. One of the subclasses is defined as follows: A p-group G is *extraspecial*, if $G' = Z(G) = \Phi(G) \cong C(p)$. Here $\Phi(G)$ denotes the *Frattini subgroup* of G, i.e. the intersection of maximal subgroups of G. Thus extraspecial means that G is two-step nilpotent and that the factor group G/G' is elementary abelian, among other things.

Theorem 10. [Brendle, 1993a,b] *Assume there is a Kurepa tree. Then there is an extraspecial p-group E of size ω_2, such that for all maximal abelian subgroups A we have $[E : A] \leq \omega_1$. In particular, the existence of a Kurepa tree implies the existence of an extraspecial p-group of size ω_2 in $\mathcal{Y}_{\omega_2} \setminus \mathcal{Z}_{\omega_2}$.*

For the proof, take a Shelah-Steprāns group of size ω_1 (a specific extraspecial p-group of size ω_1 with all abelian subgroups of size $< \omega_1$) and extend it semidirectly, using the Kurepa tree to define the automorphisms.

A *wide Kurepa tree* (also known as a *weak Kurepa tree*) is a tree of height ω_1 with at least ω_2 uncountable branches such that all levels have size $\leq \omega_1$. If CH holds, then the complete binary tree of height ω_1 is a wide Kurepa tree. To show the consistency of the non-existence of wide Kurepa trees, an inaccessible is collapsed to ω_2 (more correctly, the cardinals between ω_1 and the inaccessible are collapsed as in Mitchell's and Baumgartner's models). The Proper Forcing Axiom implies that no wide Kurepa trees exist.

Theorem 11. [Brendle, 1993a] *If there is an extraspecial p-group of size ω_2 in $\mathcal{Y}_{\omega_2} \setminus \mathcal{Z}_{\omega_2}$, then there is a weak Kurepa tree.*

The hypotheses of "extraspecial p-group" may be replaced by "(periodic) finite-by-abelian group". Similar arguments are applied in the proof of the following:

Proposition 12. [Brendle, 1993a] *The following are equivalent:*

(1) *There is a Kurepa tree.*
(2) *There is an extraspecial p-group which is $\mathcal{Z}_{\omega_1}$, but not $\mathcal{Z}_{\omega_2}$.*
(3) *There is an FC group which is $\mathcal{Z}_{\omega_1}$, but not $\mathcal{Z}_{\omega_2}$.*

Proposition 13. [Brendle, 1993a] *The following are equiconsistent:*

(1) ZFC + $\neg$KH.
(2) ZFC + *for any FC-group G and $\kappa = \omega_1$ or ω_2, if $|G/Z(G)| = \kappa$, then there is an abelian subgroup $A \leq G$, with $[G : N_G(A)] = \kappa$.*
(3) ZFC+ *any extraspecial p-group of size ω_2 has an abelian subgroup A with $[G : N_G(A)] = \omega_2$.*

Addendum

S. Todorčević has suggested (in a private communication) that the answer to our question Q3 above may be found through the use of the technique employed in [Todorčević, 1984, pp. 282–284]. Indeed there is an affirmative answer which we present here. The details have been worked out by J. Moore (a student of Todorčević). Note that a weak version of the GCH is assumed, thus the question still remains whether the same can be done without (this variant of) GCH. In fact the following holds:

Theorem. *Given a regular cardinal κ and an $\aleph$ with $\kappa^+ \leq \aleph$. Assume that a ground model satisfies $2^{<\kappa} = \kappa$. Then there is a forcing construction of a κ-Kurepa tree with exactly $\aleph$ κ-branches.*

Proof. We will define a partial order $(\mathcal{P}, \leq)$ as follows. Elements p of $\mathcal{P}$ are ordered pairs of the form (T_p, f_p) satisfying

(i) T_p is a normal subtree of ${}^{\underline{\kappa}}2$ of height $\alpha+1$ for some limit ordinal $\alpha < \kappa$ and of cardinality less than κ.
(ii) f_p is a 1-1 function from a subset of $\aleph$ onto the top level of T_p.
$p \leq q$ iff
(iii) T_p is an end extension of T_q.
(iv) $\mathrm{dom}(f_q) \subset \mathrm{dom}(f_p)$ and $f_q(\xi) \subset f_p(\xi)$ for all $\xi \in \mathrm{dom}(f_q)$.

We start with a ground model satisfying $2^{<\kappa} = \kappa$ and show that $(\mathcal{P}, \leq)$ is κ-closed (chains of length less than κ have a lower bound) and has the κ^+-c.c. (there are no antichains of size κ^+). To see that our partial order is κ-closed, let $\mathcal{C}$ be a chain of length less than κ. Define $T = \cup_{p\in\mathcal{C}} T_p$ and $f : \cup_{p\in\mathcal{C}} \mathrm{dom}(f_p) \to {}^{\underline{\kappa}}2$ by $f(\xi) = f_p(\xi)$ where $\xi \in \mathrm{dom}(f_p)$. Note that the choice of p in our definition of f can be arbitrary by the assertion in condition (iv) coupled with the fact that $\mathcal{C}$ is a chain. Since the cardinality of $\mathcal{C}$ is less than κ, (T, f) is in the partial order and clearly serves as a lower bound for $\mathcal{C}$.

To see that $(\mathcal{P}, \leq)$ has the κ^+-c.c., let $\mathcal{A} \subset \mathcal{P}$ have cardinality κ^+. We need to show that there are two elements of $\mathcal{A}$ which are compatible. Because of the assumption on cardinal arithmetic in our ground model, there are only κ many conditions T_p which satisfy (i). Thus we may find a fixed T and a subcollection $\mathcal{A}'$ of $\mathcal{A}$ of size κ^+ such that all elements of $\mathcal{A}'$ are of the form (T, f_p). The Δ-system lemma (see [Kunen, 1980] p. 49, Theorem 1.6) allows us to find a subcollection $\mathcal{A}''$ of $\mathcal{A}'$ of size κ^+ such that the domains of the f_p's form a Δ-system. Let r be the root of this Δ-system. Since $|r| < \kappa$, there are only κ many ways to define a function from this set to the top level of T. We can then pick f and g such that (T, f) and (T, g) are both in $\mathcal{A}''$ and $f(\xi) = g(\xi)$ for all $\xi \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$. It is easy now to extend (T, f) and (T, g) to some (T', h).

Two well known results of forcing state that if a partial order is κ-closed then it preserves cardinals $\leq \kappa$ (*ibid.*, Corollary 6.15, p.215), and if a partial order has the κ^+-c.c., then it preserves cardinals $\geq \kappa^+$ (*ibid.*, Lemma 6.9, p.213) (these two statements about the partial order are made in the ground model). Thus $(\mathcal{P}, \leq)$ preserves cardinals. Let G be a $\mathcal{P}$-generic filter and define $T_G = \cup_{p\in G} T_p$. It is clear that T_G is a κ-tree. Furthermore if we let $b(\xi) = \{t \in T_G : \exists p(t \subset f_p(\xi))\}$, then $b(\xi) \neq b(\xi')$ for $\xi \neq \xi'$, and thus our tree has at least $\aleph$ many branches (since by a density argument, $\mathrm{dom}(b) = \bigcup_{p\in G} \mathrm{dom}(f_p) = \aleph$).

To finish the argument we need to show that the only κ-branches are those of the form $b(\xi)$ for some ξ. First we prove the following:

Claim. If $\dot{b}$ is a name for a branch of our generic tree and $p = (T_p, f_p)$ forces $\dot{b}$ is not of the form $b(\xi)$ for some $\xi < \aleph$ then there is a condition $q \leq p$ such that for every $\xi \in \mathrm{dom}(f_p)$, q forces $f_q(\xi) \not\subset \dot{b}$.

Proof of Claim: Pick p_ξ inductively for each ξ in $\mathrm{dom}(f_p)$ such that $p_\xi \leq p_\eta$ for all $\eta < \xi$ in $\mathrm{dom}(f_p)$ and p_ξ forces $f_{p_\xi}(\xi) \not\subset \dot{b}$. This choice is made possible at limit stages because $(\mathcal{P}, \leq)$ is κ-closed. Once we have chosen our sequence of conditions, we can find a q which serves as a lower bound for the sequence (this q obviously does the job). $\square$

Let $\dot{b}$ be a name for a κ-branch of T_G and suppose for contradiction that, on the contrary, there is a condition p such that p forces $\dot{b} \neq b(\xi)$ for all $\xi < \aleph$. Now for each $n \in \omega$ pick p_n using the above claim such that $p_0 = p$, $p_{n+1} \leq p_n$ and p_{n+1} forces $f_{p_{n+1}}(\xi) \not\subset \dot{b}$ for all ξ in $\mathrm{dom}(f_{p_n})$. Let $f : \cup_{n<\omega} \mathrm{dom}(f_{p_n}) \to {}^{\underline{\kappa}}2$ be defined by $f(\xi) = \cup\{f_{p_n}(\xi) : \xi \in \mathrm{dom}(f_{p_n}) \,\&\, n < \omega\}$. Define T to be $(\cup_{n<\omega} T_{p_n}) \cup \{f(\xi) : \xi \in \mathrm{dom}(f)\}$. There is a ξ in $\mathrm{dom}(f)$ such that (T, f) forces $f(\xi) \subset \dot{b}$, since p forces $\dot{b}$ is a branch of T_G. We can pick an n such that $\xi \in \mathrm{dom}(f_{p_n})$. But p_{n+1} forces $f_{p_{n+1}}(\xi) \not\subset \dot{b}$, which is a contradiction. Thus $\dot{b}$ must be of the form $b(\xi)$ for some $\xi < \aleph$ and we are finished. $\square$

This result may be further generalized as in the following statement of S. Todorčević (private communication; note that GCH is again assumed in the ground model):

Theorem. *Suppose our ground model satisfies the* GCH *and that F and B are two class functions defined on the class of all regular cardinals with ranges in the class of all cardinals such that the following holds for all regular cardinals α and β:*

(1) *$\alpha < \beta$ implies $F(\alpha) \leq F(\beta)$.*
(2) *$cf\,(F(\alpha)) > \alpha$.*

(3) $\alpha < B(\alpha) \leq F(\alpha)$.

Then, there is an Easton-type class forcing whose forcing extension satisfies the following for all regular cardinals κ: There is a κ-Kurepa tree with exactly $B(\kappa)$ cofinal branches and $2^\kappa = F(\kappa)$.

Idea of a proof. Combine the proof of Lemma 5.4 in [Todorčević, 1981], where a finer forcing is used than above, with that of [Easton, 1970] (namely, take the products of those two forcings).

References

[1] Brendle, Jörg. "Set-theoretic aspects of periodic FC-groups — extraspecial p-groups and Kurepa trees, J. Algebra, **162** (1993), no. 1, 259–286

[2] Brendle, Jörg. "Set-theoretic aspects of non-abelian groups", Israel Math. Conf. Proc. Collection: Set theory of the reals (Bar Ilan Univ., Ramat-Gan, 1991), **6** (1993), 91–105

[3] Bukovský, Lev. "Consistency theorems connected with some combinatorial problems", Comment. Math. Univ. Carolin., **7** (1966), 495–499

[4] Chang, C.C. "Extensions of combinatorial principles of Kurepa and Prikry, Theory of sets and topology, Berlin, (1972) 49–57

[5] Cohen, Paul J. "The independence of the continuum hypothesis", I, Proc. Nat. Acad. Sci., U.S.A., **50** (1963), 1143–1148

[6] Cohen, Paul J. "The independence of the continuum hypothesis", II, Proc. Nat. Acad. Sci., U.S.A., **51** (1964), 105–110

[7] Cohen, Paul J. *Set Theory and the Continuum Hypothesis*, W. A. Benjamin, Inc., New York, 1966

[8] Cutler, Doyle and Dimitrić, Radoslav. "Valuated vector spaces, Kurepa's Hypothesis and abelian p-groups", Rocky Mountain J. Math., **23** (1993), no. 4, 1253–1266

[9] Denjoy, Arnaud. *Enumération transfinie*, Vol. I-IV, Gauthier-Villars, Paris, 1946-1954

[10] Dimitrić, Radoslav. "Academician, Professor Djuro Kurepa (1907 – 1993)", Modern Logic, **4** (1994), no. 4, 401–403

[11] Easton, William B. "Powers of regular cardinals", Annals of Math. Logic, **1** (1970), no. 2, 139–178

[12] Faber, V. and Tomkinson, M.J. "On theorems of B.H. Neumann concerning FC-groups", II, Rocky Mountain J. Math., **13** (1983), 495–506

[13] Feferman, Solomon. "Some applications of the notion of forcing and generic sets", Proc. 1963 International Symposium on the Theory of Models in Berkeley, (1965), Amsterdam

[14] Fuchs, László. *Infinite Abelian Groups*, vol.I, II, Academic Press, New York and London, 1970, 1973

[15] Fuchs, László. "Vector spaces with valuations", J. Algebra, **35** (1975), 23–38

[16] Jech, Thomas. "Trees", J. Symbolic Logic, **36** (1971), no.1, 1–14

[17] Jensen, R.B. "Souslin's hypothesis is incompatible with $V = L$", Notices Amer. Math. Soc., **15** (1968), no. 6, 935

[18] Jin, Renling. "Some independence results related to the Kurepa tree", Notre Dame J. Formal Logic, **32** (1991), no. 3, 448–457

[19] Jin, Renling and Shelah, Saharon. "A model in which there are Jech-Kunen trees but there are no Kurepa trees", Israel J. Math., **84** (1993), no. 1-2, 1–16
[20] Keef, Patrick. "On the tor functor and some classes of abelian groups", Pacific J. Math., **132** (1988), no. 1, 63–84
[21] Keef, Patrick. "A theorem on the closure of ω-pure subgroups of C_ω groups in the ω-topology", J. Algebra, **125** (1989), no. 1, 150–163
[22] Keef, Patrick. "On set theory and the balanced-projective dimension of C_ω groups", Contemp. Math., **87** (1989), 31–41
[23] Keef, Patrick. "Abelian groups and the torsion product", Abelian groups and modules. Proc. Internat. Conf., Colorado Springs, D. Arnold and K. rangaswamy, Ed., Marcel Dekker, Inc., New York, 1996, 45–66
[24] Kunen, Kenneth. *Set Theory: An Introduction to Independence Proofs*, North-Holland Publishing Company, Amsterdam, New York, Oxford, 1980
[25] Kurepa, Georges. "Ensembles ordonnés et ramifiés", Publ. Math. Univ. Belgrade, **4** (1935), 1–138
[26] Kurepa, Georges. *A propos d'une généralisation de la notion d'ensembles bien ordonnés*, Acta Math., **75** (1942), no. 1–3, 139–150
[27] Kurepa, Djuro. *Teorija Skupova*, Školska Knjiga, Zagreb, 1951
[28] Kurepa, Djuro. *Sur une propriété caractéristique du continu linéaire et le problème de Suslin*, publ. Math. Univ. Belgrade, **4** (1952), 97–108
[29] Lévy, Azriel. "Definability in axiomatic set theory", Proc. 1964 International Congress for Logic, Methodology and Philosophy of Science (Y. Bar-Hillel, ed.), Amsterdam, 1965, 127–151
[30] Mac Lane. "Triple torsion products and multiple Kunneth formulas", Math. Ann., **140** (1960), 51–64
[31] Mac Lane, Saunders. *Homology*, Springer-Verlag, Berlin, 1963
[32] Neumann, B.H. "Groups with finite classes of conjugate subgroups Math. Z.", **63** (1955), 76–96
[33] Nunke, R. "On the structure of Tor II", Pacific J. Math., **22** (1967), 453–467
[34] Nunke, R. "Homology and direct sums of countable groups", Math. Zeit., **101** (1967), 182–212
[35] Přikrý, Karel Libor. *Changing Measurable Into Accessible Cardinals*, Ph.D. Dissertation, University of California, Berkeley, 1968
[36] Ricabarra, Rodolfo A. *Conjuntos ordenados y ramificados*, Instituto de Mathematica, Universidad nacional del Sur, Bahia Blanca, 1958
[37] Rowbottom, Frederick. "Some strong axioms of infinity incompatible with the axiom of constructibility", Annals Math. Logic, **3** (1971), no. 1, 1–44 (A journal publication of the PhD dissertation, University of Wisconsin, Madison, 1964)
[38] Silver, Jack H. "The independence of Kurepa's conjecture and two cardinal conjectures in model theory", in: D.S. Scott (ed.), *Axiomatic Set Theory*, Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, R.I., **13(I)** (1971), 383–395
[39] Silver, Jack H. "The consistency of Chang's conjecture", Preprint.
[40] Silver, Jack H. "The bearing of large cardinals on constructibility", in: *Studies in model theory vol. 8*, Morley, M. (ed.), Mathematical Association of America, Buffalo, NY, 1973, 158-182

[41] Stewart, D.H. *The consistency of the Kurepa hypothesis with the axioms of set theory*, Masters Dissertation, University of Bristol, 1966

[42] Todorčević, Stevo B. "Trees, subtrees and order types", Annals Math. Logic, **20** (1981), 233–268

[43] Todorčević, Stevo B. "Trees and Linearly Ordered Sets", in: *Handbook of Set-Theoretic Topology*, K. Kunen and J.E. Vaughan (eds.), Elsevier Science Publishers B.V., (1984), 235–293

[44] Tomkinson, M.J. *FC-groups*, Pitman, London, 1984

[45] Vaught, R.L. "Models of complete theories", Bull. Amer. Math. Soc., **69** (1963), no. 3, 299–313

[46] Vaught, R.L. "The Löwenheim-Skolem Theorem", Proc. 1964 International Congress for Logic, Methodology and Philosophy of Science (Y. Bar-Hillel, ed.), North Holland, Amsterdam, 1965, 81–89

Department of Mathematics, University of California, Berkeley, CA 94720