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PEIRCE'S LOGIC OF CONTINUITY: EXISTENTIAL GRAPHS AND NON-CANTORIAN CONTINUUM

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Abstract. Peirce's systems of *existential graphs* (*Alpha*: classical propositional calculus; *Beta*: first-order purely relational logic; *Gamma*: modal calculi and second-order logic) are presented both from an historical perspective and succinctly for the modern reader. Peirce's alternative *continuum*, with its main non-Cantorian properties (genericity, reflexivity, modality), is also presented both historically and synthetically. The blend of Peirce's existential graphs and his non-Cantorian continuum gives rise to a thoroughly original logical approach to the "labyrinth of the continuum". We explain why such an approach was set aside in the main developments of logic in the xx$^{\text{th}}$ century, and we hint to a possible renewal of interest for Peirce's *continuity logic* from the viewpoint of contemporary developments in category theory and geometric logic.

CONTENTS

"La *Logique* est une science qui n'est pas des plus fortunées: car, – tandis que la plupart des philosophes préfèrent *en parler* plutôt que de l'*employer*, – la plupart des mathématiciens préfèrent l'*employer* sans *en parler* et même sans vouloir en *entendre parler*."

Alessandro Padoa (1936)

1. Introduction

Leibniz's "labyrinth of the continuum" has been explored in depth in xx$^{\text{th}}$ century mathematical logic, but mainly from a particular entrance door to the labyrinth: set theory intertwined with classical first-order logic (the system Zermelo-Fraenkel (ZF) and its extensions). The consistency (Gödel) and independence (Cohen) results on the continuum hypothesis, showed that, from a classical set-theoretic perspective, additional axioms were needed to obtain a finer grasp of the continuum. On that path, Woodin's program to unveil natural maximal principles related to the projective determinacy of $H(\omega_1)$, in order to lift them up to $H(\omega_2)$ and to force the cardinality of the continuum to be $\aleph_2$, seems to have been very successful in a descriptive understanding of many internal layers of the continuum.

Nevertheless, it should be clear that a given *particular* approach to a *generic* concept cannot hope to capture all its richness. The particularity of approaching the continuum with classical first-order set-theoretic tools – however central ZF unquestionably can be for today's mathematics – leaves aside important aspects of the continuum which those tools can, in fact, describe, but that they do not capture as *main* characteristics. The change of perspective is much more than a mere curiosity. If we understand pragmatically the continuum as a multi-layered, multi-dimensional conceptual landscape, necessarily *glued* together with the vantage points (axiomatic systems, underlying logics) which reveal its perspectives and panoramas, it becomes evident that other, *alternative*, glances at the continuum should be of great value. The continuum – a protean mathematical concept if there ever was one – gains in richness when explored from other entrance doors to the labyrinth, as the following diagram illustrates:

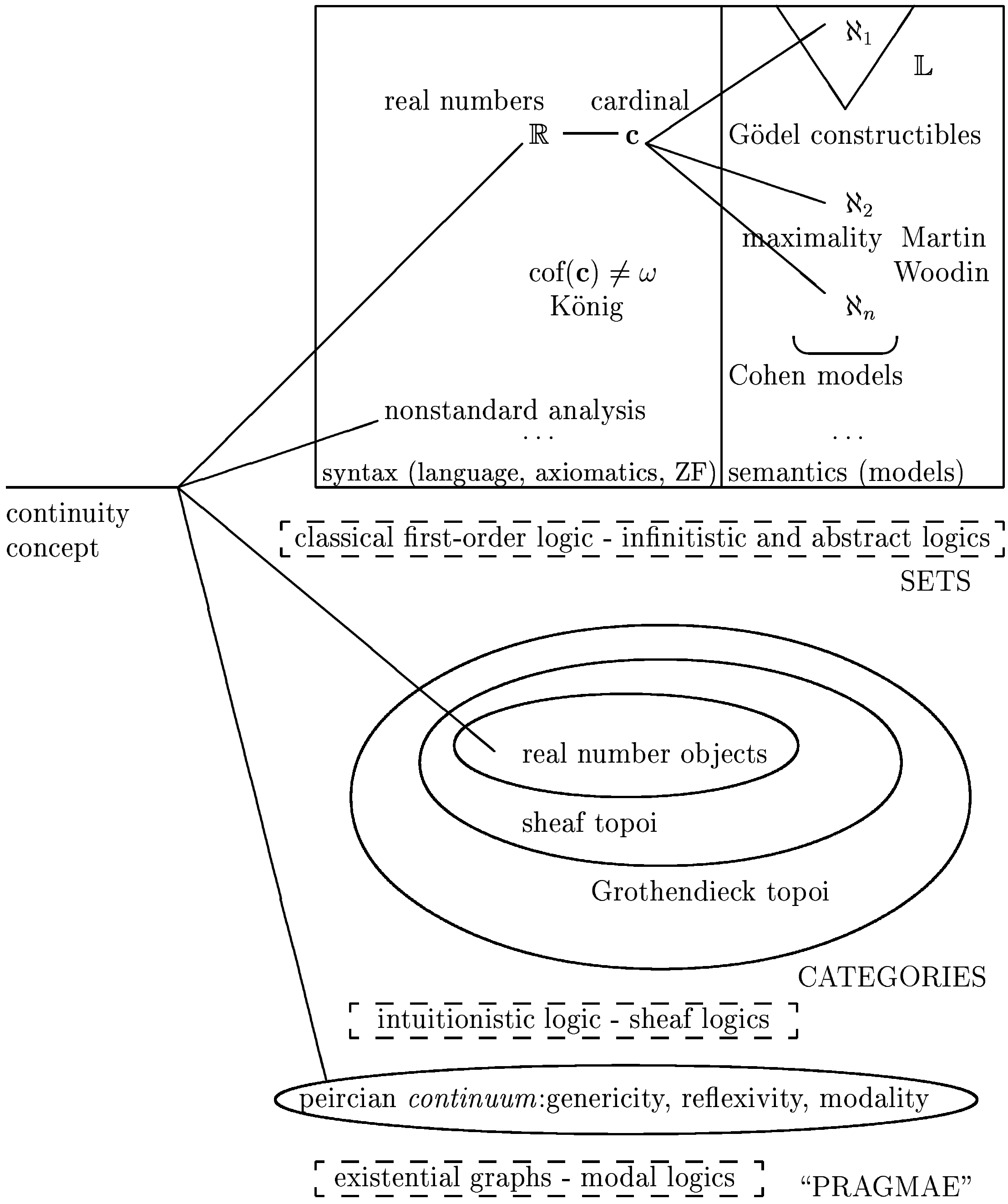

Figure 1.
Proteus: pragmatic unfolding of the continuum

In this paper, we will study a *perspective shift* on the continuum, providing an introduction to Peirce's original approach to the labyrinth of the continuum (bottom part of the preceding figure). Peirce constructed *both* a global, architectural view of a non-Cantorian continuum, *and* underlying logical systems well adapted to support it. Those logical systems, **Peirce's existential graphs**, constitute *fully formal* systems in which surprisingly deep *continuous connections* can be drawn between classical propositional calculus, classical first-order logic and some modal calculi. On the other hand, **Peirce's alternative continuum** provides an inexhaustible mine of non-formal, inspiring ideas, which concur, independently, with several developments of xx$^{\text{th}}$ century mathematical logic.

2. The Misfortune of Peirce's Late Logical Thought (1895-1910)

Why has such an important logical work – as we claim (and hope to prove in the following sections) – been so generally ignored? Several reasons emerge, but three main trends seem to carry the major weight of having blocked an understanding of Peirce's late logical ideas: 1. The solid reputation of Peirce's breakthroughs in the algebra of logic (1870-1885). 2. The posthumous and poor editing of Peirce's work on the existential graphs and the alternative continuum (1933-1935). 3. The indeterminacy of the place of Peirce's late logical ideas between philosophers, semioticians and historians of logic (1960 onward).

Paradoxically, the success of Peirce's contributions in the algebra of logic [1] (elimination and normal form procedures, algebras of relations, relative logic and quantifiers) may have hindered a fair perception of his later work in logic. On one side, the "normal" reconstruction of the history of the discipline, following Russell's authority, while magnifying Frege's place, condescended to name Peirce as co-founder of the algebra of relations (with De Morgan) and co-discoverer of modern quantifiers (with Frege). On the other side, although references to Peirce became thus "normalized" in books on the history of logic until the 1980's, the real influence of Peirce's semantical approach to logic – crucial in Löwenheim's theorem and in Tarski's relational algebras – paved the way to the emergence of modern model theory (a history recently well retold in [2]). In any case, Peirce's late logical thought (1895-1910) seemed to vanish from the panorama. Nevertheless, if we check that three fourths of the entries in the 530 pages of Peirce's *Logic Notebook* (his "journal" of logical discovery, 1865-1910) correspond to years after 1895, we can begin to measure the large amount of time and energy dedicated by Peirce to logical thought in the last years of his life (1839-1914).

While Peirce's contributions to the algebra of logic were published in a short series of polished papers (influencing Schröder, and in turn influencing Russell and Löwenheim [2]), Peirce's existential graphs and Peirce's ideas on an alternative continuum just appeared in lectures (Harvard, 1903; Lowell, 1903; National Academy of Sciences, 1906) and in a final paper (*Monist*, 1906) related to an ambitious, large-scaled and not well understood attempt to obtain a "proof of pragmaticism", showing the correctness of his entire philosophical edifice — a modal, relational, synthetical pragmatism. Thus, Peirce's late logical thought remained mostly unpublished and unknown, awaiting its posthumous appearance. This happened in Hartshorne and Weiss' edition of Peirce's *Collected Papers* [22], with the appearance of the fourth volume (1933) including a long section on existential graphs, and the fifth (1934) and sixth (1935) volumes including some paragraphs on the continuum. While the *Collected Papers* provided the invaluable service of making finally available Peirce's genius to the community of inquirers, the "cut-and-paste" editing methodology of Hartshorne and Weiss made very difficult a thorough understanding of Peirce's brilliant, but seemingly obscure, original ideas. The editors scattered a selection of Peirce's manuscripts throughout a subject-oriented compilation of their own, at first destroying (and then later attempting to reconstruct) the unity of Peirce's thought. The reception of Peirce's unpublished manuscripts on the existential graphs and the continuum went almost unnoticed by mathematical logicians, also due perhaps to a condescending review of the fourth volume of his *Collected Papers* by the young Harvard rising star in mathematical logic, W.V.O. Quine, who quickly dismissed Peirce's late logical thought as "good entertainment" [29]:

> The other material on exact logic has to do with logical graphs. A series of extensions and modifications of Euler's scheme of diagrams leads Peirce to an elaborate scheme of his own, designed for the expression of propositions involving any manner of complexity in point of relational structure, quantity and even modality. The system is intended rather for the analysis of logical structure than for the facilitation of inference; because of its cumbersomeness it is less suited to the latter purpose than is the algebraic form of logic. One questions the efficacy of Peirce's diagrams, however, in

> their analytic capacity as well. Their basic machinery is too complex to allow one much satisfaction in analyzing propositional structure into terms of that machinery. While it is not inconceivable that advances in the diagrammatic method might open possibilities of analysis superior to those afforded by the algebraic method, yet an examination of Peirce's product tends rather, apagogically as it were, to confirm one's faith in the algebraic approach. [...]
>
> The volume as a whole recommends itself to the logico-mathematical reader above its predecessors in the series. Its 600 pages contain a generous variety of good entertainment.

Many of the prejudices that would be repeated against Peirce's graphs are already present in Quine's review: their apparent "cumbersomeness", their inefficiency, their complexity, which would tend to confirm the superiority of algebraic methods. In fact, as we shall soon see, the "cumbersomeness" of the diagrams is just due to a *first glance* non-acquaintance with them, extremely easy to overcome after a little practice, a "cumbersomeness" that every logic student knows does not disappear quickly when faced with their first Hilbert-style inferences (two semester-long seminars introducing classical propositional calculus and first-order logic, one following Hilbert, the other following Peirce's graphs, have provided a remarkable success in learning logic with the graphs and *facilitating inference*; see Oostra [19]). As to the existential graphs' "efficacy", questioned by the young Quine, we will next prove very simple inferences and decision methods which show the tremendous elegance and power of the graphs. Finally, the "too complex" basic machinery of the graphs (which in fact is not so: see follow up) was directed at the analysis of *inference* (*both* propositional *and quantificational*, at the same time!) and not at the analysis of propositional structure, as Quine would think. Peirce's long efforts to construct a logic of continuity were just not understood by Quine — involved in other frames of mind and not ready to judge correctly Peirce's anticipations in topological logic. But, if Quine in the 1930s could conclude that Peirce's graphs (and his involvements with continuity) constituted just "a generous variety of good entertainment", who would have been able to assert the contrary? Nobody was, and Peirce's late logical thought remained in limbo for many more years to come.

It was only at the start of the 1960s, that two simultaneous Ph.D. theses set the way for a thorough understanding of Peirce's graphs. In 1963, Don Roberts's *The Existential Graphs of C.S. Peirce* [30], and in 1964, Jay Zeman's *The Graphical Logic of C.S. Peirce* [35], showed that Peirce's systems Alpha, Beta and Gamma served, respectively, as complete axiomatizations for the classical propositional calculus, first-order logic in a purely relational language, and intermediate modal calculi. Although the theses were outstanding pieces of work – and still provide today the best references to the subject – their influence was small, both in the philosophical community (Roberts came under the guidance of M. Fisch) and in the mathematical logic community (Zeman under A. Prior). In fact, the *middle place* of Peirce's existential graphs – neighboring both philosophy and mathematics without fully finding its *milieu* for development, riding both speculation and formalization – may have hindered its brilliant potential. Little appreciated by non technical philosophers (Ketner's [14, 15] are nice exceptions confirming the rule), rediscovered periodically by enthusiastic philosopher-logicians (Thibaud's [33], Burch's [5], Shin's [32]), and almost totally ignored by the mathematical community (although Hammer's [12] and Brady and Trimble's [3] may begin to remedy the situation), Peirce's graphs have never really been the focus of a sustained and continuous effort of a logic *school* interested in their organic development. Despite their already well established strength, *both philosophical and mathematical*, but still carrying their weight of "exotism", "curiosity" and "cumbersomeness", Peirce's late logical ideas have yet to acquire a full permit of residence, *both in philosophy and in mathematics*. In the remainder of this paper, we will emphasize the extraordinary role they can play in mathematical logic.

3. Peirce's Existential Graphs: Genetic Approach

Peirce's systematic philosophy is centered around an ample understanding of the pragmatic maxim: knowledge proceeds through representation of relations, semiotic maneuvering of the *representamens* in contexts of possibility, and reintegration of all differential information observed. Representational tools with great transformational power, such as mathematical diagrams, were then of prime importance to Peirce. As we will see in section 8, Peirce's abstract "differential and integral pragmatics" can be described through the machinery of mathematical category theory, retrieving thus Peirce's insistence that mathematical inference is, at bottom, diagrammatic:

> All deductive reasoning, even simple syllogism, involves an element of observation; namely, deduction consists in constructing an icon or diagram the relations of whose parts shall present a complete analogy with those of the parts of the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts. (1885) [22, § 3.363].

Already in 1882, in a letter to his student Mitchell, Peirce had developed a diagrammatic method to express quantification [31, p. 18]:

> The notation of the logic of relatives can be somewhat simplified by spreading the formulae over two dimensions. For instance suppose we write
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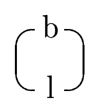
> 
>
>
> to express the proposition that something is at once benefactor and lover of something. That is,
>
> $$\sum_x \sum_y b_{xy} l_{xy} > 0 .$$

In view that his first published article on the algebraic symbolism ($\sum$, $\prod$) for quantifiers is from 1883 [27], we can see that Peirce's diagrammatic and algebraic approaches to quantification were simultaneous. The algebraic presentation probably prevailed at first in Peirce's work thanks to all the mathematical advances and analogies already naturally available in algebraic logic. In a 1906 glance back [31, p. 20], Peirce would explain how he had lost for a while the portal of the "rich treasury" of diagrammatic logic:

> The system of expressing propositions which is called Existential Graphs was invented by me late in the year 1896, as an improvement upon another system published in the *Monist* for January 1897. But it is curious that 14 years previously, I had, but for one easy step, entered upon the system of Existential Graphs, reaching its threshold by a more direct way. The current of my investigations at that time swept me past the portal of this rich treasury of ideas. I must have seen that such

a system of expression was possible, but I failed to appreciate its merits.

Nevertheless, from 1896 on, Peirce undertook a strenuous journey to unravel the "merits" of his diagrammatic systems. The unraveling was *systematic and continuous*, as witness the many entries devoted to the existential graphs in Peirce's *Logic Notebook* since 1898: nearly 50 of 250 pages, that is one fifth of the *Notebook* since the discovery of the graphs, the most entries devoted to a single subject. Peirce's 1908 well-known evaluation of his graphs [31, p. 11] — "my *chef d'oeuvre*" — is just a fair appreciation of all the energy and inventiveness he devoted to their development.

Peirce's existential graphs arise as a sharp tool to *analyze inference in the logic of relatives*. The name "existential" comes from the method of "scribing"[1] the existence of graphs on a "sheet of assertion", and from the way the graphs themselves are connected by means of "identity lines" representing existential quantification. The same idea for representing quantification in Peirce's letter to Mitchell is retained, but a new diagram for negation is introduced, with *easy transformation rules* to govern its handling, going beyond the *syntactic* limitations of a purely formal negation mentioned in the letter to Mitchell [31, p. 19] and beyond the *illative* limitations of the negative alternation introduced in his "entitative graphs" of 1896 [31, p. 26] (a "negative" version of the existential graphs, as seen from the *reverse* of the sheet of assertion).

In Peirce's last and simple solution, scribing existential graphs on the sheet amounts to their conjunction (asserting them on the sheet), drawing an "oval" around them amounts to their negation (detaching them from the sheet), and connecting them with "identity lines" amounts to their existential instantiation. As we will next see, nice diagrammatic rules — on allowed topological maneuverings of the graphs, crossings between "ovals" and "lines", and analytic continuations — provide then diverse combinatorics between conjunction, negation and existence, paving the way to the construction of amazing logical systems, in which *topological permissions* correspond to *logical deductions*.

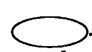
While, for Peirce, the mathematician is mainly interested in drawing inferences, the logician, in turn, is more attracted to the analysis of inferences. In his Lowell Lectures of 1903, Peirce came up with

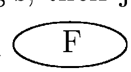
[1] *Editor's Note.* To 'scribe' is to write or draw or otherwise place a graph on [a sheet of assertion]. "Since it is sometimes awkward to say that a graph is *written* and it is sometimes awkward to say it is *drawn*, I will always say that it is *scribed*."(Peirce, MS 450, p.8 *verso*)

definitive advances on his existential graphs, analyzing them and classifying them into three major systems — Alpha, Beta, and Gamma — which retrieved diagrammatically “logical algebra” (classical propositional calculus), “relative logic” (classical first-order logic in a purely relational language) and “logical possibilities” (modal calculi). Peirce’s systems were fully organized by means of transformation rules, as in modern natural deduction systems. Alone — a recluse in his Milford country home in the last, awesomely productive, twenty years of his life — Peirce was able to leave a “rich treasury” for posterity.

4. Peirce’s Existential Graphs: Succinct Modern Presentation

For the sake of clarity, we will now present separately Peirce’s systems Alpha, Beta and Gamma, even if, as we have mentioned, Peirce constructed Alpha and Beta simultaneously. Peirce’s systems can be understood in the framework of modern logical systems, centered around the specification of a *formal language*, a set of *axioms and inference rules* and a complete *semantics*. Comparing with the usual Hilbert-style presentations of classical propositional calculus (CPC) and of first-order logic in a purely relational language ($\mathcal{L}_1^{\mathcal{R}}$), we will deal with well-formed graphs — instead of well-formed formulas, with no axioms and with natural symmetric inference rules — instead of many ad-hoc axioms and the *modus ponens* inference rule —, and with complete translations Alpha $\leftrightarrow$ CPC and Beta $\leftrightarrow$ $\mathcal{L}_1^{\mathcal{R}}$ — instead of new semantics for Peirce’s calculi —. For thorough presentations of Peirce’s graphs see the classics [31], [35] or the recent [32].

Alpha’s formal language consists of a set $\mathcal{F}$ of propositional letters (including a void) and an oval (or “cut”) . The set of Alpha’s well-formed graphs (*wfg*) is the minimal set closed recursively under the following clauses:

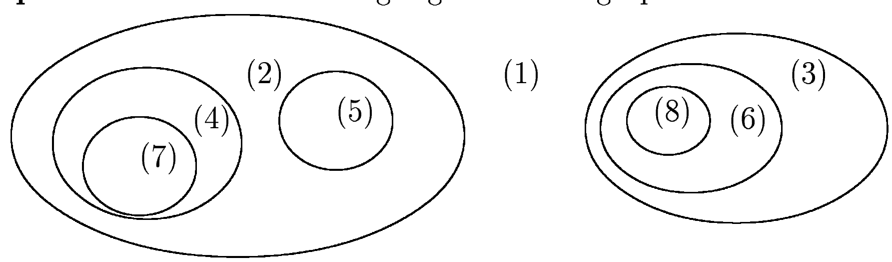
(i): Any propositional (or void) letter is a *wfg*.
(ii): If F and G are wfg’s, their juxtaposition F G is a *wfg*.
(iii): If F is a *wfg*, then 

is a *wfg*.

Alpha’s graphs can be viewed as *series of nested cuts in the plane*, where the diverse regions of the nests are marked with some (or no) propositional letters. Notice that, by the recursive definition of *wfg*’s, cuts cannot intersect between them.

A graph, or a region, inside a nested cut is called *evenly enclosed* (resp. *oddly enclosed*) if an even (resp. odd) number of cuts surrounds it (particular case: no cuts around — 0 cuts — corresponds to an evenly enclosed graph, or region). Given a graph inside a nested cut, a *strict* region for the graph is defined as any region, part of the nest and not part of the given graph, which is enclosed by more ($\geq$) cuts than the graph.

Example. Consider the following regions of the graph:



> Referring to the previous regions, any graph which would be scribed in regions (1), (4), (5) or (6) would be evenly enclosed; any graph scribed in regions (2), (3), (7) or (8) would be oddly enclosed; for any graph scribed in region (1), strict regions for that graph would be all of the 8 regions; for any graph scribed in region (2), strict regions would be (2), (4), (5) and (7) (but not (1), (3), (6) nor (8)); for any graph scribed in region (4), strict regions would be (4) or (7); for any graph scribed in region (5) — or respectively (7), (8) — its only strict region would be (5) — or respectively (7), (8). Notice also that a given region is *not* a strict region for any graph consisting of the oval enclosing that region and its subgraphs (for example, region (5) is *not* strict for the oval enclosing (5)).

Alpha's inference rules (or "illations") are the following permissions, natural and symmetric, which allow introducing, moving and forgetting information in series of nested cuts:

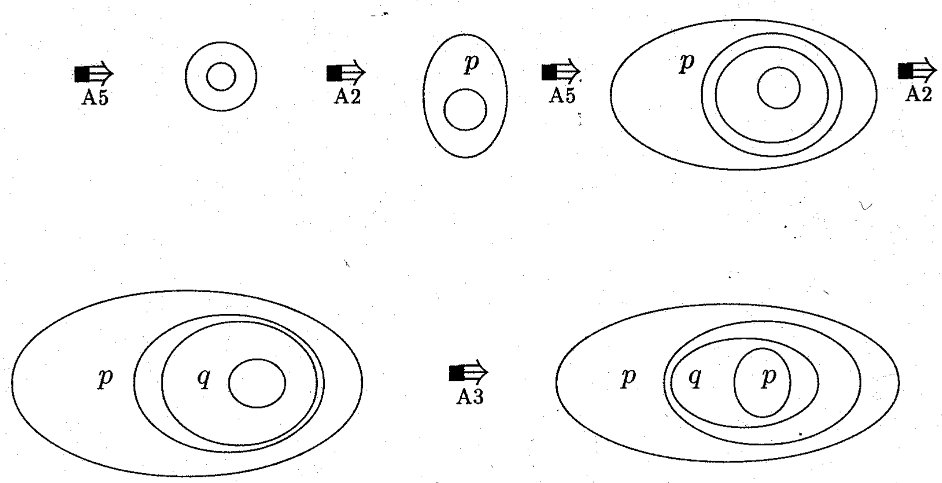
- **A1:** ("Erasure") Any evenly enclosed graph may be erased.
- **A2:** ("Insertion") Any graph may be inserted in any oddly enclosed region.
- **A3:** ("Iteration") Any graph may be iterated (*i.e.*, repeated) in a strict region for that graph.
- **A4:** ("Deiteration") Any graph whose occurrence could result from iteration may be deiterated (*i.e.*, erased).
- **A5:** ("Double cut") A double cut may be inserted or erased around any graph in any region.

With the help of the natural translation Alpha-*wfg* $\xrightarrow{*}$ CPC-*wff* defined recursively by $(void)^*$= T (with an extended language for the CPC including a syntactic sign T to represent truth), $p^* = p$, $(F\ \ G)^*$ $= F^* \wedge G^*$, and $(\textcircled{F})^* = \neg\ (F^*)$, it is easy to check that Alpha's inference rules correspond to the following valid CPC-rules:

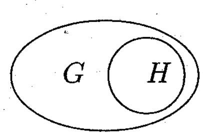
- Erasure and Insertion: rule of conjunction $p \wedge q \rightarrow p$, and contrapositive $\neg p \rightarrow \neg(p \wedge q)$.
- Iteration and Deiteration: generic intuitionistic rule of negation: $p \wedge \neg q \leftrightarrow p \wedge \neg(p \wedge q)$.
- Double Cut: classical rule of negation: $\neg\neg p \leftrightarrow p$.

To show how inferences work in Alpha, we can check that the corresponding translations $(*^{-1})$ of the usual CPC axioms are Alpha-theorems (defining an Alpha-theorem as a deduction from the *void* graph following Alpha's inference rules). We will write $F \blacksquare\!\!\Rightarrow G$ to denote that graph F illatively deduces (rules A1-A5) graph G. In case F = *void*, the notation $\blacksquare\!\!\Rightarrow G$ thus stands for "G is an Alpha-theorem".

1. Inference of $[p \rightarrow (q \rightarrow p)]^{*^{-1}}$:

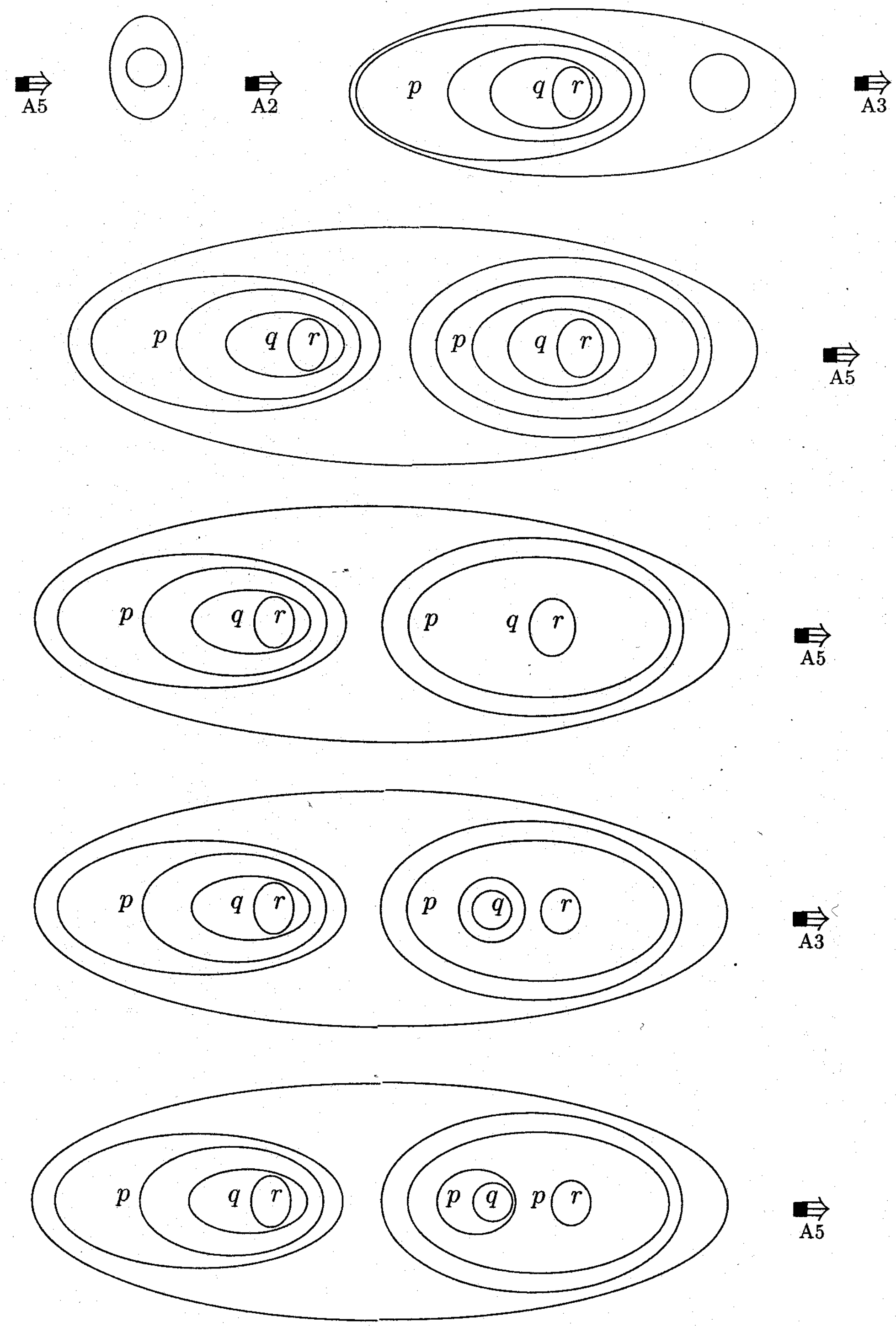


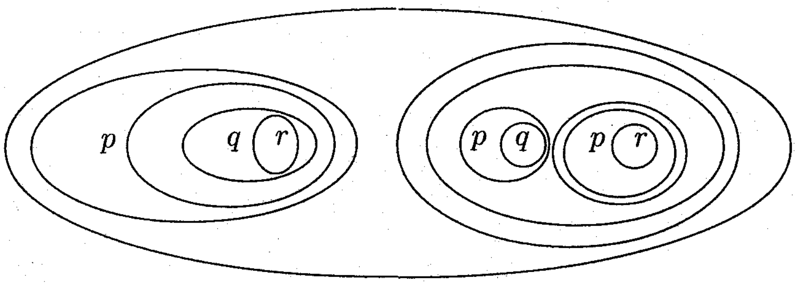
QED

Notice that the graph 

is the Alpha translation ($*^{-1}$) of $\neg(G^* \wedge \neg H^*)$, that is, of $G^* \rightarrow H^*$.

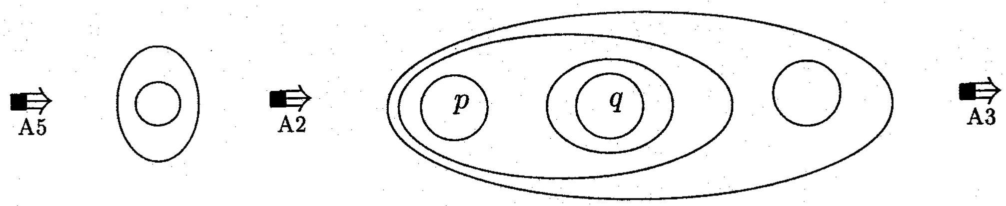
2. Inference of $[(p \to (q \to r)) \to ((p \to q) \to (p \to r))]^{*^{-1}}$:

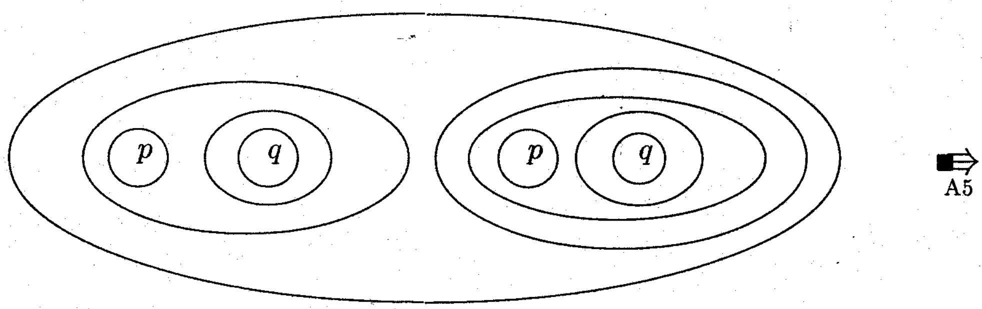


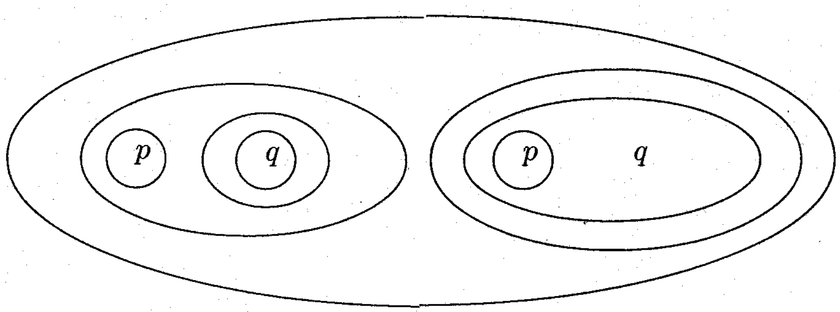


Q.E.D.

3. Inference of $[(\neg p \to \neg q) \to (q \to p)]^{*^{-1}}$:

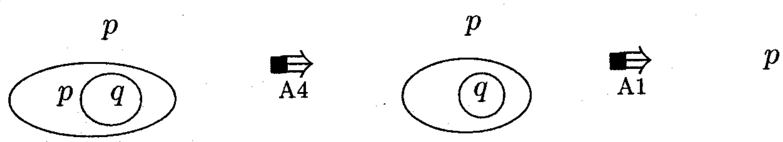




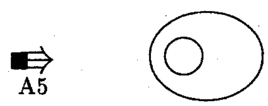


Q.E.D.

Notice that in the derivations of the CPC axioms, we have just used Alpha rules A2, A3 and A5. Rules A1 and A4 are only needed to prove the $*^{-1}$ translation of *modus ponens*:



An interesting philosophical remark immediately present in the previous inferences is that *any* Alpha theorematic derivation *must* begin introducing the double-cut

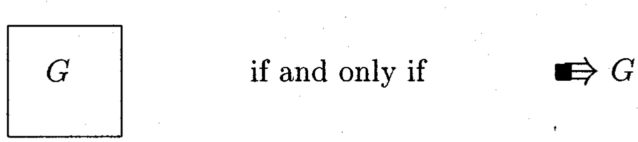


In fact, inserting *negative* information in the oddly enclosed region (A2) (to be *then* iterated (A3) towards positive regions) is the only way available in Alpha to begin an inference. This feature shows strongly how all classical derivations are inescapably relying on dichotomic subdivisions of thought and space.

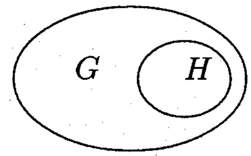
Independently of the *-translation between Alpha and CPC, Peirce introduced the *sheet of assertion* (SA) in order to provide a natural semantics for Alpha. The sheet of assertion represented for Peirce a blank region of *truth* where Alpha graphs could be scribed. If, using only rules A1-A5, the graph turned out to be scribable in SA, the graph was to be considered as true. Since the CPC *-translations of the Alpha rules preserve truth, SA — together with rules A1-A5 interpreted semantically — provides a semantics for Alpha which can be proven to be equivalent to the usual CPC truth-table semantics.

If we denote with a blank rectangle the sheet of assertion, one can prove, by induction on the complexity of *wfg*'s, the following Alpha versions of the completeness and deduction theorems. Even if the completeness theorem was not plain to Peirce (who mixed semantics and inference rules), he stated clearly the deduction theorem, both in algebraic [26, p.169] (1880) and graphical form [*Logic Notebook*, p.118r] (1898):

Completeness.



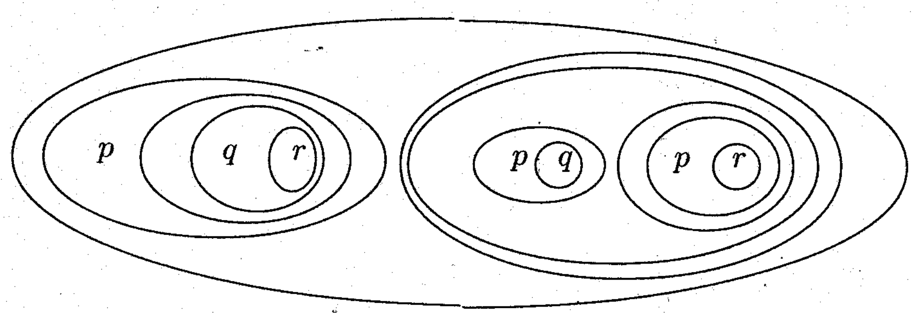
Deduction theorem.

G ➡ H if and only if ➡ 

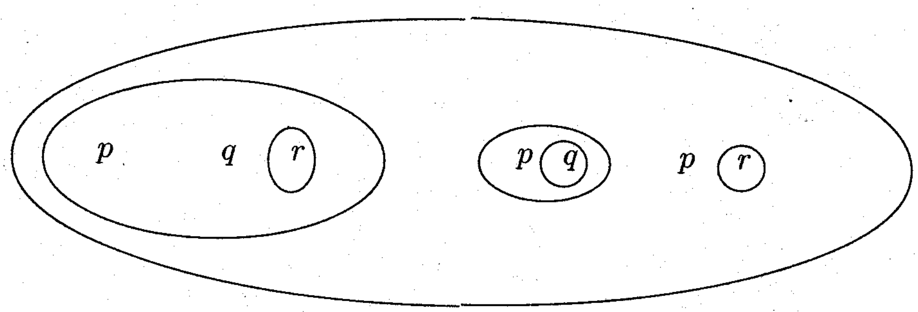
System Alpha includes very nice properties in its proof theory, such as, for example, the deep theorem that the *form* of any provable Alpha graph *completely codifies* already its Alpha proof. In fact, Alpha possesses a simple decision method, to decide whether a *wfg* is or is not an Alpha theorem. Given a *wfg* G, just:

- **(i):** *deiterate* all propositional letters up to enclosures where they can be no more deiterated;
- **(ii):** *erase* all double cuts;
- **(iii):** turn back to (i).

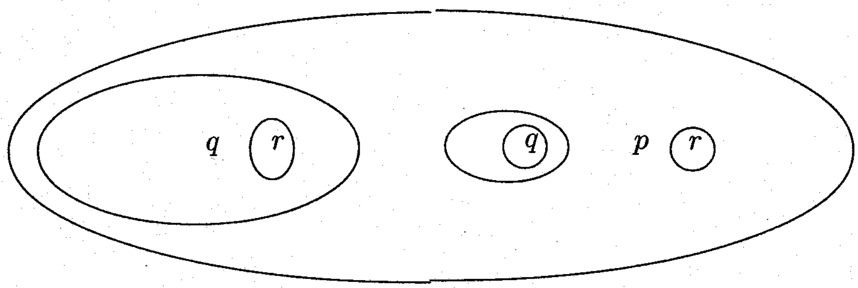
Then, G is an Alpha theorem if and only if at some stage of this recursive process G is reduced to the form  (any *wfg**).
For example, considering the graph



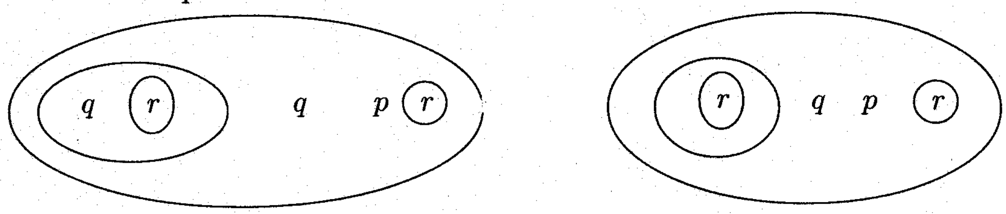
step (i) is not possible at first, but one proceeds to step (ii) to obtain:



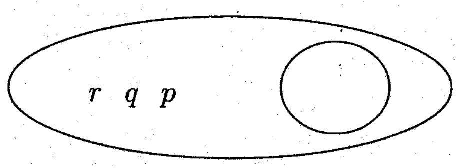
then, turning back to step (i), one can deiterate (twice) p:



then, (step (ii)) erase the double cut around q, and, again (step (i)), deiterate q:

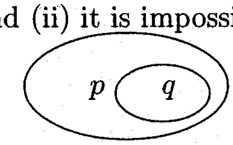
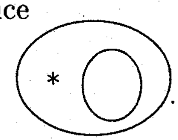


Finally, erasing the double cut around r (step (ii)) and deiterating r (step (i)), one obtains

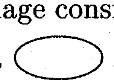


as desired. The *wfg* is thus an Alpha theorem, as we had already remarked.

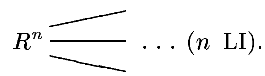
On the other hand, it is immediate to check, for example, that if we follow steps (i) and (ii) it is impossible to reduce

the graph  to a form .

Such a graph is not an Alpha theorem.

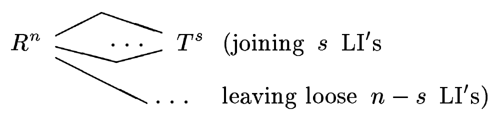
We now turn to Peirce's Beta system, an extension of the Alpha existential graphs which captures $\mathcal{L}_1^{\mathcal{R}}$: first-order logic with equality over a purely relational language. Beta's formal language consists of a set $\mathcal{R}$ of relation symbols (including a void), the cut  and the "identity line" (LI) ——. The set of of Beta's well-formed graphs (*bwfg*) is the minimal set closed recursively under the clauses:

(i): Any n-ary relation symbol joining n (≥ 1) lines of identity is a *bwfg*:



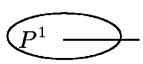
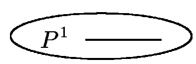
The void relation symbol produces the void *bwfg*. An LI alone is a *bwfg*.

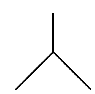
(ii): If R^n and T^s $(s \leq n)$ are relation symbols, then



is a *bwfg*.

(iii): If F and G are *bwfg*'s, then $F \quad G$ is a *bwfg*.

(iv): If F is a *bwfg*, then a graph constructed scribing a cut around any part of F is a *bwfg*. The cut can intersect identity lines but not relation symbols. As an example, both *bwfg*'s  and  scribing ovals around parts of P^1 ——— .

(v): If F is a *bwfg*, then a graph constructed with a branching of any LI in F is a *bwfg*. As an example, the *bwfg*  ("teridentity") arises from a branching of the LI ——— .

Beta's graphs are to be considered as continuous deformations of identity. In fact, Peirce's Beta system may be viewed as a topological calculus tuned to draw relational distinctions, implementing Leibniz's principle of indiscernibles ($x \neq y$ iff $\exists P \neg(P(x) \leftrightarrow P(y))$). As we will notice in sections 6 and 7, Peirce understood the logic of relatives ($\mathcal{L}_1^{\mathcal{R}}$) in a full continuity context, anticipating many fundamental contemporary ideas.

Beta's inference rules follow *exactly* the same pragmatic patterns allowed in Alpha — erasure/insertion, iteration/deiteration, double cut — but now for the *extended* language Beta. Thus, in addition to the rules A1-A5 where "graph" is now to be understood as *bwfg*, those rules provide new *derived* rules concerned with the LI's (for an extended analysis, including additions and variations, see [32, pp.134-150]). The following derived rules may be considered as the main features of the Beta system:

B1: ("Erasure") Any evenly enclosed portion of an LI may be erased.

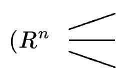
B2: ("Insertion") Any portions of LI's may be joined in an oddly enclosed region.

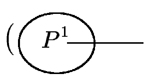
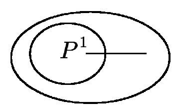
B3: ("Continuous iteration") Any LI may be extended towards strict regions. Any LI may branch in its region.

B4: ("Continuous deiteration") Any LI may be retracted towards regions with lesser cuts.

Allowing continuous deformations of identity lines — through a combinatorics of erasures, joinings, branchings, extensions and retractions — Beta can then be recognized as a *continuity* logical calculus. Compared with Alpha, where iteration and deiteration should be called "discrete", the key of Beta's system lies in the continuous spreading (extension/retraction) of the identity line.

We extend the natural translation (*) between Alpha and CPC, to a translation Beta-*bwfg* $\xrightarrow{*}$ $\mathcal{L}_1^{\mathcal{R}}$-*wff* by means of the following clauses:


$)^* = \exists x_1 \ldots \exists x_n((R(x_1, \ldots, x_n))$ (—— $)^* = \exists x \exists y \, (x = y)$


$)^* = \exists x \neg P(x)$ [thus: (

$)^* = \neg \exists \neg P(x) \equiv \forall x \, P(x)$].

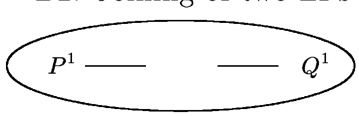
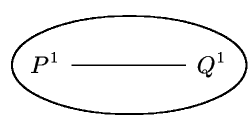
Then, the following samples of derived rules B1-B4 correspond to well-known normal procedures of the existential quantifier in first-order logic:

B1. Erasure of a portion of LI evenly enclosed:

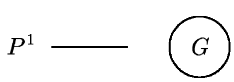
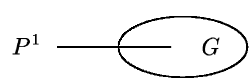
P^1 —— Q^1 ⇛ P^1 —— —— Q^1 : $\exists x \, (Px \wedge Qx) \vdash \exists x Px \wedge \exists x Qx.$

Notice that the converse does not hold, since LI's cannot be joined in even regions.

B2. Joining of two LI's oddly enclosed:

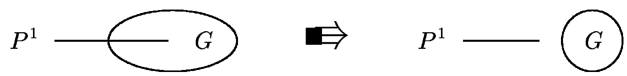

⇛

: **B1**-contrapositive

B3. Continuous extension towards strict regions:


⇛

: $(\exists x \, Px) \wedge \neg G \vdash \exists x (Px \wedge \neg G)$

Contrary to what happens in $\mathcal{L}_1^{\mathcal{R}}$, no restrictions are required in Beta on G, because all variables are automatically bounded in the *-translations of *bwfg*'s.

B4. Continuous retraction towards regions with lesser cuts:



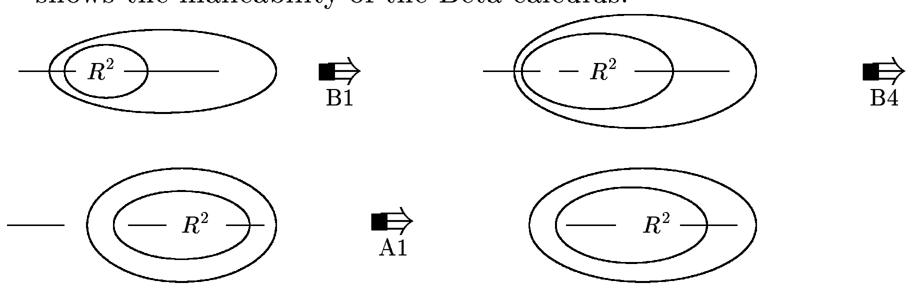
$: \ \exists x(Px \wedge \neg G) \vdash (\exists x\, Px) \wedge \neg G$

Thus, it can be seen that the erasure-insertion-iteration-deiteration Beta-calculus, applied continuously to the line of identity, corresponds to a set of first-order normal forms, interchanging quantifiers and connectives ($\exists$ and $\wedge$ — equivalently $\forall$ and $\vee$). In the enhanced setting of Peirce's continuum, we will further discuss in section 7 the plastic transformations of the existential graphs: a sort of *synthetical continuation* of logical relations, evoking (not just metaphorically, as we will see) the "analytical continuation" of holomorphic functions.

A simple example (inversion of quantifiers):

$$\exists x \forall y R(x, y) \vdash \forall y \exists x R(x, y)$$

shows the malleability of the Beta-calculus:



Notice that the converse does not hold (as expected), since the first step of the derivation cannot be reversed (the two LI's in the twice-enclosed region cannot be joined back).

Specifying a few additional Beta rules — *derived* from the *generic* understanding of continuous iteration/deiteration — and adding the existence of an identity line as an axiom (the *unique* axiom of the system), it is possible to prove the full equivalence of Beta and first-order relational logic [30, 31, 35, 33, 32]:

Completeness.

$$\underset{\text{Beta}}{\Rrightarrow} \ G \qquad \text{if and only if} \qquad \vdash_{\mathcal{L}_1^{\mathcal{R}}} \ G^*$$

As a result, we can see that we are in presence of a truly remarkable fact, unnoticed in the usual presentations of classical propositional and first-order logic: the deep observation that *both* CPC and $\mathcal{L}_1^{\mathcal{R}}$ can be axiomatized by a *same set of uniform generic* rules which govern the classical transfers of information (erasure/insertion; iteration/deiteration; double cut), providing in Alpha's context the *discrete* rules A1-A5, and providing in the extended LI-Beta language the *continuous* rules B1-B4 (as well as other minor derived rules). This is what Peirce called, with due pride, a full "apology of pragmaticism":

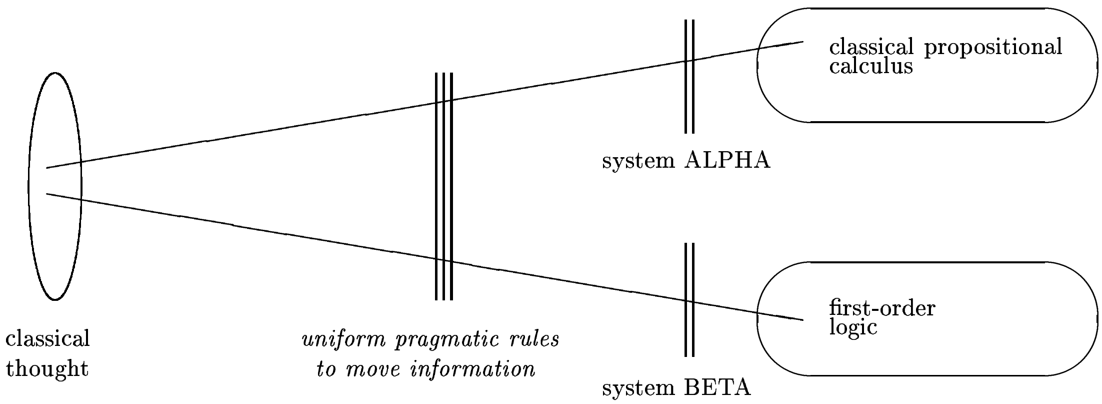
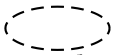
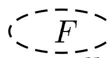


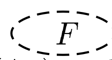
Figure 2.
"Apology of Pragmaticism": generic roots for CPC and $\mathcal{L}_1^{\mathcal{R}}$

With the Gamma existential graphs, Peirce extended the representability range of graphs, to include classical modalities, second-order logic and metalanguage. For a full treatment, see [35, 31]; here we deal only with modal Gamma. As we will see next, Peirce constructed a non-Cantorian continuum, where modal considerations were crucial. Together with the main place taken by modality in Peirce's later philosophy — supporting the edifice's architectonics and allowing the search for "proofs of pragmaticism" — the importance of the Gamma graphs can be better understood. In Peirce's words in his Lowell Lectures:

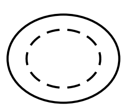
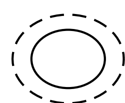
> In the gamma part of the subject all the old kinds of signs take new forms. . . . Thus in place of a sheet of assertion, we have a book of separate sheets, tacked together at points, if not otherwise connected. For our alpha sheet, as a whole, represents simply a universe of existent individuals, and the different parts of the sheet represent facts or true assertions made concerning that universe. At the cuts we pass into other areas, areas of conceived propositions which are not realized. In these areas there may be cuts where we pass into worlds which, in the imaginary worlds of the outer cuts, are themselves represented to be imaginary and false, but

> which may, for all that, be true, and therefore continuous with the sheet of assertion itself, although this is uncertain. (1903) [22, §4.512].

Gamma's modal formal language includes an additional "broken cut"  to represent contingency: a graph surrounded by a broken cut is to be understood as "conceived" but not yet realized, not yet fully scribed or discarded from the sheet of assertion. Gamma's language can extend either Alpha (to obtain intermediate modal propositional calculi) or Beta (to obtain modal first-order logic). As an extension of Alpha, the set of Gamma's well-formed graphs (*mwfg*) is the minimal set closed recursively under Alpha's clauses, and the additional clause: if F is a *mwfg*, then 

is a *mwfg*. The translation of existential well-formed graphs into Hilbert-style formulas is extended with the expected contingency interpretation: Gamma-*mwfg* $\xrightarrow{*}$ Modal-*wff* where one defines (

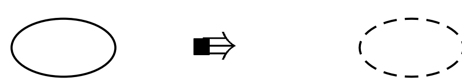
$)^* = \Diamond\neg F^*$. Assuming a classical basis, where $\neg\neg \equiv id$ (A5) and where necessity and possibility are interdefinable ($\Box \equiv \neg\Diamond\neg$), we can see that the *mixed double cuts* represent the main modalities:

 necessity  possibility.

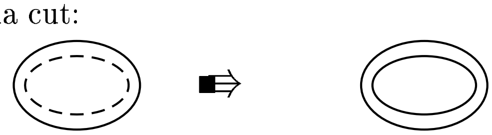
Even and odd regions are obtained as before, but now counting *indistinctly*

Alpha (full) cuts and Gamma (broken) cuts. The *generic* erasure and insertion properties provide, in the Gamma realm, new modal derived rules:

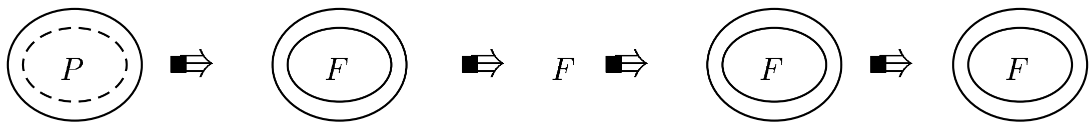
G1: In an even area, any Alpha cut may be *half-erased* to become a Gamma cut:



G2: In an odd area, any Gamma cut may be *half-completed* to become an Alpha cut:



Coupled with the Alpha double cut rule (A5), G1 provides immediately the $^{*^{-1}}$ translation of the modal T axiom $\Box p \longrightarrow p$, while G2 insures $p \longrightarrow \Diamond p$:



$$\Box\ F^* \ \vdash_T \ \neg\neg\ F^* \ \vdash_{CPC} \ F^* \ \vdash_{CPC} \ \neg\neg\ F^* \ \vdash_T \ \Diamond\ F^*$$

Additional to these Gamma cut derived rules, Zeman showed in his unpublished Ph.D. dissertation [35] that other Gamma rules — pertaining to restrictions on eventual *iterations* along *broken* cuts — give rise to intermediate modal Gamma systems, equivalent to the usual Lewis S systems. As an example, the allowance of iterating *necessary* graphs along broken cuts (together with their corresponding deiterations) gives rise to a Gamma system equivalent to S4 [35, pp.162-168].

The fact that a graphic methodology of continuous glueings can be formalized in order to capture — over *common generic pragmatic rules* — both a calculus of quantification (along joins and extensions of identity lines) and several calculi of modalities (along completions of broken cuts) turns out to be surprisingly deep. Peirce opened up for us a wide panorama that has still to be fully explored, understood and made good use of. We think that one of the keys for a better understanding of the wider picture in which fit Peirce's existential graphs lies in Peirce's non-Cantorian continuum, which we now present.

5. Peirce's Continuum: Genetic Approach

Peirce's reflections on the continuum can be traced back up to 1873, when he adopts Kant's definition — "a continuum is defined as something any part of which however small itself has parts of the same kind" — confounding it in first instance, as Kant himself did, with infinite divisibility [23, vol.3, p.103]. Nevertheless, it is around 1895 that the interest for the continuum becomes central in Peirce's large panorama of knowledge, when he begins to construct the entire architectonics of his systematic philosophy, based on his wide conception of logic as semiotics, on his full modalization of the pragmatic maxim, on his triadic classifications of sciences, and on the unveiling of his three *cenopythagorean* categories in all domains of thought and nature. Contemporary to Peirce's original readings of the Greek masters [9] —

Aristotle paving the way to modality and continuity, leading Peirce to *synechism*: the philosophy that continuity is operative in nature; Epicurus opening the crack of spontaneity, leading Peirce to *tychism*: pure chance operative in nature —, the five year period 1895-1900 becomes essential for Peirce's mature understanding of continuity [20].

Peirce's continuum is an "absolutely general" concept which intrinsically lies inside *any* other universal concept, a lean, *free* concept in the realm of the general and the possible that cannot be bounded by a given collection. Leaving free the determination contexts of the continuum — its partial extensions — and insisting on the intensionality of the continuum as a general, Peirce obtains one of the deeper peculiarities in his vision of the continuum: an original and extremely important *asymmetrization* of the abstraction principle, in which intension and extension are *not* logically equivalent, neither globally (Frege) nor locally (Zermelo). Contrary to Cantor's approach, Peirce understands *synthetically* the continuum as a *generic universal* which cannot be analytically reconstructed by an internal sum of points:

> Across a line a collection of blades may come down simultaneously, and so long as the collection of blades is not so great that they merge into one another, owing to their supermultitude, they will cut the line up into as great a collection of pieces each of which will be a line, — just as completely a line as was the whole. This I say is the intuitional idea of a line with which the synthetic geometer really works, — his virtual hypothesis, whether he recognizes it or not; and I appeal to the scholars of this institution where geometry flourishes as all the world knows, to cast aside all analytical theories about lines, and looking at the matter from a synthetical point of view to make the mental experiment and say whether it is not true that the line refuses to be cut up into points by any discrete multitude of knives, however great. (c.1897) [24, vol.3, p.96].

In fact, in a view similar to Veronese's (1891) [8], Peirce states that the continuum cannot even be captured by number lattices ("number cannot possibly express continuity" (c.1897) [24, vol.3, p.93]). Beyond the classical program of arithmetization of analysis, Peirce anticipates some sort of geometrization of logic. Not sensitive to Cantor's ordinal scale, constructed to progressively cut down the continuum, and probably not having understood Cantor's construction of the Alephs as ordinal classes, Peirce soon diverges from Cantor's approach. As we will see in section 6, Peirce insists on several fundamental characteristics that a continuum should hold, none of them shared by Cantor's real line. Against the main set-theoretic stream of the Cantorian heritage, and based on some "vague" ideas not fully transformed into mathematical material, it is no wonder that Peirce's continuum remained in oblivion. Nevertheless, profiting from many advances in xx$^{\text{th}}$ century mathematics, time may now be ripe to recover Peirce's logic of continuity, as we will hint in sections 7 and 8.

6. Peirce's Continuum: Succinct Modern Presentation

The next diagram encompasses the most salient traits of Peirce's continuum, understood unitarily as a synthetical concept where are entangled three crucial *global* properties (genericity, reflexivity, modality), three sub-determinations of those properties (supermultitudeness, inextensibility, plasticity) and four *local* methodologies (generic relationality, vagueness logic, neighbourhood logic, possibilia surgery), which can weave, in local contexts, the global architecture:

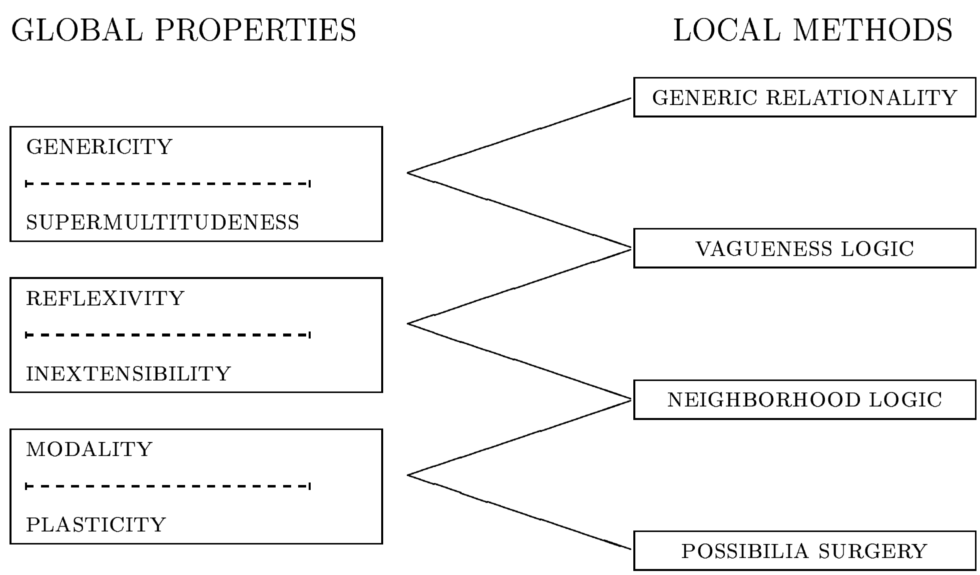


Figure 3.

The "double sigma": global and local concepts which articulate Peirce's continuum

The *double sigma* underlines some fundamental threads between global and local aspects of Peirce's continuum. The terminology tries to evoke Watson and Crick's "double helix", a double staircase of interlaced spirals where genetic information sums up. As the double helix codifies a fundamental part of the secrets of the living, the *double sigma* wishes to synthesize part of the fundamental secrets of the continuum. A vertical reading — a pragmatic reading — of the *double sigma*, gives rise to two important programs of research, which we will call *pragmae* of the continuum, and whose full elucidation would need "long duration" inquiries inside our "community of researchers": the construction of a *categorical topics*, which would systematically study the global synthetic correlations between "sites" of knowledge, and the construction of a *modal geometry*, which would study the local connection methods between those sites and detect its modal "invariants".

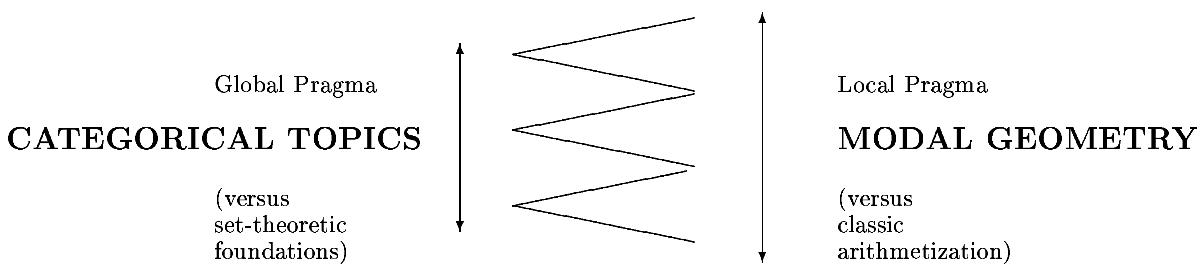


Figure 4.
"Pragmae" of the Continuum

As we will hint in section 7, xx^{th} century mathematics, independently of Peirce, will advance in the construction of a far-reaching "categorical topics", obtaining many outstanding but somewhat isolated technical results. On the other hand, the construction of a "modal geometry" is just beginning in the last decade. One of the many legacies of Peirce's continuum consists in interweaving coherently the two preceding *pragmae*, finding systematically reflections of the global into the local, and *vice versa*.

Perhaps the most salient trait of Peirce's continuum is its *general* character, with all the connotations and derivations that the term includes. In Peirce, the general includes very diverse nuances, but all united under an idea of "freeness" — whatever is free of particularizing attachments, determinative, existential or actual. The general is what can live in the realm of *possibilia*, neither determinate nor actual, and which opposes the particular mode of the existential. Generality — as a law or regularity beyond the merely individual, as a deep layer of reality beyond the merely named, as a basic weapon in the dispute between realism and nominalism — falls into Peircean thirdness and glues naturally together with the continuum. Peirce recalls several times that the continuum can be seen as a certain form of generality:

> The continuum is a General. It is a General of a relation. Every General is a continuum vaguely defined. (1902) [24, vol.3, p.925].

> The possible is general, and continuity and generality are two names for the same absence of distinction of individuals. (1897) [22, §4.172].

> A perfect continuum belongs to the genus, of a whole all whose parts without any exception whatsoever conform to one general law to which same law conform likewise all the parts of each single part. *Continuity* is thus a special kind of *generality*, or conformity to one Idea. More specifically, it is a *homogeneity*, or generality among all of a certain kind of parts of one whole. Still more specifically, the characters which are the same in all the parts are a certain kind of relationship of each part to all the coordinate parts; that is, it is a *regularity*. (1908) [22, §7.535 note 6].

The continuum is thus a *general*, where all the potentialities can fall — overcoming all determinations — and where certain modes of connection between the parts and the whole (local and global) become homogenized and regularized — overcoming and melting together all individual distinctions. The generic character of Peirce's continuum (thirdness) is thus closely woven with the overcoming of determinacy and actuality (secondness). In this process the threads of indetermination and chance (firstness) become essential, freeing the existent from its particular qualities in order to reach the generality of *possibilia*. For Peirce, the logic of relatives is the *natural filter* which allows one to free and sift out action-reaction agents, in order to melt them in a higher general continuity, because relative logic allows one to observe the individual as a "degenerate" form of relationality and the given as a degenerate form of possibility:

> Continuity is simply what generality becomes in the logic of relatives. (1905) [22, §5.436].

> Continuity is shown by the logic of relations to be nothing but a higher type of that which we know as generality. It is relational generality. (1908) [22, §6.190].

> The continuum is all that is possible, in whatever dimension it be continuous. But the general or universal of ordinary logic also comprises whatever of a certain description is possible. And thus the continuum is that which the logic of relatives shows the true universal to be. (1898) [24, vol.4, p.343].

Peirce's *dictum* '*continuity* = *genericity* via *relative logic*' is one of his most astonishing intuitions. In a first approach, it appears as a pretty cryptic, "occult" motto, but it really may be considered as a genial hypothesis (abduction), underlying the introduction of topological methods in logic and summarizing the proof that many of the fundamental theorems of the logic of relatives are no more than corresponding continuity theorems in the uniform topological space of first-order logic elementary classes. In fact, some theorems of Caicedo [6, 7] — on global continuous operations which codify structural properties of extensions of first order classical logic — yield an illuminating perspective on Peirce's fundamental weaving between generality, continuity and relative logic. Caicedo shows that general axioms in abstract logics coincide precisely with continuity requirements on algebraic operations between model spaces, and he establishes an extensive list of correspondences between topological and logical properties, many of them based on the discovery that *uniform continuity* of natural operations between structures hide strong logical contents. Caicedo's theorems elucidate Peirce's '*continuity* = *genericity* via *relative logic*' since the "general" (axioms of abstract model theory), filtered through the web of relative logic (first order classical logic), yields a natural continuum (uniform topological space by way of "local" elementary equivalence; uniform continuity of logical operations in that web: projections, expansions, restrictions, products, quotients, exponentials).

Peirce's outstanding *dictum* — clearly explicit from 1898 on, and perhaps one of the firmer expressions of Peirce's logical refinement — may have been based on two previous, crucial, logical "experiments": on one side, on his *existential graphs*, where, as we have seen, the rules of logic happen to be back-and-forth processes on the continuity of the sheet of assertion (discrete back-and-forth for the propositional calculus, and *continuous* back-and-forth for the logic of relatives — remember the continual elongations of the identity line); on the other side, on his neglected invention of *infinitesimal relatives*, in the never dried-out memory of 1870 on the logic of relatives [25] where Peirce revealed extremely interesting structural similarities between formal processes of differentiation (over the usual mathematical continuum) and operational processes of relativization (over a much more general logical continuum).

An immediate consequence of the genericity of the continuum is that the continuum must be *supermultitudinous*, in the sense that its size must be fully generic, and cannot be bounded by any other size actually determined:

> A *supermultitudinous* collection [...] is greater than any of the single collections. [...] A supermultitudinous collection is so great that its individuals are no longer distinct from one another. [...] A supermultitudinous collection, then, is no longer *discrete*; but it is *continuous*. (c.1897) [24, vol.3, pp.86-87].

> A supermultitudinous collection sticks together by logical necessity. Its constituent individuals are no longer distinct and independent subjects. They have no existence, — no hypothetical existence —, except in their relations to one another. They are no subjects, but phrases expressive of the properties of the continuum. (c.1897) [24, vol.3, p.95].

The supermultitudinous character of Peirce's continuum shows, according to Peirce, that the Cantorian real line is just "the first embryon of continuity", "an incipient cohesiveness, a germinal of continuity". Nevertheless, the cardinal indetermination ($2^{\aleph_0}$) of Cantor's continuum inside ZF — a profound discovery of xx$^{\text{th}}$ century mathematical logic that Peirce could not imagine — shows that the Cantorian model can also be considered as a valid generic candidate to capture the supermultitudeness of the continuum (even if other generic traits of Peirce's continuum, in the extensible or modal realm, do not seem capable of being modeled by the Cantorian real line). In any case, from the very beginning of their investigations, Cantor's and Peirce's paths are clearly opposed: while Cantor and, systematically, most of his followers, try to *bound* the continuum, Peirce tries to *unbound* it: to imagine a supermultitudinous continuum, not restricted in size, truly generic in the transfinite, never totally determined. It comes then, as a most remarkable fact, that many indications of the indeterminacy of the continuum found at the core of contemporary Cantorian set theory (free analysis of the set theoretic universe through disparate filters, using forcing techniques, with many *phenomena* possibly coexistent) seem to assure in retrospect the correctness of Peirce's vision. The generality of Peirce's continuum implies, as we shall now see, that it cannot be reconstructed from the "particular" or the "existent", and that it must be thought of in the true general realm of *possibilia*.

A fundamental property of Peirce's continuum consists in its *reflexivity*, a finely grained approach to Kant's conception that the continuum is such that any of its parts possesses in turn another part similar to the whole:

> A continuum is defined as something any part of which however small itself has parts of the same kind. (1873) [23, vol.3, p.103].

We will use the term "reflexivity" for the preceding property of the continuum since, following a reflection principle, the whole can be reflected in *any* of its parts:

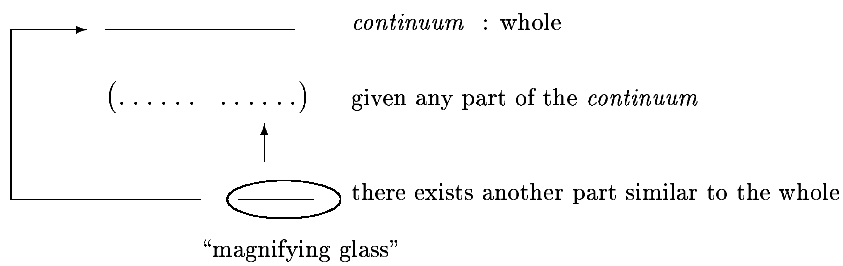


Figure 5.
The reflexivity of Peirce's continuum

As immediately infers Peirce (see next citation), reflexivity *implies* that the continuum cannot be composed of points, since points — not possessing other parts than themselves — cannot possess parts similar to the whole. Thus, reflexivity distinguishes at once the Peircean continuum from the Cantorian, since Cantor's real line *is* composed of points and *is not* reflexive. In Peirce's continuum the points disappear as actual entities (we shall see that they remain as *possibilities*) and are replaced — in actual, active-reactive secondness — by *neighbourhoods*, where the continuum flows:

> The result is, that we have altogether eliminated points. [...] There are no points in such a line; there is no exact boundary between any parts. [...] There is no flow in an instant. Hence, the present is not an instant. [...] When the scale of numbers, rational and irrational, is applied to a line, the numbers are insufficient for exactitude; and it is intrinsically doubtful precisely where each number is placed. But the environs of each number is called a point. Thus, a point is the hazily outlined part of the

> line whereon is placed a single number. When we say *is* placed, we mean *would be* placed, could the placing of the numbers be made as precise as the nature of numbers permits. [24, vol.3, pp.126-127].

Calling *inextensibility* the property which asserts that a continuum cannot be composed of points, it turns out that a continuum's reflexivity implies its inextensibility (Peirce's continuum is reflexive, thus inextensible), or, equivalently, its extensibility implies its irreflexivity (Cantor's continuum is extensible, thus irreflexive). The fact that Peirce's continuum cannot be extensible, not being able to be captured extensionally by a sum of points, retrieves one of the basic precepts of the Parmenidean *One*, "immovable in the bonds of mighty chains", a continuous whole which cannot be broken, "nor is it divisible, since it is all alike, and there is no more of it in one place than in another, to hinder it from holding together, nor less of it, but everything is full of what is" [21]. The impossibility to fully express the continuum through number grilles is a *natural limitation* which shows that, in order to obtain a finer understanding of the continuum, the program of *classical arithmetization* of the real line (Weierstrass, Cantor) should be complemented with the new "pragma" that Peirce's writings suggest: the construction of a *modal geometrization* of the continuum.

The Aristotelean influence — following Aristotle's use of a wide spectrum of possibilities to cover all reams of reality — weighs in in Peirce's approach to the continuum, when he presents the continuum as a complex modal *logos*:

> You have then so crowded the field of possibility that the units of that aggregate lose their individual identity. It ceases to be a collection because it is now a continuum. [...] A truly continuous line is a line upon which there is room for any multitude of points whatsoever. Then the multitude or what corresponds to multitude of possible points, — exceeds all multitude. These points are pure possibilities. There is no such gath. On a continuous line there are not really any points at all. (1903) [24, vol.3, p.388].

> It seems necessary to say that a continuum, where it *is* continuous and unbroken, contains no definite parts; that its parts are created in the act of defining them and the precise definition of them breaks the continuity. [...] Breaking grains of sand more and more will only make

> the sand more broken. It will not weld the grains into unbroken continuity. (1903) [22, §6.168].

The great richness of real and general possibilities far exceeds the "existent" realm and forms a "true" continuum, on which the existent must be seen as a certain type of *discontinuity.* "Existence as rupture" is another amazing Peircean intuition, which anticipates by a century Weinberg's ruptures of the symmetry principle, continuity breakdowns that help to explain in contemporary physics the cosmos's evolution. In Peirce's vision, while points can "exist" as discontinuous marks *defined* to anchor action-reaction number scales on the continuum, the "true" and steady components of the continuum are generic and *indefinite* neighbourhoods, interwoven in the realm of *possibilia* without actually marking its frontiers. The process which presupposes a general being prior to the emergence of existence seems to be akin to the genetic structure of Peirce's continuum: just like Brouwer, Peirce postulates the possibility of conceiving *previously* a global continuum ("higher generality"), on which marks and number systems are introduced *subsequently* to mimic locally the general continuum (this becomes particularly clean in Peirce's existential graphs). As Peirce clearly suggests, the infinite breaking of grains of sand never achieves them fully merging into one another: a synthetic vision of the continuum (Peirce, Brouwer) has to be given previously to its analytical composition (Cantor).

Peirce's continuum — understood as a synthetic range where whatever is possible should be able to glue — has to be a general place (*logos*), extremely flexible, plastic, homogeneous, without irregularities:

> The perfect third is plastic, relative and continuous. Every process, and whatever is continuous, involves thirdness. (1886) [23, vol.5, p.301].

> The idea of continuity is the idea of a homogeneity, or sameness, which is a regularity. On the other hand, just as a continuous line is one which affords room for any multitude of points, no matter how great, so all regularity affords scope for any multitude of variant particulars; so that the idea [of] continuity is an extension of the idea of regularity. Regularity implies generality. [22, §7.535].

> A perfect continuum belongs to the genus, of a whole all whose parts without any exception whatsoever conform to one general law to which same law conform likewise

> all the parts of each single part. Continuity is thus a special kind of generality, or conformity to one Idea. More specifically, it is a homogeneity, or generality among all of a certain kind of parts of one whole. Still more specifically, the characters which are the same in all the parts are a certain kind of relationship of each part to all the coordinate parts; that is, it is a regularity. The step of specification which seems called for next, as appropriate to our purpose of defining, or logically analyzing the Idea of continuity, is that of asking ourselves what kind [of] relationship between parts it is that constitutes the regularity a continuity; and the first, and therefore doubtless the best answer for our purpose, not as the ultimate answer, but as the proximate one, is that it is the relation or relations of contiguity; for continuity is unbrokenness (whatever that may be) and this seems to imply a passage from one part to a contiguous part. (1908) [22, §7.535, note 6].

Peirce's continuum is general, plastic, homogeneous, regular, in order to allow, in a *natural* way, the "transit" of modalities, the "fusion" of individualities, the "overlapping" of neighbourhoods. A generic *continuous flow* is present behind those transits, fusions and overlappings — ubiquitous osmotic processes. Peirce's continuum — generic and supermultitudinous, reflexive and inextensible, modal and plastic — is the global conceptual *milieu* where, in a natural way, we can construct hierarchies to bound possible evolutions and local concretions of arbitrary flow notions. In the remainder of this section we will show how to deal with those constructions, studying some of the local methods that Peirce devised to begin to control the specific "passages" of genericity, reflexivity and modality, and completing thus a succinct overview of the "double sigma" interpretation of Peirce's continuum (figure 3).

True discoverer of all the potentiality lying in the logic of relatives, Peirce applies the strength of that logical lens to the problem of approaching locally the continuum. Turning to the genericity of the continuum, Peirce notices that the "mode of connection" of the parts must be understood in full generality, involving a genuine triadic relation, and he opens thus the way to a study of *generic triadic relations*, closely tied with "general modes" of smoothness and contiguity:

> My notion of the essential character of a perfect continuum is the absolute generality with which two rules hold good, first, that every part has parts; and second,

> that every sufficiently small part has the same mode of immediate connection with others as every other has. (1908) [22, §4.642].

> No perfect continuum can be defined by a dyadic relation. But if we take instead a triadic relation, and say A is r to B for C, say, to fix our ideas, that proceeding from A in a particular way, say to the right, you reach B before C, it is quite evident that a continuum will result like a self-returning line with no discontinuity whatever... (1898) [22, §6.188].

These assertions show that Peirce is trying to find fitting reflections of the global into the local: the continuum — which in its "perfect generality" is one of the most achieved global forms of thirdness — must also embody a genuinely triadic mode of connection in the constitution of its local fragments.

Peirce's continuum, as a general, is indeterminate. Along what we could call indetermination "fibers", the general reacts antithetically with the "vague":

> Logicians have too much neglected the study of vagueness, not suspecting the important part it plays in mathematical thought. It is the antithetical analogue of generality. A sign is objectively general, in so far as, leaving its effective interpretation indeterminate, it surrenders to the interpreter the right of completing the determination for himself. "Man is mortal." "What man?" "Any man you like." A sign is objectively vague, in so far as, leaving its interpretation more or less indeterminate, it reserves for some other possible sign or experience the function of completing the determination. "This month," says the almanac-oracle, "a great event is to happen." "What event?" "Oh, we shall see. The almanac doesn't tell that." (c.1905) [22, §5.505]

We refer to the next figure for a visual image of the situation. To an important degree, the study of generality can be seen as the study of the universal quantifier ("*any* man"), while the study of vagueness is the study of the existential quantifier ("*a* great event"). As we will recall in section 7, an explicit adjunction, or evolving antithesis, between genericity ($\forall$) and vagueness ($\exists$) was to be found, and precisely studied, by another great American mathematician in the 1960s.

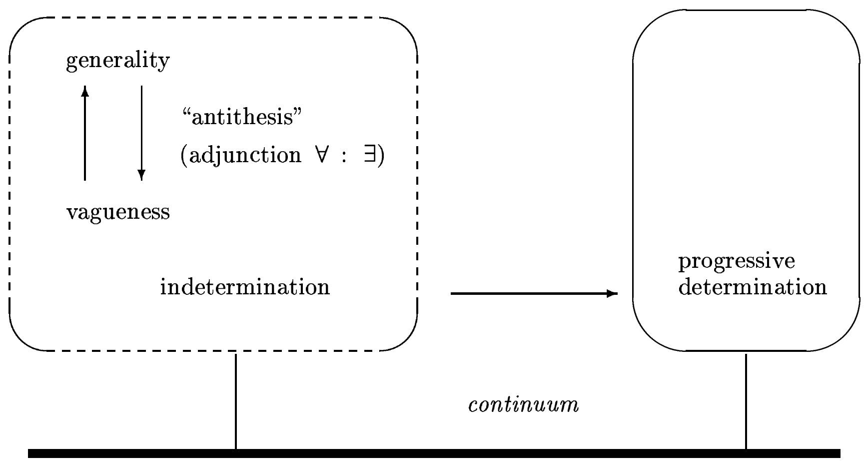


Figure 6.
Generality - vagueness "adjunction"
in the determinate "filters" of the continuum

Peirce's *logic of vagueness* [4] hopes to control the transit of the indefinite to the definite, of the indeterminate to the determinate, and to study some intermediate borders in processes of relative determination. Prior to this horizontal control, nevertheless, Peirce discovered the basic vertical antithesis '*genericity* vs. *vagueness*' whose partial resolutions were to pave the way to the construction of intermediate logical systems. The "antithesis", when applied locally to the continuum, weaves closely a scheme of general connection modes, *naturally intermediate*:

> A point of a surface may be in a region of that surface, or out of it, or on its boundary. This gives us an indirect and vague conception of an intermediary between affirmation and denial in general, and consequently of an intermediate, or nascent state, between determination and indetermination. There must be a similar intermediacy between generality and vagueness. (1905) [22, §5.450].

Mathematical logic in the XX$^{\text{th}}$ century would show that the *natural* logic associated to the connecting modes of the continuum is really an intermediate logic — the intuitionistic logic — in which the principle of excluded middle does *not* hold. It is thus amazing that Peirce — following general paths in his architectonics, very distant from the technical demands that underlie intuitionistic constructive threads — could have been able to predict that an adequate logic for the continuum would have to abandon, in fact, the law of excluded middle:

> If we are to accept the common sense idea of continuity (after correcting its vagueness and fixing it to mean something) we must either say that a continuous line contains no points or we must say that the principle of excluded middle does not hold of these points. The principle of excluded middle only applies to an individual (for it is not true that "Any man is wise" nor that "Any man is not wise"). But places, being mere possibles without actual existence, are not individuals. Hence a point or indivisible place really does not exist unless there actually be something there to mark it, which, if there is, interrupts the continuity. (1903) [22, §6.168].

> I must show that the *will be*'s, the actually *is*'s, and the *have been*'s are not the sum of the reals. They only cover actuality. There are besides *would be*'s and *can be*'s that are real. The distinction is that the *actual* is subject both to the principles of contradiction and of excluded middle; and in *one* way so are the *would be*'s and *can be*'s. In *that* way a *would be* is but the negation of a *can be* and conversely. But in another way a *would be* is not subject to the principle of excluded middle; both *would be* X and *would be not* X may be false. And in this latter way a *can be* may be defined as that which is not subject to the principle of contradiction. On the contrary, if of anything it is only true that it *can be* X it *can be not* X as well. (c.1910) [22, §8.216].

In these two quotes, Peirce points out that the logic of actuality can be approached by usual classical logic, but that the "true" logic of continuity (to be applied to the dynamical flow of potential sites and not to the static condition of points) is a logic where the principle of excluded middle fails. In his rather difficult language of "vague" modalities ("can be": $\Diamond$; "would be": $\neg\Diamond$) , Peirce also relates generality and necessity (forms of thirdness), as well as vagueness and possibility (forms of firstness), and tries to *characterize* logically the former as *failures of distribution* of the excluded middle, as well as the latter as *failures of distribution* of the contradiction principle:

> The general might be defined as that to which the principle of excluded middle does not apply. A triangle in

> general is not isosceles nor equilateral; nor is a triangle in general scalene. The vague might be defined as that to which the principle of contradiction does not apply. For it is false neither that an animal (in a vague sense) is male, nor that an animal is female. (c.1905) [22, §5.505].

Failure of Excluded Middle:	Failure of Contradiction Principle:
$p \vee \neg p$ fails for the general ($\forall$) and for the necessary ($\Box$)	$\neg(p \wedge \neg p)$ fails for the vague ($\exists$) and for the possible ($\Diamond$)
$\not\models \forall x\, P \vee \forall x \neg P$	$\not\models \neg(\exists x\, P \wedge \exists x \neg P)$
$\not\models \Box p \vee \Box \neg p$	$\not\models \neg(\Diamond p \wedge \Diamond \neg p)$

Figure 7.

Generality and Vagueness do not distribute

In Peirce's view, a generic continuum is always present in the universe, reflected in multiple layers (single continuum / line of identity / line in the blackboard) and "meta-layers" (double continuum of qualitative possibilities / sheet of assertiaon / blackboard). Through acts of "brute force" are then produced breaks on the continuum which allow one to "mark" differences: secondness, existence , discreteness, emerge all as *ruptures* of the real, the third, the continuum (the Parmenidean "*One*") — evolve towards a logic of identity, more and more determined, capable of recording differences *by means of* successive breaks, ruptures, discontinuities. Since the evolution is not absolute, but contextual, nor achievable, but partial, the counterpoint between a continuous ground and discontinuity peaks becomes *saturated* only in certain given contexts.

Peirce's continuum is formed by superposed "real" environments and neighbourhoods — modes of fusion and connection of the *possibilia.* On that continuum "ideal" points are marked — cuts and discontinuities of the actual — only to construct contrasting scales and to facilitate the "calculus". An apparent oddity, which ties the real with the possible and the ideal with the actual, is one of the radical stakes of Peircean philosophy. Indeed, the actual, the given, the present, the instant, are no more than *ideal limits*: limits of possibility neighbourhoods which contain those actuality marks, those points impossible to be drawn, those fleeting presents, those impalpable instants. Accordingly, Peirce insists that the continuum must be studied — in a coherent approach with its inextensibility — by means of a *neighbourhood logic*: an intermediate logic which would study the connecting modes of *environments* of the real, a non-classical logic which would go beyond *punctual* "positive assertion and negation":

> I have long felt that it is a serious defect in existing logic that it takes no heed of the *limit* between two realms. I do not say that the Principle of Excluded Middle is downright *false*; but I *do* say that in every field of thought whatsoever there is an intermediate ground between *positive assertion* and *positive negation* which is just as Real as they. (1909) [10, p.180].

> A *continuum* (such as time and space actually are) is defined as something any part of which however small itself has parts of the same kind. Every part of a surface is a surface, and every part of a line is a line. The point of time or space is nothing but the ideal limit towards which we approach indefinitely close without ever reaching it in dividing time or space. To assert that something is true of a point is only to say that it is true of times and spaces however small or else that it is more and more nearly true the smaller the time or space and as little as we please from being true of a sufficiently small interval. (...) And so nothing is true of a point which is not at least on the limit of what is true for spaces and times. (1873) [23, §3.106].

> A drop of ink has fallen upon the paper and I have walled it round. Now every point of the area within the wall is either black or white; and no point is both black and white. That is plain. The black is, however, all in one spot or blot; it is within bounds. There is a line of demarcation between the black and the white. Now I ask about the points of this line, are they black or white? Why one more than the other? Are they (A) both black and white or (B) neither black nor white? Why A more than B, or B more than A? It is certainly true, First, that every point of the area is either black or white, Second, that no point is both black and white, Third, that the points of the boundary are no more white than black, and no more black than white. The

> logical conclusion from these three propositions is that the points of the boundary do not exist. That is, they do not exist in such a sense as to have entirely determinate characters attributed to them for such reasons as have operated to produce the above premises. This leaves us to reflect that it is only as they are connected together into a continuous surface that the points are colored; taken singly, they have no color, and are neither black nor white, none of them. Let us then try putting "neighboring part" for point. Every part of the surface is either black or white. No part is both black and white. The parts on the boundary are no more white than black, and no more black than white. The conclusion is that the parts near the boundary are half black and half white. This, however (owing to the curvature of the boundary), is not exactly true unless we mean the parts in the immediate neighborhood of the boundary. These are the parts we have described. They are the parts which must be considered if we attempt to state the properties at precise points of a surface, these points being considered, as they must be, in their connection of continuity. One begins to see that the phrase "immediate neighborhood," which at first blush strikes one as almost a contradiction in terms, is, after all, a very happy one. (1893) [22, §4.127].

Peirce's arguments show that talking of "points" in the boundary of the ink drop is just an "ideal" postulate. There exist *really* only colored environments in the paper, of three specific kinds: black, white, or black-*and*-white neighbourhoods. Boundary "points" are characterized as ideal entities which can only be approached by neighbourhoods of the third kind. Thus, neighbourhood logic — or "continuous coloring" logic — embodies elementary forms of thirdness and triadicity, and discards immediately the law of excluded middle. It should not come as a surprise, then, that Peirce, in attempts to construct triadic connectives [10], would become the first modern logician to construct truth-tables with intermediate truth-values.

In Peirce's continuum the neighbourhoods are *possibilia* environments (Putnam's "point parts" [28]), where a supermultitude of potential "points" accumulate. In many approaches to Peirce's continuum, those *possibilia* have been described as infinitesimal monads: around an actual mark on the continuous line stands a supermultitudinous myriad of infinitesimals [13]. It will be of prime concern to construct a "local surgery" in the geometry of those possibility realms, which may involve similar techniques to Whitney's surgery techniques in differential topology, and with which germs of possibility could be glued and deployed simultaneously. That *possibilia surgery* — still fully to be developed, but implicit in Peirce's approach (for example, pretty clear in the erasure and deiteration processes in the existential graphs) — should be able to naturally interweave with Thom's cobordism techniques (a "generic cobordism" should be part of a generic third) and with Thom's call [34] for an "archetypical" continuum — a "topos" qualitatively homogeneous — similar in many ways to Peirce's continuum.

We think that Peirce's continuum hooks up perfectly with Leibniz's "maximal principle", according to which the world articulates along the simplest hypothesis and the richest phenomena. Peirce's continuum covers, in fact, a huge phenomenal range, while it articulates only three simple concepts — genericity, reflexivity, modality — from which follows a wide spectrum of global and local characteristics.

7. Peirce's Logic of Continuity

Peirce's existential graphs reflect in precise ways the genericity, modality and reflexivity of Peirce's continuum. The existential graphs' generic rules — insertion / erasure, iteration / deiteration — are at the bottom of a common presentation of classical propositional and predicate calculi. The intermediate, pointed cut regions available in Gamma open the way to modalities. The iconic reflection of the continuity of the sheet of assertion on the line of identity (see next citations) shows how the only axiom of the system (LI) codifies already, not only its semantics (SA), but even a most generic universal continuum:

> The line of identity is, moreover, in the highest degree iconic. For it appears as nothing but a continuum of dots, and the fact of the identity of a thing, seen under two aspects, consists merely in the continuity of being in passing from one apparition to another. (c.1903) [22, §4.448].

> The line of identity [...] very explicitly represents Identity to belong to the genus Continuity and to the species Linear Continuity. But of what variety of Linear Continuity is the heavy line more especially the Icon in the System of Existential Graphs? In order to ascertain this, let us contrast the Iconicity of the line with that of

the surface of the Phemic Sheet. The continuity of this surface being two-dimensional, and so polyadic, should represent an external continuity, and especially, a continuity of experiential appearance. Moreover, the Phemic Sheet iconizes the Universe of Discourse, since it more immediately represents a field of Thought, or Mental Experience, which is itself directed to the Universe of Discourse, and considered as a sign, denotes that Universe. Moreover, it is [because it must be understood] as being directed to that Universe, that it is iconized by the Phemic Sheet. So, on the principle that logicians call "the *Nota notae*" that the sign of anything, X, is itself a sign of the very same X, the Phemic Sheet, in representing the field of attention, represents the general object of that attention, the Universe of Discourse. This being the case, the continuity of the Phemic Sheet in those places, where, nothing being scribed, no *particular* attention is paid, is the most appropriate Icon possible of the continuity of the Universe of Discourse — where it only receives *general* attention as that Universe — that is to say of the continuity in experiential appearance of the Universe, relatively to any objects represented as belonging to it. (1906) [22, §4.561, note 1].

Let the clean blackboard be a sort of diagram of the original vague potentiality, or at any rate of some early stage of its determination. This is something more than a figure of speech; for after all continuity is generality. This blackboard is a continuum of two dimensions, while that which it stands for is a continuum of some indefinite multitude of dimensions. This blackboard is a continuum of possible points; while that is a continuum of possible dimensions of quality, or is a continuum of possible dimensions of a continuum of possible dimensions of quality, or something of that sort. There are no points on this blackboard. There are no dimensions in that continuum. I draw a chalk line on the board. This discontinuity is one of those brute acts by which alone the original vagueness could have made a step towards definiteness. There is a certain element of continuity in this line. Where did this continuity come from? It is nothing but the original continuity of the blackboard which makes everything upon it continuous. (1898) [22, §6.203].

Peirce's speculation on the presence of an underlying, generic, continuum under most mathematical creations can be forcefully supported by other results obtained in mathematical category theory (second half of the xx^th century). Perhaps surprisingly, the first basic and fundamental theorem in category theory, *Yoneda's Lemma*, can be seen already as one of the key vaults that support a universal presence of continuity. Yoneda's Lemma shows that *any* "small" category can be faithfully embedded in a category of presheaves (functors to sets), where "ideal" (or "non standard") objects crop up to *complete* the universe, turning it continuous:

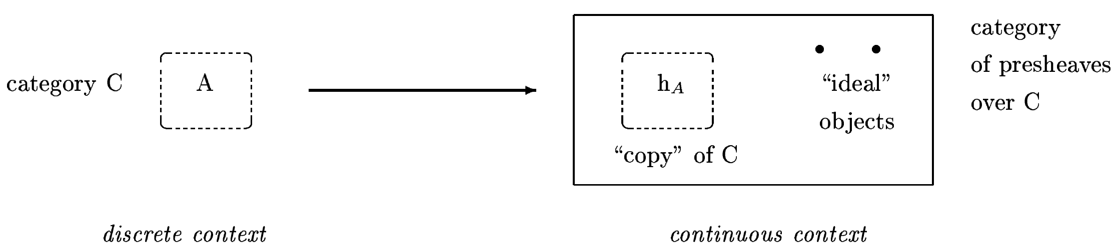


Figure 8.
Yoneda's Lemma: generic presence of the continuum

Diverse forms of continuity are hidden behind Yoneda's Lemma. *Representable functors* h_A symbolize all interrelations of A with its context and *preserve limits*: they are continuous. The presheaf category where the initial category is embedded is a *complete* category, in the sense that it possesses *all categorical limits*: it is therefore a natural continuous environment. Even deeper, Yoneda's Lemma is the natural tool to describe the *classifier objects* (semantic codifiers) in presheaf categories: the truth notions turn out to be — in a *natural* way — pragmatic notions, woven with the continuum where they lie. Explicit and unavoidable in Yoneda's Lemma, penetrating and permanent in any form of mathematical creativity, the emergence of "ideal" objects when the "real" is tried to be captured agrees with the peculiar mixture of realism and idealism present in Peirce's philosophy (and in Hilbert's late thought). The continuous bottom emerging in Yoneda's Lemma is yet another indication that Peirce's global synechism can count on amazing local reflections to support its likeliness.

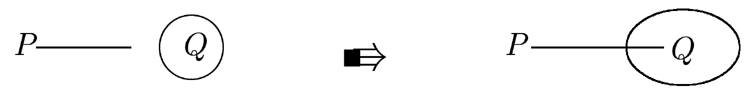
The "fundamental continuity of space" is also championed by Lawvere, one of the main contributors in the development of category theory, who remarks that "all attempts to characterize continuity in a purely intensive logical way, such as the frame algebra, leave another kind of room in spite of their profound contribution to calculations" and that "still more serious work is needed, marshalling all the achievements of subjective logic *and* objective logic, of geometry *and* algebra to hone still more realistic models of continuous spaces" [18, p.10]. If we understand "realistic" in its literal sense, to approach the "real", it is clear that Peirce's continuum — a scheme saturating all room of continuous possibilities and a key vault of Peirce's realism — should be of great interest to construct new models of continuity.

Lawvere's "geometrizing logic" program [16, 17] can be seen as a "unity of opposites" attempt, of course independent of Peirce, but very close to what we called Peirce's *pragma* of geometrizing modality. Lawvere's discovery and generic reconstruction of implicational logic (deduction theorem, *modus ponens*) as cartesian adjointness, of quantifiers ($\exists$, $\forall$) as closed cartesian adjoints (left, right) of projections, of logical algebras as subobject classifiers in elementary topoi, of modal operators as arrow topologies, show in a precise way that all main notions in propositional, predicate and modal calculi can be given a thorough geometric treatment, *precisely* what Peirce's existential graphs had also achieved in a deviant geometric direction.

Peirce's ***logic of continuity*** should be understood as a unifying program to coherently amalgamate (1) his existential graphs and some partial models of Peirce's continuum, in order to advance (2) a better understanding of geometric logic. We conjecture that category theory tools should be extremely useful achieving the first goal, and that some breakthroughs could be expected in the logic of sheaves and complex analysis when pointing to the second goal. Even if more questions (recorded in section 8) than answers are now available on these topics, in the remainder of this section we hint at some ideas that may support a fair development of Peirce's logic of continuity.

The Beta continuous iteration and deiteration of the line of identity may be seen as geometric expressions of Lawvere's categorical adjunction between existential generalization and instantiation:

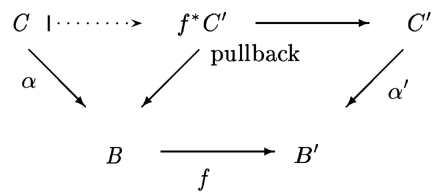
Beta iteration



$\mathcal{L}_1^{\mathbb{R}}$ $\exists$-generalization $\quad (\exists x\, Px) \wedge \neg Q \quad \vdash \quad \exists x(Px \wedge \neg Q)$

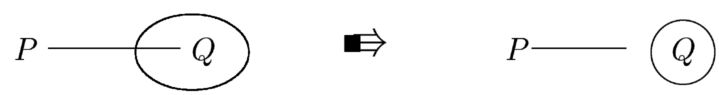
Pullback categories $\quad \exists_f\, \alpha \to \alpha' \quad \xrightarrow{\text{adjoint}} \quad \alpha \to f^*\alpha'$

where



$\exists_f\, \alpha = f \circ \alpha\,.$

Beta deiteration



$\mathcal{L}_1^{\mathbb{R}}$ $\exists$-instantiation $\quad \exists x\, (Px \wedge \neg Q) \quad \vdash \quad (\exists x\, Px) \wedge \neg Q$

Pullback categories $\quad \alpha \to f^*\alpha' \quad \xrightarrow{\text{adjoint}} \quad \exists_f\, \alpha \to \alpha'$

The heuristic categorical *motto* "existing $\equiv$ substituting", obtained from Lawvere's adjunction, corresponds thus to Beta's LI heuristics, where existing means continuously extending and retracting the line of identity, that is, substituting portions of the line on the underlying continuum of the sheet of assertion. Since the logic of quantifiers appears then deeply connected with a continuous bottom (recall also Caicedo's results [6, 7] mentioned in section 6), a study of Peirce's logic of continuity may benefit from some categorical settings where continuity crops up naturally, in a free and generic way. Such settings are obtained, for example, with Freyd's allegorical machinery [11] *(allegories* = abstract axiomatic categories of *relations*). In fact, Freyd's ubiquitous intermediate representation theorems:

	regular		$---$		regular
T *theory*	coherent	$\Longrightarrow A_T$ *allegory*	distributive	$\Longrightarrow$ $MapSplitCor(A_T)$ *category*	pre-logos
	first order		division		logos
	higher order		power		topos

show how continuity progressively *emerges* as genericity increases, yielding, in case T is pure type theory, the free topos $MapSplitCor(A_T)$ with NNO — obtained through the succesive filtrations of relationality (A_T), identity subsumption ($Cor(A_T)$), partial invertibility ($SplitCor(A_T)$) and functionality ($MapSplitCor(A_T)$) — a sort of initial *archetype* for continuity where Peirce's primigenial continuum may become partially approached in the future by nonstandard generic real number objects.

Two other trends in a further development of Peirce's logic of continuity may also be worth studying. On one side, the continuous iterations of the lines of identity may be seen as *analytic continuations* of functions of a complex variable, if we identify the sheet of assertion with the complex plane and the lines of identity with bounded affine complex functions. In this trend of thought it would have to be understood how the Alpha cuts could be interpreted as complex regions with *singularities*, allowing in some cases the analytic continuation, while prohibiting it in others. On the other side, the modal Gamma cuts may be seen as *ramification* zones, where Riemann surfaces deploy (recall Peirce's citation on a Gamma "book of separate sheets, tacked together at points"). In this way, a logical understanding of the geometry of complex variables could deliver new clues about Peirce's continuity logic (including Gamma modalities and Peirce's modal continuum), and, conversely, in what would be an original contribution of the highest order, some new developments in Peirce's continuity logic could help to uncover unsuspected logical interpretations of many theorems of complex analysis.

8. Perspectives and Future Work

In this section we list some open questions related to Peirce's logic of continuity. The questions are subdivided into three main subsections: (A). Existential graphs. (B). Peirce's continuum. (C). Blend graphs-continuum.

- **A1:** Develop the existential graphs classical calculi *in their own terms*, independently of their equivalences with classical propositional, predicate or modal calculi (*e.g.*, find a graphical decision procedure for monadic identity lines, discover graphical normal forms, unravel fully the "adjunction" insertion/iteration *vs.* erasure/deiteration, propose alternative axiomatizations, *etc.*). On the other hand, exhibit *shorter formal proofs* of the equivalences of the existential graphs calculi with their respective counterparts.

A2: Develop an *intuitionistic Alpha* (closer to the logic of continuity, since intuitionism approaches better many continuum aspects: topological representations, sheaves, topoi, *etc.*). Such an approach must leave aside the erasure of double cuts Alpha, as well as the interdefinability of classical connectives codified in Alpha, and must propose new natural symbols and rules.

A3: Develop *graphical semantics* for existential graphs, closer to the natural interpretations of the calculi and independent of the usual translations into classical propositional, predicate or modal calculi (*e.g.*, formalize Peirce's Gamma idea of a "book" of modal assertive sheets in place of Kripke's models, explore the potential of Beta's "neighbourhood logic", study a lattice-type "unwinding nests" semantics for Alpha, *etc.*).

B1: *Axiomatize* the *global* generic, reflexive and modal properties of Peirce's continuum. A towering scale of local set theoretical perspectives could be useful to obtain partial approaches to Peirce's continuum, but a category theoretic axiomatization (possibly obtained from a corresponding universal description in the free topos with NNO) would seem more feasible.

B2: Develop the *local methods* implicit in Peirce's continuum, particularly its "generic relationality" and "possibilia surgery" which could unravel nice mathematical connections with methods in differential geometry.

B3: *Compare formally* many of the features of Peirce's continuum with corresponding non-Cantorian approaches to the continuum (Veronese, Brouwer, Nelson, Conway, Vopenka, Lawvere, Thom, *etc.*).

C1: *Combine in a unified account* many of the intrinsic common characteristics of the existential graphs and Peirce's continuum (*e.g.*, the intermediate modal properties of the continuum and the intermediate "half-cut" regions in the continuous sheet of assertion, the continuum's reflexivity and the reflection of the sheet of assertion on identity lines, the genericity of the continuum and the generic "adjointness" rules for the existential calculi, *etc.*).

C2: Develop new *mathematical ideas* in Peirce's blend of existential graphs and non-Cantorian continuum (*e.g.*, the hinted "complex analysis" interpretation). In many aspects, it seems a virgin soil, strangely untouched by the amazing amount of mathematical advances obtained in the $\mathrm{xx}^{\mathrm{th}}$ century.

C3: *Formalize categorically* Peirce's existential graphs calculi and Peirce's continuum, the development of new mathematics around Peirce's logic of continuity being probably connected to its full synthetical representation in category theory.

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