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VIEWS ON THE REAL NUMBERS AND THE CONTINUUM

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1. Introduction

The history of the real numbers is related to many areas of mathematics, sometimes in a central way and sometimes in a tangential way. Such a pervasive topic is itself quite complex and can be thought of as three problems:

- What is a real number?
- Are the real numbers and the real line one and the same?
- What is a continuum?

In the first approaches to the real numbers, mathematicians used these three concepts — real number, real line, and continuum — interchangeably, as the early view of the real numbers was geometric. Mathematicians spoke of the "law of continuity" or a continuous variable or magnitude with the image of the real line in mind. They cast arguments in calculus in this geometric language. However, this dependence on geometry was unsatisfactory to some mathematicians by the early 19th century. To provide logical rigor to the proofs in calculus, some realized the need for an algebraic or arithmetic description of the real numbers. In this paper, we examine how three theories define in algebraic form a characteristic of the real number system which distinguishes it from the rational numbers, how the authors of these theories dealt with the issue of the identification of the real numbers with the real line, and finally how the authors understood the continuum.

2. Defining Characteristics of the Real Numbers

In the second half of the nineteenth century, three main theories of the real numbers were developed. These are the theories of Richard Dedekind (1831-1916), Georg Cantor (1845-1918) and Eduard Heine (1821-1881), and Karl Weierstrass (1815-1897). In each case, the authors of these theories assumed that the rational numbers were well-understood and used the rational numbers as the starting point for their theory. Each mathematician defined a real number as an infinite set of rational numbers which possessed a given property. What distinguishes one theory from the other is that given property which the set of rational numbers must satisfy. Cantor points out this commonality in the theories in Section 9 of the *Grundlagen* [3], where he critiques the three major constructions of the real numbers. He states:

> Part of the definition of an irrational real number is always a well-defined infinite aggregate of the first power of rational numbers; this is the common characteristic of all forms of definition. Their difference lies in the generative moment (*Erzeugungsmoment*) through which the aggregate is tied to the number it defines, and in the conditions which the aggregate must satisfy in order to be a suitable basis for the number definition in question. ([3, p. 80])

The particular property which the mathematician chose to define the set of rational numbers is related to his understanding of an essential characteristic of the real numbers.

2.1. **Dedekind.** Dedekind developed his theory of the real numbers in the essay "Stetigkeit und irrationalen Zahlen" (in [4]). The essay was not published until 1872, but Dedekind explained in the introduction that he had discovered the theory in the autumn of 1858. ([4, p. 1])

Dedekind realized that a "foundation for arithmetic" was needed while trying to prove to his differential calculus class that an increasing continuous magnitude has a limit. ([4, p. 1]) He found the geometric arguments for this fact were useful, but not "scientific." He complained that "differential calculus deals with continuous magnitude, and yet an explanation of this continuity is nowhere given." ([4, p. 2]) So he sought to define the real numbers and the "essence of continuity." Dedekind explained his definition of continuity was motivated by the geometric image of the continuity of the real line. ([4, pp. 9-11]) He described this *principle of continuity* in the following way:

> If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions. ([4, p. 11])

Having stated precisely what he meant by continuity, he was ready to define the irrational numbers so that the addition of these numbers to the rational numbers formed a continuous domain, satisfying the principle of continuity.

Dedekind denoted the system of rational numbers by the letter R. He presented the definition of a cut:

> If now any separation of the system R into two classes A_1, A_2 is given which possesses only this characteristic property that every number a_1 in A_1 is less that every number a_2 in A_2, then for brevity we shall call such a separation a *cut* [Schnitt] and designate it by (A_1, A_2). A cut (A_1, A_2) is said to be *produced by a rational number* if either A_1 has a greatest element or A_2 has a least element. ([4, pp. 12-13])

The system of all cuts comprises *the real numbers*. He denoted an element of this system by the symbol α; that is, he used α to represent the cut (A_1, A_2). ([4, p. 15])

Dedekind proved that there are irrational numbers, at the same time showing that rational numbers do not enjoy the property of continuity. He constructed an example of an irrational number as follows. Let D be a positive integer which is not the square of an integer. Let A_2 be the set of all rational numbers whose square is greater than D and A_1 be all other rational numbers. Then (A_1, A_2) is a cut. He showed that this cut is not produced by a rational number, establishing the existence of irrational numbers and showing that the rational numbers are not continuous. ([4, pp. 13-15])

To prove that the system of real numbers is continuous, he first defined an order on the set of cuts. Then he named four properties which the real numbers satisfy:

I Transitivity of order.

II Between two different numbers there exists infinitely many different numbers.

III If α is any real number, then all the real numbers fall into two classes: U_1 those less than α and U_2 those greater than α. The number α can be arbitrarily assigned to either set. Every number in the first set is less than every number in the second set. We say that this separation is produced by the number α.

IV The *continuity* property (which we call the Dedekind Property): If the system of real numbers breaks up into two classes U_1, U_2 such that every number α_1 of the class U_1 is less than every number $\alpha_2 \in U_2$, then there exists one and only one number α by which this separation is produced.

He proved this last property, which for him represented an essential characteristic of the real numbers.

After defining the real numbers, Dedekind turned to a discussion of infinitesimal analysis. He used the continuity property to prove the theorem which gave rise to his interest in the foundations of arithmetic: Every continuously increasing bounded magnitude has a limit. Early in the paper, he had described this result as "sufficient basis for infinitesimal analysis." ([4, p. 2]) Thus he saw the reliance of the rigor of calculus on a precise definition of the real numbers.

2.2. **Cantor-Heine.** Just as Dedekind was led to a consideration of the foundations of arithmetic of the real numbers by his interest in a theorem from another area of mathematics, so too was Cantor drawn into this problem. Cantor wanted to extend his results on the uniqueness of the coefficients of two trigonometric series which converged to the same sum for all values x. He had already generalized the result to the case for which there were only a finite number of points at which either one of the series did not converge or the two series did not produce the same sum. In his work to include an infinite number of such points, he needed an algebraic characterization of the real numbers. Cantor published his extension of his theorem as well as a short description of the real numbers in 1872 in [1].

In this same year, Heine published a careful and complete description of the real numbers in [6]. Heine, as Dedekind, felt that function theory was built on a shaky foundation as long as there was no arithmetic definition of the irrational numbers. He criticized the geometric view of the real numbers as a line. Without a solid definition of the irrational numbers, the truth of some theorems could be questioned. He stated:

> Das Fortschreiten der Functionenlehre ist wesentlich durch den Umstand gehemmt, dass gewisse elementare Sätze derselben, obgleich von einem scharfsinnigen Forscher bewiesen, noch immer bezweifelt werden, so dass die Resultate einer Untersuchung nicht überall als richtig gelten, wenn sie auf diesen unentbehrlichen Fundamentalsätzen beruhen.
>
> ...
>
> Ihre Wahrheit beruht aber auf der nicht völlig feststehenden Definition der irrationalen Zahlen, bei welcher Vorstellungen der Geometrie, nämlich über die *Erzeugung* einer Linie *durch Bewegung*, oft verwirrend eingewirkt haben. *Die Sätze sind für die unten zu Grunde gelegte Definition der irrationalen Zahlen gültig, bei welcher Zahlen*

> *gleich genannt werden, die sich um keine noch so kleine angebbare Zahl unterscheiden, bei welcher ferner der irrationalen Zahl eine wirkliche Existenz zukommt,* so dass ein einwerthige Function für jeden *einzelnen* Werth der Veränderlichen, sei er rational oder irrational, gleichfalls einen *bestimmten* Werth besitzt. ([6, p. 172])
>
> Translation:
> Progress in function theory is essentially hampered due to the condition that certain elementary theorems, although proven by a clever researcher, will always be questioned, so that the results of an inquiry are not considered as totally correct, if they rest on these absolutely necessary fundamental theorems.
>
> ...
>
> Their truth rests on the not completely rigorous definition of the irrational numbers, which has often been incorrectly influenced by a geometric representation, namely through their generation by means of movement along a line. *The theorems are valid by reason of the definition of irrational number given below, by which numbers are meant those which differ from themselves by an arbitrarily small number, and by which furthermore the existence of the irrational numbers comes,* so that a *single* valued function has for each single value of the variable, be it rational or irrational, a *certain* value.

Heine stated that Weierstrass had developed all the fundamental theory in his lectures. While transcripts of the lectures had been prepared, neither had Weierstrass himself published his ideas nor were the ideas developed in one place. ([6, p. 172]) Thus, encouraged by others, Heine wrote the paper. He expressed his appreciation to Cantor who influenced Heine's development and gave him helpful input.

First, we address Cantor's work. In Section 9 of the *Grundlagen* ([3, pp. 81-84]), Cantor elaborated on his theory first explained in [1] and introduced the terminology with which we are more familiar today. Cantor used the term *fundamental sequence of the first order* for an aggregate (a_ν) of rational numbers such that "after the choice of an arbitrarily small rational number ϵ a finite number of members of the aggregate can be separated off, so that those remaining have pairwise a difference which in absolute terms is smaller than ϵ. " ([3, p. 81]) (Today, we also call such a sequence a Cauchy sequence.) To the fundamental sequence, he attached the number b, which is defined by that sequence. The collection of all these fundamental sequences of the first order form the real numbers.

It was very important to Cantor to point out the logical rigor of his definition. In both papers ([1, 3]), he emphasized that the symbol b was just a representation for the fundamental sequence. He indicated that he was not assuming that the limit of the sequence existed nor defining b as the limit of the sequence. In fact, after defining the basic arithmetic operations on the fundamental sequences and the notion of order, he stated that

> now we get the first *rigorously* provable theorem that if b is the number determined by a fundamental sequence (a_ν), then ...
>
> $$\lim_{\nu=\infty} a_\nu = b. \qquad ([3, \text{p. } 82])$$

Cantor perhaps felt he had belabored his point, because he stated

> May I be forgiven my thoroughness which I motivate with the perception that most people pass by this unpretentious detail and then easily get entangled in doubts and contradictions with respect to the irrational which, by observing the particulars emphasized here, they could have been spared entirely, for they would then recognize clearly that the irrational number, by virtue of the *characteristics given to it by the definitions*, is just as definite a reality in our mind as the rational number, even as the whole rational number, and that one need not first *obtain* it by a limiting process but on the contrary — through its *possession* one is convinced of the feasibility and evident admissibility of the limiting processes. ([3, p. 83])

In his discussion of Weierstrass's treatment of the real numbers, Cantor also credits Weierstrass with avoiding the logical trap of defining the real number as the limit of the sequence. He stated

> I believe that this logical mistake, which was first avoided by Herr Weierstrass, was committed almost universally in previous times, and not noticed because it belongs among those rare cases in which actual mistakes cannot cause any significant damage to the calculus.
>
> I am nonetheless convinced that all the difficulties which have been found in the concept of the irrational are linked to this mistake, whereas when this mistake

> is avoided, the irrational number will implant itself in our mind with the same determinateness, distinctiveness, and clarity as the rational number. ([3, p. 81])

While Cantor did not give an example of a fundamental sequence which does not converge to a rational number, he stated that when one says that a polynomial has a real root, that means there is a fundamental sequence which represents the root. ([1, p. 125]) He also stated that to every rational number corresponds to a fundamental sequence, but not every fundamental sequence corresponds to a rational number. ([1, p. 126]) In a short paper which is a response to a criticism of the Weierstrass-Cantor theories of the irrational numbers, Cantor does give the example of the sequence

$$(1.7, 1.73, 1.732, \dots)$$

which represents $\sqrt{3}$. ([2, p. 476])

Once the irrational numbers were defined, Cantor moved on to the issue of the defining properties of the real numbers. Given the fundamental sequences of the first order, one can form the set of *fundamental sequences of the second order* which is the set of fundamental sequences of numbers of the first order. The process can be repeated, and one calls the n-th iteration of this process the set of *fundamental sequences of n-th order.* Cantor pointed out that the spaces of order greater than one are essentially the same: "All these fundamental sequences accomplish exactly the same thing for the determination of a real number b as the fundamental sequences of the first order, the only difference consisting of the more complicated and broader form in which they are given." ([3, p. 83]) Thus we see one arithmetic property of the real numbers, *viz.*, that iterations of the process of forming fundamental sequences do not produce new numbers, other than in a formal sense.

In the second section of the 1872 paper, Cantor begins a topological characterization of the real numbers, a theme he will return to later in discussing the notion of continuum. Let P represent a set of numbers. Cantor defined a *limit point* of P and the *derived set* of P. The *first derived set* of a set P, denoted P', is the set of limit points of the set. One can indefinitely repeat this process, forming the sequence of derived sets: $P, P', P'', \dots, P^{\nu}, \dots$. Included in the examples Cantor provided is the case where P is the set of rational numbers in the interval (0,1). The derived set, denoted P', is the closed interval of real numbers [0,1]. He added that all $P^{\nu} = P'$, for all natural numbers ν ([1, p. 126].) In Section 10 of the *Grundlagen*, Cantor used the term *perfect* to describe a set whose derived set is itself ([3, p. 86]). Thus, Cantor also provided a topological characterization of the real numbers as a perfect set.

Heine's concept of the real numbers was the same as Cantor's; however, his presentation differed from Cantor's in two ways. First, Heine's presentation was very much like a modern text — in the style of definition, remark, theorem, proof. Secondly, Heine also more carefully developed the equivalence relation on the set of fundamental sequences. Heine defined an *elementary sequence* to be one whose terms are smaller than any given number as the index increases without bound.([6, p. 174]). The symbol for an elementary sequence is 0. ([6, p. 176]) Two sequences are said to be *equal* if their difference is an elementary sequence. ([6, p. 175]) An arbitrary sequence of rational numbers $a, b, c, \ldots$ is associated to the number symbol $[a, b, c, \ldots]$. Two symbols for a number are the *same or interchangeable* if the sequences to which they belong are equal. ([6, p. 176]) Heine defined the sets of fundamental sequences of n-th order as Cantor had and concluded that all those beyond the second order yield no new numbers. ([6, p. 180])

2.3. **Weierstrass.** Weierstrass did not publish his theory of the real numbers, so we must rely on manuscripts of his students. Some of these manuscripts have been published. The notes of Moritz Pasch, Wilhelm Killing, Georg Hettner, and Adolf Hurwitz are available. Victor Dantscher published a book on Weierstrass's theory of the irrational numbers. The remarks below are based mainly on the published notes of Hurwitz ([8]). (For a discussion of the development of some of Weierstrass's ideas on the fundamentals of analysis according to several of the manuscripts of his students, see [5].)

Weierstrass's development of the real numbers began with a short discussion of equality, addition, and multiplication on the set of multiples of a unity (what we would call the natural numbers). ([8, p. 4]) He next developed the operations on positive rational multiples of a unity. This section begins with a definition of *exact parts of unity* which are numbers of the form $\frac{1}{n}$ where $n \cdot \frac{1}{n} = 1$. By a *number (Zahlgröße)* is meant a set consisting of a finite number of the unity and exact parts of unity. ([8, p. 4]) There are two transformations which can be applied to a set without changing the number:

- any n elements $\frac{1}{n}$ can be replaced with 1, and
- any element can be expressed through its exact parts, for example 1 by $n \cdot \frac{1}{n}$ or $\frac{1}{m}$ by $n \cdot \frac{1}{m \cdot n}$. ([8, p. 5])

After giving a definition of order, addition, and multiplication on these numbers with finitely many elements, Weierstrass considered numbers with infinitely many elements. So that one can have an exact representation of these numbers, it is necessary that the elements are selected from the previously defined numbers according to a certain law. He gave the example of

$$1 + \frac{1}{3} + \frac{1}{3 \cdot 3} + \frac{1}{3 \cdot 3 \cdot 3} + \cdots . \quad ([8, \text{p. } 7])$$

To determine if two such numbers are equal a third rule for transformation is needed. Let a and a' be two numbers, with a' having a finite number of elements. We say a' is a *component* of a if a' can be transformed into a'' so that the same elements of a'' appear as often in a as in a'' and in addition a either has other elements or has the same elements but in greater number. Then two numbers a and b are *equal* if each component of a is one of b and *vice versa*. Moreover, we say $b > a$ if every component of a is a component of b but not *vice versa*. ([8, pp. 7-8])

While any two numbers with a finite numbers of elements can be added or multiplied, this is not the case for numbers with an infinite number of elements. To recognize those numbers for which these operations are possible, Weierstrass first defined finite numbers. A number a with infinitely many elements is said to be *finite* if there exists a number consisting of finitely many elements which is greater than a. Addition and multiplication are defined for these numbers. At this point, Weierstrass explained that the numbers developed thus far are inadequate for subtraction and division. So he carefully and completely constructed the negative numbers and reciprocals. The end result of all these constructions is the set of real numbers, a set on which the operations of addition, subtraction, multiplication, and division (by non-zero numbers) are possible and satisfy the commutative, associative, and distributive laws. ([8, p. 17, 20])

The question for us now is how Weierstrass distinguished the real numbers from the rational numbers. While there is a parenthetical remark that infinitely many numbers cannot be expressed with a finite number of elements, a proof is not given. ([8, p. 8]) In another parenthetical remark, Weierstrass said, "One is brought to the extension of the number systems if one encounters an impossible operation, for example, squareroots." ([8, p. 9]) There is an algorithm to find the square root of numbers which have a rational square root, but for those numbers without a rational square root one is led to an infinite decimal expansion.

> Definiert war dadurch z.B. $\sqrt{2}$ jedenfalls, indem man durch besagten Algorithmus für jede (Decimal-)Stelle eine bestimmte Zahl findet. Danach konnte man sagen:

> Es giebt freilich keine rationale Zahl, die mit sich selbst multipliciert 2 ergiebt, aber man kann doch eine Reihe von rationalen Zahlen aufstellen, von denen jede spätere dieser Eigenschaft näher kommt als eine frühere. Dieses läßt sich auch für die Wurzeln einer Gleichung sagen.
>
> Translation:
> For example, $\sqrt{2}$ was defined in this way, in which through the aforementioned algorithm, one finds one certain number for each decimal place. From this, one can say: There is certainly no rational number, which multiplied by itself yields 2, but one can still construct a series of rational numbers such that each later term of the series comes closer to this property than an earlier. This can also be said for other roots. ([8, p. 9])

In this parenthetical remark, Weierstrass established that the set of real numbers was larger set than that of the rational numbers, because the set of real numbers contains all the roots.

In the ninth chapter of the book, Weierstrass gave a characterization of the real numbers in topological terms. It is in this chapter that we find the Bolzano-Weierstrass Theorem which is equivalent to a set satisfying the Dedekind property or being perfect. Weierstrass defined the upper bound (*obere Grenze* (for us the supremum)) of a set. First, he proved that *every bounded set of real numbers has a supremum and an infimum.* He did this by construction, breaking the proof into two parts.

- First he showed that if $a_0, a_1, \ldots$ is a non-decreasing bounded sequence of real numbers, then letting $b_1 = a_1 - a_0$, $b_2 = a_2 - a_1$, $\ldots$, $b_\nu = a_\nu - a_{\nu-1}$, the number

$$b = \sum_{\nu=1}^{\infty} b_\nu$$

  is finite.
- Now dealing with an arbitrary set of positive real numbers that is bounded above, he created a sequence of nested intervals such that
  - the sequence of lengths of the intervals tends to zero,
  - the right endpoint of each interval is an upper bound for the set,
  - and in each subinterval, there is an element of the set.

The differences of successive endpoints of these intervals are used to create a non-decreasing bounded sequence and the supremum is constructed as indicated in the first part of the proof. He showed that the number b is indeed the supremum of the set. ([8, pp. 84-86]) The above result is a lemma for the *Bolzano-Weierstrass Theorem: In any set which has infinitely many points, there exists at least one point which is distinguished through the property that any so small neighborhood of that point contains infinitely many points of the set.* ([8, pp. 86-87]) Again, he used the nested interval technique and proof by construction. Thus, Weierstrass has provided the foundation for his analysis by carefully defining the real numbers and proving the Bolzano-Weierstrass Theorem.

3. Relation between the real numbers and the real line

After developing a theory of the real numbers, each of the authors came back to the issue of the relation between the real numbers and the real line. They recognized that identifying the real numbers with the real line was indeed different from constructing the real numbers.

Dedekind stated that while one may say that to every point, one can attach a number, it must be taken as an axiom that to each number there exists a point on the line. He added that if a line exists, it does not have to be continuous. If we found a line was not continuous, we could create new points to fill it up. ([4, pp. 11-12]) Cantor agreed with Dedekind but talked about these ideas in a more concrete fashion. He explained the correspondence between the real numbers and real line in Section 2 of [1]. Imagine a straight line with a fixed origin O, fixed unit length, and positive and negative directions. Mark a distance on the line. If it corresponds to a rational number, one is finished. He continued:

> In the other case, if the point is *known* through a construction, it is always possible to develop a sequence
>
> $$a_1, a_2, a_3, \ldots, a_n, \ldots,$$
>
> satisfying the conditions of Section 1 and having with the distance in question the relation that the points of the line corresponding to the distances $a_1, a_2, a_3, \ldots, a_n, \ldots$ move infinitely nearer with increasing n .
>
> We express this by saying: *The distance of a specified point from the point O is equal to b where b is the number corresponding to the above sequence.* ([1, p. 127])

On the other hand, it is an axiom that to each number there is a point on the line whose coordinate is that number. ([1, p. 128])

As is typical of Weierstrass, he went into more detail on this issue. After completing his development of the real numbers, he provided in the fourth section of Chapter 2 a geometric illustration of the real numbers. He explained how to determine

- the sum of a finite number of lengths,
- a multiple of a length, and
- an exact part of a length.

Weierstrass next showed exactly how to compare two lengths. Given two lengths a and b, if one is a multiple of the other, then one is finished. Suppose, however, that one can lay off m copies of a against b so that the remaining piece is less than a. So one can write $b = ma + b_1$. Next one compares b_1 with some part of a, say $\frac{a}{10}$. Then either b_1 is a multiple of $\frac{a}{10}$ or not. If it is, then one is done; if not, one continues the procedure. Either it eventually terminates or one has an infinite series which converges to b. ([8, pp. 21-22])

Then, assuming the continuity of the line, he showed how one can represent a number by a length. He says

> Haben wir eben gezeigt, daß das Verhältnis zweier Strecken $b : a$ (so nenne ich die in obiger Weise durch b gegebene Zahl, wenn ich a als Einheit auffasse) eine Zahl mit endlich oder unendlich vielen Elementen sein kann, so wollen wir jetzt umgekehrt nachweisen, daß jede aus unendlich vielen Elementen zusammengesetzte Zahlgröße sich als eine Strecke darstellen läßt, wenn wir eine feste Strecke als Einheit annehmen, und die erwähnte Zahlgröße in unserem Sinne einen endlichen Werth besitzt.
>
> Translation:
> We have just shown that the ratio of two lengths $b : a$ (so I name b in the above way the given number if a is understood as the unit) can be a number with finitely or infinitely many elements. Thus, we want to prove the opposite, that each number with infinitely many elements can be represented by a length if we fix a unit length and the number has a finite value in our sense. ([8, p. 22])

He does this by example, showing how to mark off a point for the expansion of

$$1 + \frac{1}{2!} + \frac{1}{3!} + \cdots \quad .$$

First one marks off the unit length, denoted by PQ. Then to each partial sum there is a length. Now one marks off a point X so that $s_n > PX$ for all n. Any point to the left of X has this same property. Because the sum is finite, there must be a point Y so that $PY > s_n$ for all n. All points to the right of Y have the same property. Since the points X and Y form a continuous line of points, there must be a point X_0 which has the property that PX_0 is greater than any partial sum but that at the same time, if to PX_0 is added some small element representing X_0Y_1, PY_1 is no longer contained in

$$1 + \frac{1}{2!} + \frac{1}{3!} + \cdots \ .$$

So PX_0 is the desired length.([8, p. 22]) Weierstrass summarized saying

> Jede Zahlgröße läßt sich durch das Verhältnis zweier Strecken repräsentieren und das Verhältnis irgend zweier Strecken durch eine Zahlgröße ausdrücken.
>
> Translation:
> Each number can be be represented as the ratio of two lengths and the ratio of any two lengths can be represented by a number. ([8, p. 22])

So Weierstrass did not make as careful a distinction as did Cantor and Dedekind, but his goal in that section was to produce a meaningful picture of the real numbers.

4. Continuum

The last problem to consider is the relationship between the continuum and the system of real numbers. For Dedekind, the issue of continuity is one and the same as the real numbers. However, Weierstrass and Cantor recognized that a continuum required a definition distinct from the real numbers.

Dedekind expressed his views both in his essay *Stetigkiet und irrationalen Zahlen* and in correspondence with Cantor. The title of his essay itself suggests that he considered continuity (Stetigkiet) and the real numbers one and the same. Throughout the essay, he referred to his search for the real numbers as a way to understand a continuous domain. In the preface to *Stetigkiet und irrationalen Zahlen*, Dedekind stated,

> The statement is so frequently made that the differential calculus deals with continuous magnitude, and yet an explanation of its continuity is nowhere given; even the most rigorous expositions of the differential calculus do

> not base their proofs upon continuity but, with more or less consciousness of the fact, they either appeal to geometric notions or those suggested by geometry, or depend upon theorems which are never established in a purely arithmetic manner. ... It then only remained to discover its true origin in the elements of arithmetic and thus at the same time to secure a real definition of the essence of continuity. ([4, p. 2])

He wished to capture arithmetically the continuity that is modeled by an unbroken line.

> If now, as is our desire, we try to follow up arithmetically all phenomena in the straight line, the domain of rational numbers is insufficient and it becomes absolutely necessary that the instrument $\mathbb{R}$ constructed by the creation of the rational numbers be essentially improved by the creation of new numbers such that the domain of numbers shall gain the completeness, or as we may say at once, the same *continuity*, as the straight line.

Having proved that the real numbers as constructed by himself satisfied the law of continuity, Dedekind believed he had characterized continuity completely.

4.1. **Dedekind and Cantor Correspondence.** For Cantor there was more to the concept of a continuum than a characterization of the real numbers. He discussed the issue with Dedekind in a series of letters. The discussion began with a letter from Cantor to Dedekind. At issue was Dedekind calling Property IV the "law of continuity." Dedekind replied on May 11, 1877, asking for clarification of the objection. ([7, pp. 21-22]) Cantor responded on May 17. His main concern was that Dedekind equated continuity with Property IV, a property which the highly discontinuous whole numbers possess. Cantor wrote:

> Indessen bitte ich Sie mir die Bemerkung zu erlauben, ob nicht vielleicht doch die Betonung, welche Sie an verschiedenen Stellen Ihrer Schrift *ausdrücklich* auf die Eigenschaft IV, als *auf das Wesen* der Stetigkeit legen, zu Missverständnissen Gelegenheit geben muss, welche ohne jene Hervorhebung von IV (als das eigentliche Wesen) an Ihre Theorie, meiner Ansicht nach, nicht herantreten könnten. Im Besonderen sagen Sie in dem Vorworte, dass das von mir bezeichnete Axiom vollständig mit dem übereinstimmt was Sie in §3 als *Wesen* der

> Stetigkeit angeben. Darunter verstehen Sie aber dieselbe Eigenschaft, welche auf Seite 25 unter IV genannt ist; diese Eigenschaft kommt aber auch dem System aller ganzen Zahlen zu, welches doch als ein Prototyp von Unstetigkeit betrachtet werden kann.
>
> Translation:
> Permit me to make this remark: the emphasis that you put in different spots expressly on property IV, as being the essence of continuity, could cause misunderstanding of your theory. In particular you say in the preface that the axiom presented by me (the Axiom of Continuity) is equivalent to that which you present in §3 as the essence of continuity. By the latter you understand however the same property, which is named on p.25 under IV; but this property belongs also to the system of all whole numbers, which could be considered the prototype of a discontinuous space. ([7, p. 22])

Cantor concluded the letter with a request that Dedekind examine his reservations more closely. In a postscript to the letter, he added in a conciliatory tone,

> Ich erkläre mir, warum Sie auf IV einen besonderen Nachdruck legen, dadurch, dass in dieser Eigenschaft dasjenige liegt, was das vollständige Zahlgebiet unterscheidet von dem Gebiet aller rationalen Zahlen; und dennoch scheint es mir aus obigen Gründen, dass man der Eigenschaft IV nicht den von Ihnen gebrauchten Namen "Wesen der Stetigkiet" beilegen kann.
>
> Translation:
> I understand that you insist particularly on property IV because it is this property which distinguishes the reals from the rationals. However, it seems to me for the reasons given in the letter that one cannot give property IV the name "essence of continuity." ([7, p. 23])

On May 18, 1877, Dedekind responded to Cantor:

> Nach Ihrem letzten Briefe schient es mir, als ob wir Gefahr liefen, mehr um Worte als um Dinge zu streiten. Jeder aufmerksame Leser meiner Schrift wird meine Meinung über die Stetigkiet gewiss so verstehen: Gebiete mit einer Gegen- sätzlichkeit und Vollständigkeit ihrer Elemente, wie sie durch I und II in §1 S. 14, §2 S.

> 15, §5 S.25 ausgedruckt ist (III ist eine Folge von I und nur deshalb hinzugefügt, um auf IV vorzubereiten), sind darum noch nicht nothwendig stetige Gebiete; die Eigenschaft der Stetigkeit erhalten solche Gebiete durch die Hinzufügung der Eigenschaft IV (auf S. 18 ohne Nummer, und auf S. 25) und nur durch diese Eigenschaft. Und insofern ist diese Eigenschaft als das Wesen der Stetigkeit bezeichnet. ... Sie theilen mir nun durch Ihre Karte vom 10. d. M. mit, dass meine Definition der Stetigkeit nicht vollständing ist, und machen einen Verbesserungsvorschlag, um diesem Mangel abzuhelfen. Darauf lehne ich diesen Vorschlag ab, indem ich Sie auf II aufmerksam mache, worin das von Ihnen Vermisste enthalten ist. Hierauf geben Sie in Ihrem letzten Briefe zu, das in meiner Definition eigentlich Nichts übersehen ist; wenn ich z. B. sage: "Gebiete, welche die Eigenchaften I und II bestizen, heissen stetige, wenn sie zugleich die Eigenschaft IV besitzen", so werden Sie, wenn ich Ihren letzten Brief recht versthehe, gegen die *Völlständigkeit* einer solchen Erklärung Nichts einzuwenden haben.
>
> Translation:
> According to your last letter, it seems to me that we run the risk of discussing words rather than ideas. Every attentive reader of my work understands my opinion on continuity as such: the domains of which the elements possess the property expressed as I and II (III is a consequence of I and is only added to prepare for IV) are not yet necessarily continuous. Such domains obtain the property of continuity by the addition of property IV and only by the addition of IV.
>
> You tell me that my definition of continuity is not complete and make a suggestion to improve it. I decline your suggestion, in drawing your attention to property II, which contains that which you think is lacking. You tell me in your last letter that nothing will in effect be missing in my definition if I say for example "Domains which possess properties I and II are called continuous if they also possess property IV." ([7, p. 23])

He added that all attentive readers will understand what his point of view is and that the example of the natural numbers would not occur as an objection. He regarded his work as a progression from the rationals to the reals and as such it would suffer if he moved property II, as the system of the rational numbers, a discontinuous field, already possesses II. He had no objection to the legitimacy of rephrasing the definition. But his original formulation pleased him most. In any case, he objected to the necessity of changing the definition. He concludes that they agree in content, and that the debate would, if continued, not produce much. ([7, p. 24]) Dedekind's view of the progressive nature of his work and his understanding that the system of real numbers is the continuous domain seem to have precluded the possibility of other continuous domains. The letter of May 18th was their last exchange on the topic.

Joseph Dauben in the book *Georg Cantor: His Mathematics and Philosophy of the Infinite* discusses how Cantor wanted an algebraic definition of continua. We merely include a few remarks of Cantor that appear in the *Grundlagen*. In Section 10 of the *Grundlagen*, Cantor explained his understanding of the continuum. He stated that there has been confusion about the concept of a continuum since the time of the Greeks. Thus, he concluded,

> I see myself obliged only to develop the concept of the continuum here as briefly as possible, in as logically sober a fashion as I must grasp it and as I need it in the theory of manifolds, and, also, only with respect to the *mathematical* theory of aggregates. This treatment was not so easy for me, for among mathematicians whose authority I like to call upon, not a single one has dealt closely with the continuum in the sense that I am in need of here.
>
> Indeed, taking one or several real or complex continuous magnitudes (or, what I take to be the more correct expression, continuous sets of magnitudes) as a basis, the concept of a continuum depending on them either univocally or multivocally — *i.e.*, the concept of a continuous function — has been shaped out in the best possible way and in the most varied directions.
>
> ...
>
> However, the independent continuum itself has merely been presupposed by the mathematical authors in that most simple manifestation and has not been subjected to any more thorough consideration.
>
> ...

> Thus I am left with no choice but to attempt with the aid of the real number concepts defined in Section 9, as general as possible a definition of a purely arithmetical concept of a point continuum. ([3, p. 85])

To give the definition of continuum, Cantor first recalled the notion of perfect. But perfect alone is not sufficient for a definition of a continuum, because there are perfect sets which are not everywhere dense. So another concept is required, namely that of connected. ([3, p. 86]) His definition reads,

> We call T a *connected* point aggregate if for any two points t and t' of the latter and for a pre-assigned arbitrarily small number ϵ, there always exists a *finite* number of points
>
> $$t_1,\ t_2,\ \ldots,\ t_\nu$$
>
> of T in multiple ways so that the distances $tt_1,\ t_1t_2,\ \ldots,\ t_\nu t'$ are all smaller than ϵ. ([3, p. 86])

Cantor presented his definition that a *continuum* is a set that is both perfect and connected. ([3, p. 86]) Having given his definition, he criticized the definitions of Bolzano and of Dedekind. Both lack the notion of connected. Of Dedekind, he said

> Similarly it also appears to me that in the essay of Herr Dedekind (*Continuity and Irrational Numbers*) only one *different* property of the continuum has been stressed one-sidedly, *viz.* the property which it shares with *all* "perfect" aggregates. ([3, p. 87])

4.2. **Weierstrass.** Weierstrass also discussed the continuum in Chapter 9. Suppose an infinite set of points, call it X, is selected from the real numbers. If the set can be represented by the points of a line, continuously following one after another, we say they form a *continuum.* This is analytically defined as follows: Let a be a point in a set X and the points in any arbitrarily small neighborhood of a lie in the set X. In a neighborhood of a point a of a set X is another point a_1 so that all the points in the interval a to a_1 belong to the set. If one can continue in this way to $a_2, a_3, \ldots a_n$ then we say a *continuous path* from a_1 to a_n is possible. ([8, pp. 83-84]) Thus, he included the notion of connected, just as Cantor did.

* * *

In their efforts to answer questions about the foundations of mathematics as well as other areas of mathematics, mathematicians came to an understanding of the distinctions between the real numbers, the continuum, and the real line. This journey was not linear nor focused only on the real numbers, but wandered into other branches of mathematics and created new branches.

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