Modern Logic
Volume 8, Number 3 & 4 (May 2000-October 2001), pp. 121–126.

Review of

STEWART SHAPIRO, *PHILOSOPHY OF MATHEMATICS. STRUCTURE AND ONTOLOGY.*

New York/Oxford: Oxford University Press, 1997
x + 279 pp. ISBN 0-19-509452-2

ENRICO MORICONI

The background to the view developed in this book is constituted by the form that classic issues in the philosophy of mathematics — What is the ontological status of mathematical objects? Do numbers, sets, and so on, exist? What is the semantical status of mathematical statements? What do mathematical statements mean? Are they literally true or false, are they vacuous, or do they lack truth-values altogether? — received in the two famous papers Benacerraf devoted to *What Numbers Could Not Be* ([1]), in 1965, and to the notion of *Mathematical Truth* ([2]), in 1973. Both papers stress some difficulties concerning the *realistic* conception, which is taken in both the ontological sense (mathematical *objects* — numbers, functions, and the like — do exist), and in the semantical sense (each well-formed, meaningful sentence is determinately true or false). Difficulties stem from combining these theses with the requirement that mathematical statements have the same semantics as ordinary statements and with the so-called *causal* theory of knowledge: if mathematical objects are outside the causal nexus, how can we know anything about them? How is it possible to guarantee an epistemic access to a causally inert, eternal, and detached mathematical realm? Moreover, confronted with *different* ways to set-theoretically model virtually every kind of mathematical object, we are left without an answer to the question of which kind of objects *are*, for instance, the natural numbers, with the possible conclusion that numbers are not objects at all.

Contrary to what Benacerraf seems to suggest, Shapiro thinks that *antirealist* philosophies of mathematics don't fare any better: in fact, his structuralist program is a realism in ontology and a realism in truth-value. Preliminary to the characterization of his program is the emphasis on the importance of philosophy for mathematics. Dealing with the relationship between the practice of mathematics and the philosophy of mathematics, Shapiro rejects both the *philosophy-first principle* and the *philosophy-last-if-at-all principle*. Besides other criticisms, decisive against the philosophy-first-principle is its not being true to the history of mathematics. The axiom of choice represents an interesting case study: this principle was not accepted because realism sanctions it, but because it is needed. The opposite thesis, one version of which is Quine's naturalism, maintains the irrelevance of philosophy to mathematics. Shapiro doubts that this position is healthy, on balance, for either mathematics or philosophy. There are important questions which are not entirely mathematical matters. One job of the philosopher is to give an account of mathematics and its place in our intellectual lives. What is the subject matter of mathematics? What is the relationship between the subject matter of mathematics and the subject matter of science that allows such extensive application and cross-fertilization? How do we manage to do and know mathematics? How can mathematics be taught? How is mathematical language to be understood? How is mathematics applied in the sciences? (see p. 32).

After these preliminary clear-cut settlements, which are the topics of the first two chapters forming Part I of the book, "Perspective", in the three chapters of Part II, "Structuralism", Shapiro characterizes his structuralist program by holding, first, that a nonalgebraic field like arithmetic is about a realm of objects — numbers — that exists independently of the mathematician, and, second, that there is no more to the individual numbers "in themselves" than the relations they bear to each other. (Needless to say, a non-algebraic field, like arithmetic, analysis, and perhaps set theory, is intended to be about a single structure, or isomorphism type. On the contrary, algebraic fields, like group theory, field theory, topology, and so on, are not about a single structure that is unique up to isomorphism. Rather they are about a class of related structures.) The notion of *structure* is defined as the abstract form of a *system*, which, in its turn, is defined to be a collection of objects with certain relations. "Abstract" means that any features of the interrelated objects that do not affect how they relate to other objects in the system are ignored.

From an epistemological point of view, Shapiro mentions three main ways to apprehend a particular structure: the first is through a process of pattern recognition, or abstraction. A second way is through a direct description of it. The third way is given by the possibility of describing a structure as a variation of a previously understood structure. To this subject the fourth chapter, "Epistemology and Reference", is devoted. The third chapter, "Structure", concerns the ontological perspective. On this front, a first group of issues pertains to the status of the whole structure, such as the natural-number structure, as well as a baseball defense. The other group concerns the status of mathematical objects: natural numbers, players and so on. Beginning from the latter group, Shapiro states that "each mathematical object is a place in a particular structure" (p. 78). The natural-number structure may be exemplified by many different systems, and in any exemplification different objects can play the role of 2; what does not change is the *office*: the number 2 is the second place in the structure. On this basis, Shapiro thinks it possible to resolve Frege's *Caesar problem* ("to determine how and why each number is the same or different from *any object whatsoever*", p.78), and counteract criticism by Benacerraf and Kitcher against the thesis that numbers are objects. The solution, somewhat reminiscent of Carnap's *Empiricism, Semantics and Ontology* ([3]), is to consider that questions like "Is Julius Caesar = 2?" or "Is $1 \in 4$ (as for von Neumann), or not (as for Zermelo)?" are not *good* questions. "Good" questions concern relations between natural numbers *which can be defined in the language of arithmetic.* For instance, "Is $1 < 4$?" or "Can 5 be evenly divided by 3?". Asking whether 1 is an element of 4 cannot be answered, just like asking if 1 is braver than 4. The treatment of the Caesar problem is analogous: structuralism answers that anything can "be" 2, *i.e.*, $\{\{\emptyset\}\}$, $\{\emptyset, \{\emptyset\}\}$, Julius Caesar, anything can occupy that place in a system exemplifying the natural-number structure. Just as anybody prepared to play ball *can* be a shortstop, meaning that anybody can occupy that place, or play that role, in an infield system.

The previously recorded slogan, "mathematical objects are places in structures", is susceptible of two readings: the first is called the *places-are-offices* perspective, the second the *places-are-objects* perspective. We practice the former when we say, for instance, that the man currently occupying the office of vice president is more intelligent than his predecessor, or that "the von Neumann 2 has one more element than the Zermelo 2" (p. 82). When practicing the latter we treat the places of a given structure as objects in their own right. We may say that the vice president, regardless of who is the (wo)man who occupies that place, is president of the Senate. While the previous stance presupposed a background ontology capable to supply objects to fill the places of the structure, now the statements are about the respective structure as such, independent of any exemplifications it may have. The contrast places-are-offices/places-are-objects is related to the distinction between the two interpretations of the copula: as identity or as predication. When we adopt the first point of view and say, for instance, that 7 *is* the largest prime less than 10, we use the "is" of identity. Alternatively, if we say that $\{\emptyset, \{\emptyset\}\}$ *is* 2, we are of course not saying that $\{\emptyset, \{\emptyset\}\}$ "is identical to" 2, but that *it plays the rôle* (occupies the office) of 2 in the system of finite von Neumann ordinals. In this sense, we use the "is" of predication (relative to a system which exemplifies the structure).

As a technical tool to reason about systems, Shapiro thinks that isomorphism is a too demanding one since, for instance, being not isomorphic, we can not say, much as it seems natural to do, that the natural numbers with addition and multiplication and the natural numbers with addition, multiplication and less-than are systems which exemplify the same structure. More useful, and more suitable to express the "sameness of structure" relationship among systems, is judged the equivalence relation between systems (and structures) formulated by Resnik in "Mathematics as a Science of Patterns: Ontology and Reference" ([5]). Two systems M and N are defined to be *structure-equivalent* if there is a system R such that M and N are each isomorphic to full subsystems of R. And a system P is defined to be a *full subsystem* of R if they have the same objects and if every relation holding in R can be defined in terms of the relations of P. So, being less-than definable in terms of addition ($x < y$ iff $\exists z\,(z \neq 0 \;\&\; x + z = y)$), the natural numbers with addition and multiplication are a full subsystem of the natural numbers with addition, multiplication and less-than. Therefore, the two systems are trivially structure-equivalent: just let the natural numbers with addition, multiplication and less-than play the role of "R" in the previous definition. Shapiro notes that, being characterized in terms of definability, the notion of structure-equivalence, and hence a number of ontological matters, are dependent on the linguistic resources available in the background metalanguage. Strangely enough, instead of furthering the application of this notion to the analysis of some concrete systems, Shapiro devotes himself to outline an axiomatic treatment of structures. We are given an axiom of Infinity (there is at least one structure that has an infinite number of places), three axioms concerning substructures (Subtraction, Subclass, and Addition), two axioms aimed at assuring the existence of large structures (Powerstructure and Replacement), one axiom, Coherence, assuring that any coherent theory characterizes a structure, or a class of structures, and lastly a Reflection axiom which states that if ϕ, then there is a structure S that satisfies the other axioms of structure theory and ϕ (ϕ being any first- or second-order sentence in the language of structure theory). One realizes the impact of the last axiom if he considers that, letting ϕ be a tautology, Reflection entails the existence of a structure the size of an inaccessible cardinal. As regards Coherence, it is worth stressing that because second-order language is used, simple (proof-theoretic) consistency is not sufficient to guarantee that a theory describe a structure or class of structures. At pages 134-136, Shapiro dwells upon the possibility of equating coherence and consistency (where the latter notion is defined as the nonexistence of a sort of deduction, *i.e.*, the deduction of a contradiction; meaning, of course, not the lack of concrete tokens for the relevant deduction, but the nonexistence of a certain type) concluding that satisfiability (the assertion of the existence of a model) is a better analogue than consistency for coherence. This must be intended as meaning that satisfiability is a rigorous mathematical notion that provides a good mathematical *model* of coherence. As an analogy: "satisfiability is to coherence pretty much as recursiveness is to computability" (p.135).

The fifth and concluding chapter of the second part, "How we got here", retraces in a lucid and interesting way the steps of the idea that mathematics is the science of structures, mainly following the debates on the foundations of geometry which involved, among others, Helmholtz, Hilbert, Frege, Russell and Poincaré.

Part III of the book, "Ramifications and Applications", consists of three chapters and is devoted to evaluate the extension of structuralism to other aspects of mathematics and to science and ordinary language. A characteristic feature of this part of the book is that direct exposition of the various themes is largely interwoven in a penetrating survey of the relevant literature: Quine, Dummett, Field, Hellmann, Resnik, and others, are extensively discussed. In the reviewer's opinion, just this capacity of keeping together various threads, developing them from both an historical and a theoretical point of view, is what mainly shapes Shapiro's book and makes it an absolutely remarkable tool for anybody working in the foundations of mathematics.

References

[1] Benacerraf P., "What Numbers Could Not Be", *Philosophical Review* **74** (1965), pp. 47-73.

[2] ———, "Mathematical Truth", *Journal of Philosophy* **70** (1973), pp. 661-679.

[3] Carnap R., "Empiricism, semantics, and ontology", *Rev. Int. de Phil.* **4** (1950), pp. 20-40.

[4] Kitcher P., *The Nature of Mathematical Knowledge*, New York: Oxford Univ. Press, 1983.

[5] Resnik M., "Mathematics as a Science of Patterns: Ontology and Reference", *Nous* **15** (1981), pp. 529-550.

Università di Pisa, Dipartimento di Filosofia, Piazza Torricelli 2, 56126 Pisa, Italy
E-mail address: moriconi@fls.unipi.it