*The Review of Modern Logic*
Volume 9 Numbers 1 & 2 (November 2001–November 2003) [Issue 29], pp. 81–94.

# META INDUCTION IN OPERATIONAL SET THEORY

LUIS E. SANCHIS

**§1.Introduction.** Set theory is usually organized as a first order theory where the variables range over a collection or universe of sets. This universe is assumed to be a well defined totality, for the theory involves global quantifiers $(\forall Y)$ and $(\exists Y)$ that range over the universe. The global quantifiers *induce local quantifiers* $(\forall Y \in X)$ and $(\exists Y \in X)$.

On the other hand, operational set theory (see [6]) rejects the universe of sets as a well defined totality and in principle questions the legitimacy of the global quantifiers. Noting that a first order theory without global quantifiers is difficult to handle, we have tried to avoid a complete rupture by allowing global quantifiers under some restrictions. For example, in [6] we restrict the global quantifiers by imposing a general control of such quantifiers under intuitionistic logic. We do not support anymore this type of restriction, as we prefer to preserve classical logic for the whole system, and we have chosen to impose restrictions on the axiomatic structure of the theory.

The situation of the local quantifiers (that in standard set theory are induced by the global quantifiers) is different. In fact, we assume, and we require, that each set in the universe is a complete totality that supports local quantification. As we explain below, there is a price to pay for this requirement, concerning how sets in general are allowed to enter the theory.

**1.1.** We are now in position to give a rough description of what we understand by operational set theory. It is a first order theory involving set operations, set predicates, classical connectives, local quantifiers and global quantifiers. Furthermore, the axioms of the theory are *operational*, and this means that each axiom is a closed formula of the form $(\forall Y_1)...(\forall Y_n)\phi$ is where $\phi$ is a local formula (no global quantifiers) and $0 \leq n$. Usually, we identify the axioms via the local formula $\phi$.

**1.1.1.** We require that the sets in the theory be introduced with (operational) axioms, but we also require that whenever a set is introduced some argument or construction is provided that shows that the set is a well defined totality that supports local quantification. For example, the traditional separation rule provides a procedure where a set is introduced as a subset of another given and available set. Since the given set is a well defined totality, it follows that the new set is also a well defined totality and supports local quantification (compare with 2.2.2).

**1.2.** The purpose of this note is to describe a general technique that we call *metainduction*, where new sets can be introduced in operational set theory by a construction that shows that each such set is a well defined totality, and in fact describes explicitly this totality. This technique is a generalization of the approach in [6].

**1.3.** Local quantifiers appear frequently in contemporary set theory, most of the time as derivatives of local quantifiers, as in [1]. Enderton in [2] provides a good introduction to standard set theory. Ershov in [3] deals with both global and local quantifiers, but the approach is essentially semantical. Reviews of [6] have been positive but still very critical (see [4] and [5]).

In the description of metainduction and its applications we assume a fixed system of operational set theory, that we call **OST**. Each application of metainduction culminates with the introduction in **OST** of a set or operation (and in one case a predicate) together with operational axioms derived from the metainductive construction. In particular, we assume that the classical separation rule is available in **OST**. Furthermore, standard operations in set theory are assumed available in **OST** with the usual axioms. For example, we have an operation $\{X\}$ with the axiom $Y \in \{X\} \equiv Y = X$.

## §2. Meta Induction.

Metainduction is a construction that depends on a given set operation $\rho$ and a given set $Z$, both available in **OST**. The construction itself is independent and takes place outside **OST**.

**2.1.** A *$\rho$-branch for $Z$* is a (potentially infinite) sequence of sets $Z_0, Z_1, ..., Z_k, ...$ ($k$ a numeral), where $Z_0 \in \rho(Z)$ and $Z_{k+1} \in \rho(Z_k)$. While $\rho$ is a fixed given operation, $Z$ is not necessarily a fixed set, and can be assumed to be an arbitrary set. On the other hand $\rho$ is not necessarily a unary operation, and may contain extra arguments that behave as *parameters* in the construction. To make explicit this situation we should write $\rho(Z, \mathbb{X})$ where $\mathbb{X}$ is a list of extra variables $X_1, ..., X_k$. In general we do not take advantage of this notation, but the reader should keep in mind this more general formulation. In one application below one extra parameter $W$ is in fact brought explicitly into the notation (see 5.1.1).

**2.1.1.** Note that, essentially, the definition says that $Z_{k+1}$ is determined by *choice* among the elements of $\rho(Z_k)$. So here we are assuming a weak informal version of the axiom of choice, in fact close to the usual axiom of dependent choices. This does not contradict our basic assumptions, because the choices take place one by one and in each case over a given set $\rho(Z_k)$. *This process does not assume the universe of sets as a complete totality.*

**2.1.2.** If $\rho(Z) = \emptyset$ then $Z_0$ is undefined and the $\rho$-branch is *empty*. In general, if for some $k$, $Z_k$ is defined and $\rho(Z_k) = \emptyset$, then $Z_{k+1}$ is undefined and we say that the $\rho$-branch *halts* at $k$. Otherwise, $Z_{k+1}$ is defined. We say that the sets $Z_0, Z_1, ...$ are *$\rho$-residuals* of $Z$. The set $Z$ is not necessarily a residual.

**2.1.3.** The structure determined by the totality of all $\rho$-branches we call the *$\rho$-tree* for $Z$ induced by $\rho$. More precisely, we mean that the $\rho$-tree consists of all the $\rho$-branches when we assume they are generated simultaneously. The $\rho$-residuals in the branches are also $\rho$-residuals in the $\rho$-tree. We say that $\rho$ is the *handle* for the $\rho$-branches and also for the $\rho$-tree. From now on we omit the prefix $\rho$ unless it is necessary or convenient.

**2.2.** At this stage we postulate a *completion* of the $\rho$-tree, that contains all the residuals and all the branches, even if some of them are infinite. More precisely, the completion looks at the $\rho$-tree as a *complete totality* ( exactly in the way we claim it is not legitimate to look at the universe of sets). *Still, we claim that the completion of the tree does not involve the universe as a complete totality.*

**2.2.1.** From the completion of a $\rho$-tree we can derive sets, operations and predicates, as we show in the applications of metainduction, where from a given operation $\rho$ and a set $Z$ we introduce the $\rho$-tree construction, take the completion and define new operations (eventually sets) and predicates. These operations and predicates are defined in the context determined by the $\rho$-tree, so they are not meaningful definitions inside **OST**. Rather, we derive axioms from the construction, these axioms are formulas in **OST**, and in fact become axioms there.

This is a two-step process where first we define operations or predicates in the metainductive construction, and then write axioms for the operations or predicates in **OST**. In every application the axioms are operational in the formal sense explained above (see 1.1).

**2.2.2.** On the other hand, the construction itself makes explicit the extension of the sets introduced by the metainduction application, and in this way satisfies the requirement in 1.1.1.

§**3. Transitive closure.** In the first application of metainduction we assume an operation $\rho$ (in **OST**) and define a new metainductive operation $\mathbf{tc}_\rho$.

$\mathbf{tc}_\rho(Z)$**:** = the set of all $\rho$-residuals for $Z$, where $Z$ is an arbitrary set in the universe of operational set theory.

We call $\mathbf{tc}_\rho$ the *transitive* $\rho$*-closure* operation. The set $\mathbf{tc}_\rho(Z)$ comes essentially from the completion of the metainduction construction induced by the operation $\rho$ and the set $Z$. It is in fact a definition *in the construction* and it is meaningless outside the construction (see 2.2.1). Note that the metainduction construction actually defines explicitly the extension of the set $\mathbf{tc}_\rho(Z)$ (see 1.2 and 2.2.2).

**3.1.** The next step introduces the operation $\mathbf{tc}_\rho$ (in fact a symbol) into **OST** with the following axioms, where $V, Z$ are arbitrary sets in **OST**:

**TC 1:**

$$\rho(Z) \subseteq \mathbf{tc}_\rho(Z)$$

**TC 2:**

$$(\forall X \in \mathbf{tc}_\rho(Z))\rho(X) \subseteq \mathbf{tc}_\rho(Z)$$

**TC 3:**

$$\rho(Z) \subseteq V \wedge (\forall X \in V)\rho(X) \subseteq V. \rightarrow .\mathbf{tc}_\rho(Z) \subseteq V$$

Here, as in every application of metainduction, we must show that the axioms follow indeed from the definition of $\mathbf{tc}_\rho$. This argument takes place outside **OST**, although it may involve known properties of the operation $\rho$, that come from **OST**. For example, in the example above concerning the operation $\mathbf{tc}_\rho$, we note that if $X \in \rho(Z)$, then certainly $X$ is a residual in some $\rho$-branch, hence $\rho(Z) \subseteq \mathbf{tc}_\rho(Z)$ and **TC 1** holds. Furthermore, if $X$ is a residual and $Y \in \rho(X)$, then $Y$ is also a residual, hence **TC 2** holds. Finally, if $V$ is a set that satisfies the closure properties in axiom **TC 3**, then in the process that generates a $\rho$-branch for $Z$ each generated residual is an element of $V$. We conclude that $\mathbf{tc}_\rho(Z) \subseteq V$, so **TC 3** holds.

Technically, the axioms are the universal closure of such assertions. Still, we refer to them as axioms. The whole application involves the introduction of a new primitive operation $\mathbf{tc}_\rho$ and three new axioms in **OST**. We call these axioms the *Peano axioms* for the operation $\mathbf{tc}_\rho$. As usual, from the Peano axioms we can derive a rule of proof by residual induction.

**3.1.1.** In order to prove the rule of residual induction we assume that the three Peano axioms are valid in **OST** for an operation $\mathbf{tc}_\rho$ and arbitrary sets $Z, V$. Le $p$ be a set predicate in **OST**. We define a predicate $p^*$ with the axiom $p^*(X) \equiv (\forall Y \in \rho(X))p(Y)$. For a fixed set $Z$ we assume that the following conditions are satisfied:

**(i):** $p^*(Z)$
**(ii):** $(\forall X \in \mathbf{tc}_\rho(Z))(p(X) \to p^*(X)$

We claim that from these conditions it follows that:

**(iii):** $(\forall X \in \mathbf{tc}_\rho(Z))p(X)$

To prove that **(iii)** follows from **(i)** and **(ii)** in **OST** we use the separation rule (available in **OST** see 1.3) to introduce a set $V$ with the axiom:

$$X \in V \equiv X \in \mathbf{tc}_\rho(Z) \wedge p(X)$$

We note now that from **(i)** and **(ii)** it follows that $V$ satisfies the conditions in axioms **TC 3**, hence $\mathbf{tc}_\rho(Z) \subseteq V$ and $(\forall X \in \mathbf{tc}_\rho(Z))p(X)$ holds.

**3.1.2.** The preceding argument depends only on the axioms of **OST**, so we have proved in **OST** that the implication $(\mathbf{i}) \wedge (\mathbf{ii}) \to (\mathbf{iii})$ is valid for every set $Z$, any operation $\rho$ in **OST**, and any predicate $p$ defined in **OST**.

This result supports an inductive procedure, where in order to prove that $p(X)$ holds whenever $X \in \mathbf{tc}_\rho(Z)$, we need only to prove that **(i)** and **(ii)** hold. In such a proof **(i)** is usually given as an assumption. To prove **(ii)** typically we assume that $p(X)$ holds for some $X \in \mathbf{tc}_\rho(Z)$ (the induction hypothesis) and show that $p(Y)$ holds for every $Y \in \rho(X))$ (here the proof usually depends on properties of the predicate $p$). Since we have **(i)** and **(ii)** we conclude that **(iii)** holds.

**3.2.** The classical example of transitive closure is the set $\omega$, with the usual meaning in standard set theory (see [6], 4.2 and 7.1). The notation for the natural numbers can be chosen in many different ways. The following is very convenient in the frame provided by metainduction. We set $X^+ = \{X\}$. We represent 0 with $\emptyset^+$, and represent $n+1$ with $X^+$ when $X$ is the representation of $n$. We define $\rho(X) = X^{++}$ and set $\omega = \mathbf{tc}_\rho(\emptyset)$. Note that $\emptyset \notin \omega$.

The $\rho$-tree induced by $\rho$ consists of only one infinite branch of the form: $\emptyset^+, \emptyset^{++}, \emptyset^{+++}, ...,$ and the completion of this tree is the set $\omega$ of natural numbers.

The Peano axioms for the set $\omega$ (see 3.1) take the following traditional form:

**NT 1:**

$$\emptyset^+ \in \omega$$

**NT 2:**

$$X \in \omega \to X^+ \in \omega$$

**NT 3:**

$$\emptyset \in V \wedge (\forall X \in V)X^+ \in V. \to .\omega \subseteq V$$

.

In these axioms the variables $X$ and $V$ are arbitrary sets. It follows from residual induction in 3.1.2 that in order to prove $p(X)$ for every $X \in \omega$ we need only to prove $p(\emptyset^+)$ and $p(X) \to p(X^+)$ for every $X \in \omega$. As usual, we refer to this procedure as *mathematical induction.*

**3.2.1.** On the other hand, the transitive closure construction can be applied with any operation $\rho$ available in **OST**, including with operations that have been introduced before via transitive closure. For example, it is possible to take $\rho$ to be the power set operation, provided it is available in **OST** (see 1.1.1).

§**4. Bar Induction.** Bar induction is, in some sense, the inverse of residual induction. In fact, it is quite different from the residual construction, because it introduces a predicate, rather than an operation. Still, it is another application of metainduction.

**4.1.** Again, we start with a set operation $\rho$ from **OST**, and $Z$ is an arbitrary set. We define a predicate $\mathbf{WF}_\rho$ as follows:

$\mathbf{WF}_\rho(Z)$**:** $\equiv$ every $\rho$-branch for $Z$ halts.

We read $\mathbf{WF}_\rho(Z)$ as $Z$ is $\rho$-well-founded.

**4.1.1.** As usual, we have to write axioms in order to introduce this predicate in **OST**.

**Closure Axiom:**

$$\mathbf{WF}_\rho(Z) \to (\forall X \in \rho(Z))\mathbf{WF}_\rho(X)$$

**Foundation Axiom:**

$$\mathbf{WF}_\rho(Z) \wedge Z \in V. \to .(\exists Y \in V)V \cap \rho(Y) = \emptyset$$

**4.1.2.** These are axioms in **OST**, intended to be valid for arbitrary sets $V, Z$. Still, we must show that they indeed follow from the metainductive construction. The closure axiom is trivial noting that if for some $X \in \rho(Z)$, $X$ is not well-founded, then there is a non-halting branch for $X$. Clearly, this means there a non-halting branch for $Z$, contradicting $\mathbf{WF}_\rho(Z)$.

The foundation axiom is more involved. Assume given sets $V, Z$, where $\mathbf{WF}_\rho(Z)$ holds and $Z \in V$. We want to show there is $Y \in V$ such that $V \cap \rho(Y) = \emptyset$. If $V \cap \rho(Z) = \emptyset$, we take $Y = Z$ and we have finished. Otherwise, we generate a $\rho$-branch for $Z$, say $Z_0, Z_1, ..., Z_k, ...$

but we impose the following condition for any $k$: if $V \cap \rho(Z_k) \neq \emptyset$, then $Z_{k+1} \in V \cap \rho(Z_k)$ (and otherwise, $Z_{k+1} \in \rho(Z_k)$, as usual). Now we note that there is a $k$ such that $\rho(Z_k) = \emptyset$ (the branch halts at $k$). It follows that there is a first $k$ with the property that $Z_k \in V$ and $V \cap \rho(Z_k) = \emptyset$. Hence, we take $Y = Z_k$.

**4.1.3.** The argument in 4.1.2 that validates the foundation axiom takes place in the metainduction construction and depends on the definition of the predicate $\mathbf{WF}_\rho$. Still, the axiom itself is a formula in **OST**, which now becomes an axiom in **OST**.

With the two new axioms in **OST**, we can use them to prove (in **OST**) formal properties of bar induction, noting that residual induction (see 3.1.1) is always available. For example, using residual induction we can prove the following theorem: $\mathbf{WF}_\rho(Z) \rightarrow (\forall X \in \mathbf{tc}_\rho(Z))\mathbf{WF}_\rho(X)$. This follows by residual induction, taking as $p$ the predicate $\mathbf{WF}_\rho$ and using the closure axiom. On the other hand, we can prove also a rule of proof by bar induction.

**4.2.** The general rule of proof by bar induction involves an arbitrary predicate $p$ (from **OST**, compare with 3.1.1). As in 3.1.1 we define $p^*(X) \equiv (\forall Y \in \rho(X))p(Y)$. For a given set $Z$ we assume the following conditions are satisfied:

**(i):** $\mathbf{WF}_\rho(Z)$
**(ii):** $(\forall X \in \mathbf{tc}_\rho(Z))(p^*(X) \rightarrow p(X)$

It follows from these conditions that:

**(iii):** $(\forall X \in \mathbf{tc}_\rho(Z))p(X)$

In order to prove (in **OST**) that (**i**) $\wedge$ (**ii**) implies **(iii)** we assume **(i)** and **(ii)** and introduce by separation the following set $V$:

$$X \in V \equiv X \in \mathbf{tc}_\rho(Z) \wedge \neg p(X).$$

We shall show that $V = \emptyset$, so **(iii)** holds. To get a contradiction, let $X \in V$. Noting that $\mathbf{WF}_\rho(Z)$ holds by (**i**), it follows from 4.1.3 that $\mathbf{WF}_\rho(X)$ also holds, hence by the foundation axiom there is $X' \in V$, such that $\mathbf{WF}_\rho(X')$ also holds, and $\rho(X') \cap V = \emptyset$. Furthermore, $\rho(X') \subseteq \mathbf{tc}_\rho(Z)$ holds, hence $(\forall Y \in \rho(X'))p(Y)$ also holds. From the definition of $p^*$ and **(ii)**, it follows that $p(X')$ holds, and this is a contradiction because $X' \in V$.

**4.2.1.** The preceding result supports another induction procedure to prove assertions of the form: $(\forall X \in \mathbf{tc}_\rho(Z))p(X)$ for a given predicate $p$ and a given set $Z$. In fact, if we know $\mathbf{WF}_\rho(Z)$ holds, we need only to assume $p^*(X)$ and prove $p(X)$ for any arbitrary $X \in \mathbf{tc}_\rho(Z)$. The assumption $p^*(X)$ is here the *induction hypothesis* (compare with 3.1.2)

§**5. Numerical Functions.** We have discussed two applications of metainduction: transitive closure and bar induction. In the first we introduce an operation $\mathbf{tc}_\rho(Z)$ and prove that an inductive procedure is available to prove statements of the form $(\forall X \in \mathbf{tc}_\rho(Z))p(X)$. In the second we introduce a predicate $\mathbf{WF}_\rho(Z)$ and show that a different inductive procedure is available to prove a similar statement under the assumption that $\mathbf{WF}_\rho(Z)$ holds.

**5.1.** Now we extend the scope of metainduction with a different application where we are concerned with *numerical functions*, which are functions in the usual sense of set theory: sets of single-valued ordered pairs. The formal definition is available in **OST** via the predicate $\mathbf{NF}(F, W)$ that means $F$ *is a numerical function over the set* $W$:

$$\mathbf{NF}(F, W) \equiv \mathbf{FU}(F) \wedge \mathbf{do}(F) = \omega \wedge \mathbf{ra}(F) \subseteq W$$

For the notation in this definition see [6], Chapter 3. Note that $F$ is an ordinary set symbol (like $X, Y, V$,...), that we intend to use to denote functions.

**5.1.1.** We want to introduce in **OST**, via metainduction, a set operation $\mathcal{F}$ such that the following axiom is satisfied for any sets $W, F$:

$$F \in \mathcal{F}(W) \equiv \mathbf{NF}(F, W)$$

The basic idea is quite simple. We identify the numerical functions in $\mathcal{F}(W)$ with the $\rho$-branches generated by a handle $\rho$, defined in such a way that every branch determines exactly one numerical function over $W$, and every numerical function is determined exactly by one branch. Via the completion of the $\rho$-tree we get the set of all $\rho$-branches, hence the set of all numerical functions over $W$. The set $W$ is a parameter in the construction, that appears as an argument in the operation $\rho$, so we have now a binary operation $\rho(Z, W)$ (see 2.1). In this section we assume that $W \neq \emptyset$.

In the following the symbol "$n$" is a set variable that denotes an element of the set $\omega$ when used in positions that require a set argument. Otherwise, it is a syntactical symbol that denotes a numeral.

**5.1.2.** We take as initial set $Z = \emptyset$ and define $\rho(\emptyset, W) = \{\emptyset^+\} \times W$. Hence $Z_0 = \langle 0, w \rangle$, where $w \in W$.

To complete the definition we define $\rho(Z, W)$ for the case $Z = \langle n, w' \rangle$, where $n \in \omega$ and $w' \in W$. We set $\rho(\langle n, w' \rangle) = \{n + 1\} \times W$. For example, $\rho(\langle 4, w' \rangle) = \{5\} \times W$. Clearly, $\rho$ is an operation in **OST**. It follows that a $\rho$-branch under these definitions takes the form:

$$\langle 0, w_0 \rangle, \langle 1, w_1 \rangle, \langle 2, w_2 \rangle, \cdots, \langle n, w_n \rangle, \cdots$$

where $w_0, w_1, w_2, ..., w_n, ...$ are arbitrary elements of $W$. Note that when $n$ occurs as a subscript it is a numeral, and not an element of $\omega$.

The completion of the $\rho$-tree induces the completion of each $\rho$-branch, which is a set of ordered pairs, in fact it is a numerical function over the set $W$. So the completion of the tree induces the set of all such functions. *It should be clear that this construction does not involve the universe of sets.*

**5.1.3.** We denote by $\mathcal{F}(W)$ the set of all numerical functions on $W$, as determined by the preceding metainductive construction. This construction is a $\rho$-tree, where the behavior of the operation $\rho$ is determined by the rules above. The completion of the tree is an objective structure where each numerical function occurs as a $\rho$-branch (more precisely, occurs as the set of all residuals in some particular branch).

**5.2.** To introduce $\mathcal{F}$ in **OST** we must determine the (operational) axioms that control the operation $\mathcal{F}$. Note that the argument to support the axioms takes place in the metainduction construction, although the axioms are written in the language of **OST**. This is, of course, the same situation we have found in the applications above (see 3.1 and 4.1.2)

The first axiom is a trivial consistency property, and simply says that the elements of the set $\mathcal{F}(W)$ are numerical functions on $W$. This follows because the elements of $\mathcal{F}$ are (the completions of) $\rho$-branches, and each one of these is a numerical function by construction.

**Consistency Axiom:**

$$F \in \mathcal{F}(W) \rightarrow \mathbf{NF}(F, W)$$

This axioms is necessary, for example, whenever we introduce a set $F$ with the assumption $F \in \mathcal{F}(W)$, as it provides a minimal basic information about the meaning of such characterization.

**5.2.1.** In the other direction, a second axiom appears to be necessary where if a set $F$ is found to have the necessary properties, we can assert that it is an element of $\mathcal{F}(W)$. We start with a weak version of this condition.

**Weak Completeness Axiom:**

$$\mathbf{NF}(F, W) \rightarrow F \in \mathcal{F}(W)$$

Clearly, we need this axiom, but it is weak in the sense that it determines that $F \in \mathcal{F}(W)$, but it fails to take advantage of the metainductive construction that supports the set $\mathcal{F}(W)$. For example, the fact that the elements of $\mathcal{F}(W)$ are actually generated by free choices (see 2.1.1) plays no rôle in the axiom. So, we propose next a stronger second version of the axiom, where under given circunstances we assert the existence of some $F \in \mathcal{F}(W)$, that satisfies an extra condition that depends on a given binary predicate $p$ (from **OST**).

**5.2.2.** The strong completeness axiom requires a technical construction, available in **OST** (see [6], definition 2.4.1). If $F$ is a numerical function, and $n$ is a natural number, then $F{\restriction}n$ is also a function where $\mathbf{do}(F{\restriction}n) = n$ and for $i < n$, $F{\restriction}n(i) = F(i)$. This construction is usually referred to as the *restriction of* $F$ *to* $n$.

The strong completeness axiom takes the following form, where $p$ is a given predicate. Note that this is in fact an operational axiom, the predicate $p$ is given, and $W$ is an arbitrary set in the universe of **OST**.

**Strong Completeness Axiom:**

$$(\exists F \in \mathcal{F}(W))(\forall n \in \omega)[(\exists X \in W)p(F{\restriction}n, X) \to p(F{\restriction}n, F(n))]$$

This axiom is operational, and it is intended to be valid for any non-empty set $W$. As explained above, the predicate $p$ is given, so here we have an axiom schema that becomes an axiom once the predicate $p$ has been determined.

**5.2.3.** The argument that supports this axiom involves a simple application of the definition of metainduction. We assume the predicate $p$ and a non-empty set $W$, and want to prove that there is $F \in \mathcal{F}(W)$ that satisfies the axiom. To prove there is such a function $F$ we generate a branch $\langle 0, w_0\rangle, \cdots, \langle n, w_n\rangle, \cdots$ as follows. Suppose that at $n$ there is in fact some $X \in W$ such that $p(F{\restriction}n, X)$ holds. If this is the case we proceed to set $w_n = X$, where $X$ is arbitrarily chosen among those sets in $W$ that satisfy the condition. If it is not the case, we take $w_n$ arbitrarily. By applying systematically this strategy we get a numerical function $F$ that satisfies the axiom.

Note that the operation $\mathcal{F}$ (that comes from the metainduction construction) is essential for the argument. For this operation provides the local existential quantifier that it is required for the axiom to be operational.

**5.2.3.1.** Furthermore, the argument in 5.2.3 is essentially recursive, for the value of $F(n)$ depends on the values $F(0), ..., F(n-1)$, although it is not completely determined by such values. In fact, the determination involves a choice, so we have a combination of recursion with choice.

**5.2.4.** The weak completeness axiom follows (in **OST**) from the strong completeness axiom. To prove this, assume $F'$ is a numerical function over $W$, so the predicate $\mathbf{NF}(F', W)$ holds. We want to show, using the strong completeness axiom, that $F' \in \mathcal{F}(W)$. To apply the strong completeness axiom we take a predicate $p$ defined:

$$p(Y, X) \equiv F'(\mathbf{do}(Y)) = X.$$

where $\mathbf{do}(Y)$ is the domain of $Y$ in case $Y$ is a function (otherwise it does not matter). From the strong completeness axiom it follows that there is $F \in \mathcal{F}(W)$ such that whenever for some $n$ there is $X \in W$ such that $p(F\lceil n, X)$ holds, then $p(F(\lceil n, F(n))$ holds. Hence (noting that $\mathbf{do}(F\lceil n) = n$), it follows that $F'(n) = F(n)$. Now, using induction on $n$, it follows that $F'(n) = F(n)$ for every $n$, hence $F' = F$ and $F' \in \mathcal{F}(W)$. This proof can be formalized in **OST** using mathematical induction on $n$ from 3.1.2.

**5.3.** The set $P(\omega)$, the power set of $\omega$, can be derived from the operation $\mathcal{F}$. For example, we can introduce this set with the following axioms, where $h$ is an auxiliary operation:

$$X \in h(F) \equiv X \in \mathbf{do}(F) \wedge F(X) = 0 \tag{1}$$

$$Z \in P(\omega) \equiv (\exists F \in \mathcal{F}(\{0,1\}))Z = h(F) \tag{2}$$

The operation $h$ follows by the standard separation rule, which is available in **OST**. On the other hand, the operation $P(\omega)$ follows by the general separation rule, proposed in [6] as the replacement rule.

**5.3.1.** The set $P(\omega)$ in the axiom above is actually generated via the strong completeness axiom, and this means that it can be forced to satisfy a previous inductive condition defined by a predicate $p$. As explained in 5.2.3.1, this process is essentially recursive, although it involves an element of choice.

On the other hand, the set $W$ in this application is actually $\{0, 1\}$ and this means that the tree in 2.1.3 is simply a binary tree where each branch is an infinite sequence of 0's and 1's.

§**6. Recursion.** Whenever there is a rule of proof by induction we have an associate recursion rule, where an operation $f$ can be defined (in the metainduction construction) and the values $f(X)$ of the operation are derived in a manner similar to the proof of $p(X)$ in the corresponding induction rule. Such a rule may not be well defined in somes cases.

**6.1** The first rule we must consider is residual induction in 3.1.1. Here, the purpose of the recursion rule is to introduce an operation $f$ with domain $\mathbf{tc}_\rho(Z)$ where the value of $f(Y)$ for $Y \in \mathbf{tc}_\rho(Z)$ is determined by the values of $f(X)$ whenever $Y \in \rho(X)$. In principle, such a rule is not well defined because for a given $Y$ there are many residuals $X'$ such that $Y \in \rho(X')$ and the value of $f(Y)$ will depend on which $X'$ is used. In other words, a recursion rule in this situation is not single-valued.

**6.1.1.** Obviously, we can overcome the difficulty in the preceding paragraph by considering a case where $\rho$ is strongly single-valued. This happens when $\mathbf{tc}_\rho(Z)$ is in fact the set $\omega$ in 3.2. Here the recursion rule takes the following form: the value of $f(X^{++})$ is determined from the value of $f(X^+)$, and the value of $f(\emptyset^+)$ is given.This is, of course, the well-known rule of *primitive recursion.*

**6.1.2** In the case of bar induction, the order of dependency is reversed, and there is no ambiguity in the recursion. Here the value of $f(X)$ is determined from the values of $f(Y)$ when $Y \in \rho(X)$. This form of recursion is known in the literature as *bar recursion.*

**6.1.3** The application of metainduction to numerical functions induces a form of induction combined with recursion, via the strong completeness axiom (see 5.2.3.1). This axiom depends on a given predicate $p$, so it is essentially inductive. At the same time it asserts the existence of a function $F \in \mathcal{F}(\{0,1\})$, where the value of $F(n)$ is determined, up to some point, from the values in $F\lceil n$, and this is essentially recursion.

**6.2.** Operational set theory by itself, as explained in the introduction, is an extremely weak theory, where not even the existence of an infinite set can be proved. By allowing metainductive constructions we move to a different system where a substantial portion of standard set theory can be formalized.

From metainduction, we get not only the set $\omega$, but also the set $P(\omega)$, hence the set $P(\omega \times \omega)$, which is the set of all countable relations, in particular the countable well-orders. Finally, we get the set of all countable ordinals, hence the ordinal $\omega_1$, the first non-countable ordinal. Still, the general power set operation is missing.

**7. Why Operational?** We have explained before our reservations concerning the notion of a universe of sets as a complete totality, and the need for restrictions where the extension of sets is determined independently of the universe (see 1.1.1).

We discuss now a pair of examples where sets are introduced in violation of this condition and in one case the result is the actual inconsistency of the theory. We note first, that in most systems of set theory, the introduction of a set $V$ with the properties of the universe (say via the axiom $(\forall Y)Y \in V$) is inconsistent.

**7.1.** The first example is the Russell paradox, where a set $R$ is introduced with the axiom:

$$Z \in R \equiv Z \notin Z$$

We know that the set $R$ is inconsistent, but here we are interested in the fact that the axiom involves a quantification over the universe. More precisely, in order to determine the extension of the set $R$ we must check every set $Z$ in the universe, to determine whether $Z \notin Z$. For example, if we write $(\exists Y \in R)\phi$ for some formula $\phi$, it may appear that we are using a local quantifier, but in fact to determine whether such $Y$ exists we must search over the whole universe.

Of course, it is not worthwhile to discuss the quantification involved in the definition of $R$ because it happens that $R$ is inconsistent. Still, one cannot help considering to what extent the two phenomena are related, and the universe is intrinsically inconsistent.

**7.2.** The second example is more important, because it involves the power set axiom, which is crucial in classical set theory.

We write $P(X)$ to denote the power set operation, which can be defined with the the following operational axiom:

$$Z \in P(X) \equiv (\forall Y \in Z) Y \in X.$$

The problem with this axiom is essentially the same: the axiom fails to determine the extension of the set $P(X)$, or more precisely describes the extension implicitly via the universe of sets.

On the other hand, the axiom is universally assumed to be consistent, and there is no expectation, in classical or non-classical set theory, that the axiom may eventually turn out to be inconsistent.

**7.2.1.** There are two general principles of set existence in standard set theory. One is the separation axiom, that requires that a new set should be expressed as a subset of an already available set. The second is the replacement axiom ([2],page 179), which is necessary, for example, in order to prove the principle of transfinite recursion. The power set axiom violates both principles. It does not involve separation and it does not involve replacement. Not only that, the power set operation is a crucial element in applications of separation, as in many cases it provides the extra set required by the rule.

It appears to us that the status of the power set axiom has not been properly examined. The axiom is included in most systems of set theory, apparently because it has always been there, and because it is essential for some constructions. Without the power set axiom more than half of standard set theory collapses.

**7.3.** We think that, at this stage, the power set axiom is not consistent with the operational program. Still, we are prepared to allow the axiom if an adequate construction is proposed. One such construction is proposed above, in 5.3, for the set $P(\omega)$. On the other hand no similar approach seems to be available for $P(P(\omega))$.

A more general construction was proposed in [6], Chapter 8. Unfortunately, we must withdraw this construction, now that we think it involves the universe as a complete totality (see also Section 8.5 in [6]).

## References


[1] Barwise, J., *Admissible Sets and Structures*, Berlin/Heidelberg/New York: Springer Verlag, 1975.

[2] Enderton, H.B., *Elements of Set Theory*, New York: Academic Press, 1977.

[3] Ershov, Y. L., *Definability and Computability*, New York: Consultants Bureau, 1996.

[4] Holmes, M.R., Review of [6] in *Mathematical Reviews*, **98** (1998), pp.1378-1379.

[5] Mayberry, J., Review of [6] in *The Journal of Symbolic Logic* **63** (1998), pp.751-752.

[6] Sanchis, L.E., *Set Theory – An Operational Approach*, Amsterdam: Gordon and Breach Science Publishers, 1997.



Department of Electrical Engineering and Computer Science, Syracuse University, Syracuse, NY 13244-4100

http://www.cis.syr.edu/~sanchis

*E-mail address*: Sanchis@top.cis.syr.edu