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A counterexample for the geometric traveling salesman problem in the Heisenberg group

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Abstract

We are interested in characterizing the compact sets of the Heisenberg group that are contained in a curve of finite length. Ferrari, Franchi and Pajot recently gave a sufficient condition for those sets, adapting a necessary and sufficient condition due to P. Jones in the Euclidean setting. We prove that this condition is not necessary.

Introduction

In the Euclidean setting a subset $E \subset \mathbb{R}^n$ is said to be d-rectifiable if there exists a countable family of Lipschitz maps $(f_k)_{k\in\mathbb{N}}$ from $\mathbb{R}^d$ to $\mathbb{R}^n$ such that $\mathcal{H}^d\left(E \setminus (\bigcup_{k=1}^{+\infty} f_k(\mathbb{R}^d))\right) = 0$, where $\mathcal{H}^d$ is the d-dimensional Hausdorff measure. Federer generalized this notion in [4] with the d-rectifiable metric spaces (X, ρ). These spaces have to be covered, up to a set of $\mathcal{H}^d_\rho$-measure 0 by $\bigcup_{k=1}^{+\infty} f_k(U_k)$ where f_k is Lipschitz, $U_k \subset \mathbb{R}^d$ and $\mathcal{H}^d_\rho$ is the d-dimensional Hausdorff measure with respect to distance ρ. Unfortunately, as observed by Ambrosio and Kirchheim [1], in the Heisenberg group with its Carnot-Carathéodory distance $(\mathbb{H}, d_c)$ this definition does not make much sense for dimensions $d \geq 2$. Indeed for these dimensions, d-rectifiable metric spaces included in $(\mathbb{H}, d_c)$ have vanishing d-Hausdorff measure. It should be noticed that a definition of rectifiable set in codimension 1 has been proposed by Franchi, Serapioni and Serra-Cassano [7, 8] in connection with sets of finite perimeter and BV functions. The case $d = 1$ is particular. Indeed, any rectifiable curve in a metric space can be parametrized by arclength and is the Lipschitz image of an interval of $\mathbb{R}$. Hence there is a lot of non-trivial 1-rectifiable metric spaces included in $(\mathbb{H}, d_c)$.

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A more quantitative study of rectifiability properties of subsets of the complex plane has been introduced by P. Jones in connection with problems in harmonic analysis and complex analysis (L^2-boundness of the Cauchy operator on Lipschitz graphs, geometric characterisation of removable sets for bounded analytic functions in $\mathbb{C}$). This study has been pursued by David and Semmes in general spaces and has led to the notion of uniform rectifiability [3]. From the work of P. Jones arises the following problem that is known now as the geometric traveling salesman problem or analyst's traveling salesman problem: under which condition is a compact set E in a metric space (X, ρ) contained in a rectifiable curve? In the complex plane, P. Jones gives a complete characterisation of such sets by introducing β numbers. These quantities measure how well the set E is approximated by straight lines at each scale and each place.

In [5] Ferrari, Franchi and Pajot adapted the β number of P. Jones to $\mathbb{H}$ and proved that a condition similar to that of P. Jones is sufficient for being contained in a rectifiable curve. In this paper, we prove that this condition is not necessary. Our counterexample is a curve $\omega([0,1]) \subset \mathbb{H}$ of finite length. This curve is constructed in an iterative way and Figure 1 represented the projection on $\mathbb{C}$ of the first three curves (where $\mathbb{H}$ is seen as $\mathbb{C} \times \mathbb{R}$). The construction is inspired by the construction of the classical Koch snowflake.

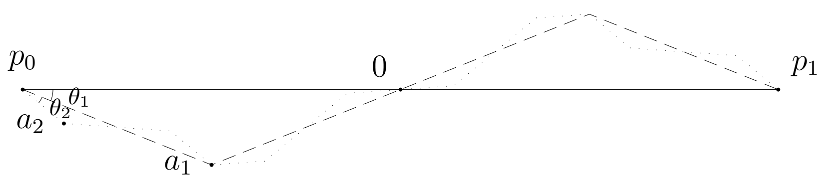

Figure 1: The counterexample curve.

0.1. Definitions

In order to give the characterization of P. Jones (and then of Ferrari, Franchi and Pajot), we must first define what is a dyadic net of a compact subset E in a metric space (X, ρ). It is an increasing sequence $(\Delta_k)_{k\in\mathbb{Z}}$ of subsets of E such that for any $k \in \mathbb{Z}$,

- for all $x_1, x_2 \in \Delta_k$, we have $x_1 = x_2$ or $\rho(x_1, x_2) > 2^{-k}$,
- for any $y \in E$ there exists $x \in \Delta_k$ such that $\rho(y, x) \leq 2^{-k}$.

Actually for any compact set E, there exists such a dyadic net $(\Delta_k)_{k\in\mathbb{Z}}$. In this paper the results are independent of the choice of the dyadic net. We define

$$B_X^\Delta(E) = \sum_{k\in\mathbb{Z}} 2^{-k} \sum_{x\in\Delta_k} \beta_X^2(x, A\cdot 2^{-k})(E) \tag{0.1}$$

where $A > 1$ is a constant to be specified (we will only assume that it is greater than 5) and $\beta_X(x,r)(E)$ depends on the ambient space. In the Euclidean case,

$$\beta_{\mathbb{R}^n}(x,r)(E) = \min_{l \text{ is a line}} \frac{\max_{y\in E\cap\mathcal{B}(x,r)} \rho(y,l)}{r}.$$

We consider the maximum distance to an Euclidean line l of the points of E that are included in $\mathcal{B}(x,r)$, the close ball of center x and radius r. The minimum of this quantity over l is $\beta_{\mathbb{R}^n}(x,r)(E)$. A set that is "flat" around x at scale r will have a small β number. We give a version of P. Jones' theorem as it is formulated in the survey [16]. The original theorem is given for dyadic squares instead of a dyadic net. Moreover the result proved in $\mathbb{R}^2$ by P. Jones [12] has actually been generalized by Okikiolu [15] who gave the reverse implication for the Euclidean spaces of greater dimensions.

Theorem 0.1. ([12, 15]) *There exists a constant $C > 0$ (independent of the dyadic net Δ) such that for any compact subset $E \subset \mathbb{R}^n$ with $B_{\mathbb{R}^n}^\Delta(E) < +\infty$, there are Lipschitz curves $\Gamma = \gamma([0,1]) \supset E$ satisfying the following inequality*

$$\mathcal{H}^1(\Gamma) \le C\left(\operatorname{diam}(E) + B_{\mathbb{R}^n}^\Delta(E)\right)$$

and for each Lipschitz curve Γ containing E

$$B_{\mathbb{R}^n}^\Delta(E) \le C\mathcal{H}^1(\Gamma).$$

In [17], Schul proved that in the previous result, one can find a constant C that is independent of the dimension n while it was not the case in the original proof of Theorem 0.1, where the β numbers are taken on dyadic squares (C depends exponentially on the dimension). It permitted him to prove a similar theorem for separable Hilbert spaces. From there it is natural to try to prove the same type of result in other metric spaces. In general metric spaces (X,ρ) there is an article by Haolama [10] where the author uses the Menger curvature in the definition of the β_X numbers. It seems actually not possible to define these β_X's as distance to some special lines of (X,ρ). In the case of the first Heisenberg group $\mathbb{H}$, Ferrari, Franchi and Pajot [5] obtain the exact counterpart of the beginning of Theorem 0.1

by using $\mathbb{H}$-lines (see Subsection 1.4 for the definition) in the definition of $\beta_{\mathbb{H}}(x,r)$. Precisely

$$\beta_{\mathbb{H}}(x,r)(E) = \min_{l \text{ is a } \mathbb{H}\text{-line}} \frac{\max_{y\in E\cap\mathcal{B}^{\mathbb{H}}(x,r)} d_c(y,l)}{r}$$

where the balls $\mathcal{B}^{\mathbb{H}}(x,r)$ are the close balls of $\mathbb{H}$.

The authors show that if the quantity $B^{\Delta}_{\mathbb{H}}(E)$ of (0.1) is finite, there is a rectifiable curve γ covering E. Note that as a rectifiable curve, γ has a Lipschitz parametrization on $[0,1]$. We give here a discrete version of the theorem – in the original theorem $B_{\mathbb{H}}$ is defined by integrating continuously the $\beta^2_{\mathbb{H}}$ on $\mathbb{H}\times\mathbb{R}^+$.

Theorem 0.2. ([5]) *Let E be a compact subset of $\mathbb{H}$ and Δ a dyadic net. Then if $B^{\Delta}_{\mathbb{H}}(E) < +\infty$ there is a Lipschitz curve $\Gamma = \gamma([0,1])$ such that $E\subset\Gamma$. Moreover, Γ can satisfy*

$$\mathcal{H}^1(\Gamma) \le C\left(\operatorname{diam}(E) + B^{\Delta}_{\mathbb{H}}(E)\right)$$

where the constant C is independent of E and of its dyadic net.

They also prove that for regular enough curves of finite length, $B^{\Delta}_{\mathbb{H}}$ is finite. We will define $\mathbb{H}$ and the horizontal curves in Section 1. For now consider that $\mathbb{H}$ is $\mathbb{C}\times\mathbb{R}$ with a special distance d_c.

Proposition 0.3. ([5]) *Let $\gamma : [0,1]\to\mathbb{H}$ be $\mathcal{C}^{1,\alpha}$-curve, i.e. a horizontal curve such that the projection on $\mathbb{C}$, $Z(\gamma)$ is a $\mathcal{C}^{1,\alpha}$-curve. Then*

$$B^{\Delta}_{\mathbb{H}}(\gamma([0,1])) < +\infty.$$

The previous theorem suggests that it should be possible to characterize any compact set E contained in a rectifiable curve by the condition $B_{\mathbb{H}}(E) < +\infty$. This would in particular happen for rectifiable curves themselves. Our curve $\omega([0,1])$ is a counterexample to this statement.

Theorem 0.4. *There is a Lipschitz curve $\omega : [0,1]\to\mathbb{H}$ such that for any dyadic net Δ of the set $\Omega = \omega([0,1])$,*

$$B^{\Delta}_{\mathbb{H}}(\Omega) = +\infty.$$

We introduce the Heisenberg group in the first section. In the second part of this paper, we complete our description of curves of $\mathbb{H}$ and we state two useful lemmas that estimate the distance of points to $\mathbb{H}$-lines. The third part is the construction of the curve Ω and in the fourth one we use the lemmas of Section 2 for reducing the problem to a planar geometry question and proving Theorem 0.4. The appendix about Dido's problem is important for the comprehension of Proposition 1.4 and Section 2.

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1. Horizontal curves in $\mathbb{H}$

The counterexample of this article is a horizontal curve. In this section we define horizontal curves and give their main properties. By curve we mean the continuous image of a closed interval of $\mathbb{R}$.

1.1. Definition of $\mathbb{H}$

We give a naive presentation of $(\mathbb{H}, d_c)$, the (first) Heisenberg group equipped with the Carnot-Carathéodory metric. As a set $\mathbb{H}$ can be written in the form $\mathbb{R}^3 \simeq \mathbb{C} \times \mathbb{R}$ and an element of $\mathbb{H}$ can also be written as (z,t) where $z := x + \mathbf{i}y \in \mathbb{C}$. The group structure of $\mathbb{H}$ is given by

$$(z,t)\cdot(z',t') = (z+z', t+t' - \frac{1}{2}\Im(z\overline{z'}))$$

where $\Im$ denotes the imaginary part of a complex number. $\mathbb{H}$ is then a Lie group with neutral element $0_{\mathbb{H}} := (0,0)$ and inverse element $(-z,-t)$.

Throughout this paper, $\tau_p : \mathbb{H} \to \mathbb{H}$ will be the left translation

$$\tau_p(q) = p \cdot q.$$

For $\lambda > 0$, we denote by δ_λ the dilation

$$\delta_\lambda(z,t) = (\lambda z, \lambda^2 t)$$

where $p, q \in \mathbb{H}$ and $\lambda \geq 0$.

We also introduce the rotations

$$\rho_\theta(z,t) = (\mathrm{e}^{\mathbf{i}\theta} z, t)$$

for any $\theta \in \mathbb{R}$.

1.2. Lifts and projections between $\mathbb{H}$ and $\mathbb{C}$

We first introduce the complex projection Z from $\mathbb{H}$ to $\mathbb{C}$ defined by

$$Z : (x,y,t) \mapsto (x + \mathbf{i}y).$$

A curve $\gamma(s) = (\gamma_x, \gamma_y, \gamma_t)$ of $\mathbb{H} = \mathbb{R}^3$ is said to be *horizontal* if it is absolutely continuous and

$$\dot{\gamma}_t(s) = \frac{\mathrm{d}}{\mathrm{d}s}\left[\mathcal{A}\left(Z(\gamma)\right)\right](s)$$

for almost every point s. Here $\mathcal{A}\left(Z(\gamma)\right)$ is the algebraic area swept by the curve $\alpha = Z(\gamma)$. It is uniquely defined by $\mathcal{A}(\alpha)(s_0) = 0$ where s_0 is the initial time and by the relation

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\mathcal{A}(\alpha)\right](s) = \frac{1}{2}(\alpha_y\dot{\alpha_x} - \alpha_x\dot{\alpha_y})$$

for almost every s.

Similarly we call *planar* curves the absolutely continuous curves of $\mathbb{C}$. The complex projections of horizontal curves γ are in particular planar curves. Moreover if one knows γ at the initial time and the complex projection $\alpha = Z(\gamma)$, it is possible to recover the whole horizontal curve by the formula giving the third coordinate:

$$\gamma_t(s) = \alpha_t(s_0) + \frac{1}{2}\int_{s_0}^{s}(\alpha_y\dot{\alpha_x} - \alpha_x\dot{\alpha_y}). \tag{1.1}$$

Thus we have the following proposition.

Proposition 1.1. *Let p be a point of $\mathbb{H}$. We denote by Υ_p the set of horizontal curves α such that α starts in p, and $\Upsilon_{p^\mathbb{C}}$ the set of planar absolutely continuous curves starting in $p^\mathbb{C} = Z(p)$. The projection Z is a bijection from Υ_p to $\Upsilon_{p^\mathbb{C}}$.*

We denote by Lift_p the inverse of Z from $\Upsilon_{p^\mathbb{C}}$ to Υ_p. We call it the $\mathbb{H}$-*lift* starting from p.

1.3. Direct similitudes

We introduce the complex direct similitudes

$$\begin{aligned} \delta_\lambda^\mathbb{C}(z) &= \lambda z \\ \tau_{a+\mathbf{i}b}^\mathbb{C}(z) &= a + \mathbf{i}b + z \\ \rho_\theta^\mathbb{C}(z) &= \mathrm{e}^{\mathbf{i}\theta}z. \end{aligned}$$

The complex projection Z almost commutes with δ_λ, τ_p and ρ_θ: we have to replace them by their corresponding complex similitudes. Precisely

$$\begin{aligned} Z(\delta_\lambda(z,t)) &= \delta_\lambda^\mathbb{C}(z) \\ Z(\tau_p(z,t)) &= \tau_{Z(p)}^\mathbb{C}(z) \\ Z(\rho_\theta(z,t)) &= \rho_\theta^\mathbb{C}(z). \end{aligned}$$

One can easily check that the transformations δ_λ, τ_p and ρ_θ conserve the horizontality of curves. As a consequence for Lift_p, we have the relations:

$$\begin{aligned}\delta_\lambda(\mathrm{Lift}_p(\alpha)) &= \mathrm{Lift}_{\delta_\lambda(p)}(\delta_\lambda^{\mathbb{C}}(\alpha))\\ \tau_p(\mathrm{Lift}_q(\alpha)) &= \mathrm{Lift}_{p\cdot q}(\tau_{Z(p)}^{\mathbb{C}}(\alpha))\\ \rho_\theta(\mathrm{Lift}_p(\alpha)) &= \mathrm{Lift}_{\rho_\theta}(p)(\rho_\theta^{\mathbb{C}}(\alpha)).\end{aligned}$$

1.4. Carnot-Carathéodory distance and geodesics

We define now the metric aspect of $\mathbb{H}$.

Definition 1.2. The length of a horizontal curve α of $\mathbb{H}$ is the length in $\mathbb{C}$ of the projected curve $Z(\alpha)$.

As a consequence of Subsection 1.3, the transformations δ_λ multiplies the length of a horizontal curve by λ. This quantity does not change under the action of ρ_θ and τ_p.

Definition 1.3. The Carnot-Carathéodory distance from $p \in \mathbb{H}$ to $q \in \mathbb{H}$ is the infimum of the length over the horizontal curves going from p to q.

Then the Carnot-Carathéodory distance between two points is invariant under the action of ρ_θ and τ_p. It is multiplied by λ if the points are dilated by δ_λ.

This infimum in Definition 1.3 is in fact a minimum and the minimizing curve is a $\mathbb{H}$-line or a $\mathbb{H}$-circle as we will see in Proposition 1.4. By $\mathbb{H}$-*line* we mean the $\mathbb{H}$-lift of a line of $\mathbb{C}$. Similarly a $\mathbb{H}$-*circle* is the $\mathbb{H}$-lift of a circle of $\mathbb{C}$. Here by circles and lines we don't mean the sets but the curves. In [5], the authors define the $\mathbb{H}$-lines as the left-translations $\tau_p(l_0)$ of the lines l_0 going through $0_{\mathbb{H}}$ in the plane $\mathbb{C} \times \{0_{\mathbb{R}}\}$ (actually the $\mathbb{H}$-lines going through $0_{\mathbb{H}}$). One can easily check that both definitions coincide.

Proposition 1.4. *For any two points p and q of $\mathbb{H}$, there is a shortest horizontal curve from p to q. It is the $\mathbb{H}$-lift of a line or of a circle arc.*

Proof. The horizontal curves from $p = (z_p, t_p)$ to $q = (z_q, t_q)$ are exactly the $\mathbb{H}$-lifts starting in p of those absolutely continuous planar curves connecting $z_p = Z(p)$ to $z_q = Z(q)$ that enclose an algebraic area $t_q - t_p$. Minimizing the length of these curves is the same as minimizing the length in this family of planar curves. This variational problem is strongly related to Dido's problem and its dual problem that are described both in the appendix. The main difference is that the planar curves we are considering here are not closed (except if $z_p = z_q$). Nevertheless the difference of algebraic area to the curves closed with the segment $[z_p, z_q]$ is a constant. Indeed it is up to a sign the area of the planar triangle $0_{\mathbb{C}} z_p z_q$. ∎

Remark 1.5. We will not prove the following important facts that are widely broadcasted. The Carnot-Carathéodory distance is a true distance. It provides the usual topology of $\mathbb{R}^3$. It is a geodesic distance and the length of the curves is also the length one can define from this distance (it is true for the curves of finite length and the curves of infinite length as well). See [6, 9, 13] for classical presentations introducing the subRiemannian structure.

1.5. Closed horizontal curves

If α and β are two curves such that the end point of α is the starting point of β, we define $\alpha\beta$ as the concatenation of the two curves. For α defined on $[a, b]$, let the reverse curve $\bar{\alpha}$ be defined on $[-b, -a]$ by $\bar{\alpha}(s) = \alpha(-s)$.

Lemma 1.6. *Let $z \in \mathbb{C}$, $z' \in \mathbb{C}$ and (α_1, α_2) two planar curves going from z to z', defined respectively on $[a_1, b_1]$ and $[a_2, b_2]$. Then the algebraic area swept by the concatenation $\overline{\alpha_2}\alpha_1$ is equal to the third coordinate of*

$$[\mathrm{Lift}(\alpha_1)(b_1) - \mathrm{Lift}(\alpha_2)(b_2)] - [\mathrm{Lift}(\alpha_1)(a_1) - \mathrm{Lift}(\alpha_2)(a_2)]$$

for any $\mathbb{H}$-lift $\mathrm{Lift}(\alpha_1)$ *and* $\mathrm{Lift}(\alpha_2)$ *of α_1 and α_2 respectively.*

Proof. We first assume that both $\mathbb{H}$-lifts $\mathrm{Lift}(\alpha_1)$ and $\mathrm{Lift}(\alpha_2)$ start in a same point p with $Z(p) = z$. Then $\overline{\mathrm{Lift}(\alpha_2)}\,\mathrm{Lift}(\alpha_1)$ is a $\mathbb{H}$-lift of $\overline{\alpha_2}\alpha_1$ and it follows that the planar curve encloses an algebraic area equal to the third coordinate of

$$\begin{aligned}[\mathrm{Lift}(\alpha_1)(b_1) - \mathrm{Lift}(\alpha_2)(b_2)] - [0] =& [\mathrm{Lift}(\alpha_1)(b_1) - \mathrm{Lift}(\alpha_2)(b_2)] \\ & - [\mathrm{Lift}(\alpha_1)(a_1) - \mathrm{Lift}(\alpha_2)(a_2)].\end{aligned}$$

The third coordinate difference between two $\mathbb{H}$-lifts of a same planar curve is a constant because of equation (1.1). The conclusion follows by making a vertical translation of $\mathrm{Lift}(\alpha_1)$ or $\mathrm{Lift}(\alpha_2)$. ■

2. Geometric Lemmas

In this section we will often use the exponent $^{\mathbb{C}}$ for $Z(\cdot)$. For example, we will write $l^{\mathbb{C}}$ and $p^{\mathbb{C}}$ for the complex projections of l and p respectively.

The orthogonal projection on a line of $\mathbb{C}$ has no obvious counterpart in $\mathbb{H}$. Let p be a point of $\mathbb{H}$ and l a $\mathbb{H}$-line. Starting from p and lifting horizontally the orthogonal segment from the $p^{\mathbb{C}}$ to the line $l^{\mathbb{C}}$, we will not reach l. Moreover the point $p^l \in l$ such that $Z(p^l)$ coincides with the orthogonal projection of $p^{\mathbb{C}}$ on $l^{\mathbb{C}}$ is not the metrically closest point of l to p. Hence we need special definitions:

Definition 2.1. Let $p \in \mathbb{H}$ and l be a $\mathbb{H}$-line. The $\mathbb{C}$-*projection* of p on l is the unique point $p^l \in l$ such that $p^{l,\mathbb{C}} := Z(p^l) = (p^l)^{\mathbb{C}}$ is the orthogonal projection of $p^{\mathbb{C}}$ on $l^{\mathbb{C}}$.

Now, let ζ be a planar line. There is a unique the point $p^\zeta \in \mathbb{H}$ such that

- $p^{\zeta,\mathbb{C}} := (p^\zeta)^{\mathbb{C}}$ is the orthogonal projection of $p^{\mathbb{C}}$ on the line ζ,
- p and p^ζ are on a $\mathbb{H}$-line.

We call p^ζ the *lifted-*$\mathbb{C}$*-projection* of p on ζ.

We give an example. The line of equation

$$x = 2 \quad \text{and} \quad t = 3 + y$$

is a $\mathbb{H}$-line. Its complex projection is $x = 2$. The $\mathbb{C}$-projection of the origin $0_{\mathbb{H}} = (0, 0, 0)$ on this line is $(2, 0, 3)$. The lifted-$\mathbb{C}$-projection on $x = 2$ is $(2, 0, 0)$ because $y = t = 0$ is a $\mathbb{H}$-line and its complex projection is orthogonal to $x = 2$.

Notice that like in the previous example, for a given $\mathbb{H}$-line l and a point $p \in \mathbb{H}$, the point $p^{l^{\mathbb{C}}} = p^{Z(l)}$ is a well-defined point of $\mathbb{H}$ and that it is not always on l. If it is, then $p^{l^{\mathbb{C}}} = p^l$ and this point also realizes the distance of p to l. In the Lemma 2.2, we give pieces of information about the metric projection of a point to a $\mathbb{H}$-line in the general case.

Lemma 2.2. *Let p be a point of $\mathbb{H}$ an l a $\mathbb{H}$-line. There is a point q on l that minimizes the distance to p. At the point $q^{\mathbb{C}}$ the Z-projection of the unique geodesic between p and q make a right angle with $l^{\mathbb{C}}$.*

Proof. It is easier to convince oneself with a look at Figure 2. It represents the situation seen from above, which is equivalent to the planar figure obtained by Z-projection. Nevertheless the names of the points and curves are the names of the figure in $\mathbb{H}$. There are many analytical or geometric ways to convince that the distance of p to a point of the $\mathbb{H}$-line tends to ∞ at the ends of this line. With a standard compactness argument, there is a point q on l (maybe not unique) that minimizes the distance to p and the geodesic from p to q is (one of) the shortest path from p to l. For now let q be any point of l and γ a horizontal curve from p to q. We will apply Lemma 1.6 in order to find a q and a γ minimizing the length of γ. For the first curve α_1, we start from $\alpha := \gamma^{\mathbb{C}}$ and continue it with a part of $l^{\mathbb{C}}$ going from $q^{\mathbb{C}}$ to $p^{l,\mathbb{C}} = p^{l^{\mathbb{C}},\mathbb{C}} \in l^{\mathbb{C}}$, the orthogonal projection of $p^{\mathbb{C}}$ on $l^{\mathbb{C}}$. The second curve (α_2 in Lemma 1.6) is the segment line from $p^{\mathbb{C}}$ to $p^{l,\mathbb{C}}$. The lemma brings us the following information: our closed curve $\overline{\alpha_2}\alpha_1$ encloses an algebraic area whose value $\mathcal{T}$ is the difference between the third coordinates of p^l and

$p^{l^{\mathbb{C}}} = p^{Z(l)}$. Indeed starting from p, the $\mathbb{H}$-lift of α_1 is γ and then a part of l until p^l. The $\mathbb{H}$-lift of α_2 is a segment of $\mathbb{H}$-line leading from p to $p^{l^{\mathbb{C}}}$. Using Lemma 1.6 with starting points p^l and $p^{l^{\mathbb{C}}}$ we can actually be more precise: the $\mathbb{H}$-lift starting in p of a planar curve α will reach the $\mathbb{H}$-line l if and only if the algebraic area enclosed by the concatenation of α, the segment $q0_{\mathbb{C}}$ and $q_{\mathbb{C}}p_{\mathbb{C}}$ is $\mathcal{T}$.

The following argument and the dual of Dido's problem conclude the proof: using the symmetry with respect to the line $l^{\mathbb{C}}$, the shortest curve from $p^{\mathbb{C}}$ to its symmetric point that covers the area $2\mathcal{T}$ is a circle arc. It is a symmetric curve with respect to $l^{\mathbb{C}}$ such that half of this curve is a solution α of our variational problem. The right angle in $q^{\mathbb{C}}$ is then a consequence of the symmetry. Note that the solution is unique if $p^{\mathbb{C}} \notin l^{\mathbb{C}}$. ■

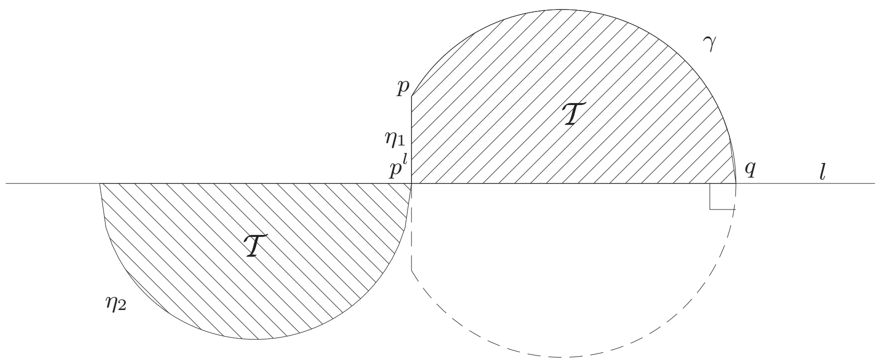

Figure 2: Projection lemmas

Remark 2.3. Another proof could use the Heisenberg gradient of the distance [2, 14].

We estimate now the distance of a point to a $\mathbb{H}$-line.

Lemma 2.4. *Let p be a point of $\mathbb{H}$ and l a $\mathbb{H}$-line. Then the distance of p to the line l is comparable to the Euclidean distance between the projections $p^{\mathbb{C}}$ and $l^{\mathbb{C}}$ plus the distance of the point $p^{l^{\mathbb{C}}} = p^{Z(l)}$ obtained by lifted-$\mathbb{C}$-projected to l. In fact*

$$\max\left(d_c(p^{\mathbb{C}}, l^{\mathbb{C}}), \frac{d_c(p^{l^{\mathbb{C}}}, l)}{\sqrt{2}}\right) \le d_c(p,l) \le d_c(p^{\mathbb{C}}, l^{\mathbb{C}}) + d_c(p^{l^{\mathbb{C}}}, l).$$

Proof. We use the same notations as in Lemma 2.2. We have in fact to compare the length of γ to the sum of the lengths of two curves: η_1, the

$\mathbb{H}$-line segment from p to $p^{l^{\mathbb{C}}} = p^{Z(l)}$ and η_2 one of the two possible shortest curves from $p^{l^{\mathbb{C}}}$ to l. The concatenation η of the two η_i's goes from p to l. It follows that the length of η is greater than the one of γ. For the other estimate, we just need to remark than each of the η_i is up to a constant smaller than γ. It is obvious for η_1 with constant 1. For η_2 we require one more time Lemma 1.6 and the dual of Dido's problem with a symmetrization in a similar way as in Lemma 2.2. We observe that $\eta_2^{\mathbb{C}}$ describes a half circle and encloses an algebraic area $\mathcal{T}$ as it is represented on Figure 2. We obtain that η_2 has a length smaller than $\sqrt{2}$ times the one of α i.e. the ratio is smaller than the quotient of the lengths of a circle and a half circle (solution of Dido's problem) with the same area. Indeed when we symmetrize $\eta_2^{\mathbb{C}}$ we obtain a circle of area $2\mathcal{T}$ such that the length of $\eta_2^{\mathbb{C}}$ is $2\sqrt{\pi\mathcal{T}}$. The curve $\alpha = \gamma^{\mathbb{C}}$ connected with its symmetrization enclose the same area such that the length of α is greater than $\sqrt{2\pi\mathcal{T}}$ (equality if α is a quarter circle). ■

We estimate the distance of two points to a $\mathbb{H}$-line.

Lemma 2.5. *Let p_1 and p_2 be two points that lie on a same $\mathbb{H}$-line and denote another $\mathbb{H}$-line by l. Then*

$$d(p_1, l) + d(p_2, l) \geq \frac{d(p_1^{\mathbb{C}}, l^{\mathbb{C}}) + d(p_2^{\mathbb{C}}, l^{\mathbb{C}}) + \sqrt{|\mathcal{U}(p_1^{\mathbb{C}} p_1^{l,\mathbb{C}} p_2^{l,\mathbb{C}} p_2^{\mathbb{C}})|}}{2}$$

where $\mathcal{U}(p_1^{\mathbb{C}} p_1^{l,\mathbb{C}} p_2^{l,\mathbb{C}} p_2^{\mathbb{C}})$ is the algebraic area of the trapezoid $p_1^{\mathbb{C}} p_1^{l,\mathbb{C}} p_2^{l,\mathbb{C}} p_2^{\mathbb{C}}$.

Proof. First of all $d(p_i^{\mathbb{C}}, l^{\mathbb{C}}) \leq d(p_i, l)$ for $i \in \{1, 2\}$ and we can sum these two relations. It is then enough to prove $d_c(p_1, l) + d_c(p_2, l) \geq \sqrt{|\mathcal{U}(p_1^{\mathbb{C}} p_1^{l,\mathbb{C}} p_2^{l,\mathbb{C}} p_2^{\mathbb{C}})|}$. For that we use Lemma 1.6 where we consider the two following curves (in fact their complex projections): On the one hand the $\mathbb{H}$-line segment of l from p_1^l to p_2^l and on the other hand the $\mathbb{H}$-polygonal line from $p_1^{l^{\mathbb{C}}} = p_1^{Z(l)}$ to $p_2^{l^{\mathbb{C}}} = p_2^{Z(l)}$ going through p_1 and p_2. Then the algebraic area of the trapezoid is the third coordinate of

$$[p_1^{l^{\mathbb{C}}} - p_1^l] - [p_2^{l^{\mathbb{C}}} - p_2^l].$$

Let $\mathcal{T}_i$ be the third coordinate of $[p_i^{l^{\mathbb{C}}} - p_i^l]$ for $i \in \{1, 2\}$ and write simply $\mathcal{U}$ instead of $\mathcal{U}(p_1^{\mathbb{C}} p_1^{l,\mathbb{C}} p_2^{l,\mathbb{C}} p_2^{\mathbb{C}})$. Then there is a i such that $|\mathcal{T}_i| \geq \frac{|\mathcal{U}|}{2}$. For this i we know that the distance of $p_i^{l^{\mathbb{C}}}$ to l is $\sqrt{2\pi|\mathcal{T}_i|}$ (Dido's problem or see the end of Lemma 2.4). Therefore and because of Lemma 2.4, we have $d_c(p_i, l) \geq d_c(p_i^{l^{\mathbb{C}}}, l)/\sqrt{2}$ and finally

$$d_c(p_1, l) + d_c(p_2, l) \geq \frac{d_c(p_i^{l^{\mathbb{C}}}, l)}{\sqrt{2}} \geq \frac{1}{\sqrt{2}}\sqrt{2\pi|\mathcal{T}_i|} \geq \frac{1}{\sqrt{2}}\sqrt{2\pi\frac{|\mathcal{U}|}{2}} \geq \sqrt{|\mathcal{U}|}.$$

■

3. Construction of $\omega([0,1])$

As we saw in Section 1, the $\mathbb{H}$-lift provides a direct link between the horizontal curves of $\mathbb{H}$ and the absolutely continuous curves of $\mathbb{C}$. We will describe our curve ω as the $\mathbb{H}$-lift starting in $\omega(0) = (-1,0,0)$ of a planar curve $\omega^{\mathbb{C}}$. This curve is a Koch snowflake-like fractal with finite length that we obtain as a limit of certain polygonal lines $(\omega_n^{\mathbb{C}})_{n\in\mathbb{N}}$ (see Figure 1 for a representation of $\omega_0^{\mathbb{C}}$, $\omega_1^{\mathbb{C}}$ and $\omega_2^{\mathbb{C}}$). Before we explain the recursive way to build the curves, we precise that ω and the ω_n will go from $(-1,0,0)$ to $(1,0,0)$. The direct consequence is that $\omega^{\mathbb{C}}$ and the $\omega_n^{\mathbb{C}}$ go from -1 to 1 in $\mathbb{C}$.

For the construction of $(\omega_n^{\mathbb{C}})_{n\in\mathbb{N}}$, we require a sequence $(\theta_n)_{n\geq 1}$ of nonnegative angles that tends to 0. We start from the simple line segment $\omega_0^{\mathbb{C}} : s \in [0,1] \mapsto (-1+2s, 0, 0)$ and we obtain $\omega_{n+1}^{\mathbb{C}}$ from $\omega_n^{\mathbb{C}}$ in the way we describe below. The curve $\omega_n^{\mathbb{C}}$ is made of 4^n segments having the same length. Let us denote this length by l_n and the total length by $L_n = 4^n \cdot l_n$. On the $(n+1)^{\mathrm{st}}$ step we change every segment line by a polygonal line made of 4 segments, having the same beginning and the same end. These four segments have length $\frac{l_n}{4\cos\theta_{n+1}}$ and all make with the former line segment an angle θ_{n+1} (see Figure 1). There are two ways to respect these conditions. However, the construction is unique if we precise the orientation: when the time grows the first of the 4 small segments makes a negative angle with respect to the segment of length l_n.

The important remark is that replacing the segment by the polygonal line of 4 segments, we do not change the swept algebraic area. Namely this area will be modified by the area of two equal isosceles triangles. One of them will be sumed with a positive sign and the other with a negative sign.

Let us define the value of the angles θ_n. In all this construction, it will be $\theta_n = \frac{C}{n}$ where $C = 0.2$. We prove now that $\omega^{\mathbb{C}}$ is well-defined as the limit of $\left(\omega_n^{\mathbb{C}}\right)_{n\in\mathbb{N}}$ where each $\omega_n^{\mathbb{C}}$ is parametrized with constant speed on $[0,1]$.

Proposition 3.1. *The sequence of curves $(\omega_n^{\mathbb{C}})_{n\in\mathbb{N}}$ tends to a rectifiable curve $\omega^{\mathbb{C}} : [0,1] \to \mathbb{H}$ parametrized with constant speed.*

Proof. The speed of the curves $\omega_n^{\mathbb{C}}$ is exactly the length L_n and they are L_n-Lipschitz. Let us prove the uniform convergence. The curves $\omega_n^{\mathbb{C}}$ and $\omega_{n+1}^{\mathbb{C}}$ coincide at every time $\frac{\sigma}{4^n} \in [0,1]$ where $\sigma = 0, \cdots, 4^n$. Between two subsequent meetings the curve ω_{n+1} always repeats the same motion pattern while ω_n is a segment. On $[\frac{\sigma}{4^n}, \frac{\sigma+1}{4^n}]$ the curves are the more distant at the end of the first segment, exactly at time $\frac{\sigma}{4^n} + \frac{1}{4^{n+1}}$. The maximum distance is also attained at time $\frac{\sigma}{4^n} + \frac{3}{4^{n+1}}$. From this observation we deduce

$$\|\omega_n^{\mathbb{C}} - \omega_{n+1}^{\mathbb{C}}\| = (\sin\theta_n) l_{n+1}.$$

The quotient between l_n and l_{n+1} is $\frac{1}{4\cos(\theta_{n+1})}$. Because all θ_n have a cosine greater than 0.5, this quotient is smaller than 1/2. We conclude that the series

$$\sum_{n=0}^{+\infty} \|\omega_{n+1}^{\mathbb{C}} - \omega_n^{\mathbb{C}}\| \leq \sum_{n=0}^{+\infty} (\sin\theta_n) l_0 \cdot 2^{-n}$$

converge.

In the next lemma we prove that $L := \limsup_{n\to+\infty} L_n < +\infty$. As a direct consequence $\omega^{\mathbb{C}}$ will be L-Lipschitz. We recall that $\theta_n = \frac{C}{n}$ where $C = 0.2$ and with a little trigonometry we see that $L_n = \frac{2}{\prod_{m=1}^{n}\cos\theta_m}$.

Lemma 3.2. *We have $L \leq 2.4 = 1.2 \cdot L_0$. Moreover, L is the Lipschitz constant and the length of $\omega^{\mathbb{C}}$.*

Proof. Because of the convexity of log, if $(1-x) \in [\mathrm{e}^{-1}, 1]$, then

$$\log(1-x) \geq \frac{-x}{1-\mathrm{e}^{-1}} \geq -2x.$$

It is possible to apply it to $x = \theta^2/2$ because $\theta \leq C \leq \sqrt{2-2\mathrm{e}^{-1}}$. Then we have

$$\begin{aligned}\log\left(\frac{1}{\prod_{n=1}^{N}\cos\theta_n}\right) &= -\sum_{n=1}^{N}\log(\cos\theta_n) \\ &\leq -\sum_{n=1}^{N}\ln(1-\frac{\theta_n^2}{2}) \leq \sum_{n=1}^{N}\theta_n^2 \leq C^2\frac{\pi^2}{6} \leq 0.08.\end{aligned}$$

Then we have $L \leq L_0\exp(0.08) \leq 1.2 \cdot L_0$.

We prove that L is the Lipschitz constant for $\omega^{\mathbb{C}}$. Indeed for $m \geq n$ the distance between $\omega^{\mathbb{C}}(\frac{\sigma}{4^n})$ and $\omega^{\mathbb{C}}(\frac{\sigma+1}{4^n})$ is $L_n/4^n$ because

$$\omega^{\mathbb{C}}(\frac{\sigma}{4^n}) = \omega_m^{\mathbb{C}}(\frac{\sigma}{4^n}) = \omega_n^{\mathbb{C}}(\frac{\sigma}{4^n}).$$

It follows also from the same observation that L is the length of $\omega^{\mathbb{C}}$. ■

We defined ω as the $\mathbb{H}$-lift of $\omega^{\mathbb{C}}$ starting from $(-1,0,0)$ and ω_n the one of $\omega_n^{\mathbb{C}}$ starting from $(-1,0,0)$. All these curves are parametrized with constant speed on $[0,1]$.

Lemma 3.3. *The curves ω_n and ω_{n+1} exactly coincide at the points $\frac{\sigma}{4^n}$ for $\sigma = 0,\ldots,4^n$.*

Proof. The property is surely true for $\sigma = 0$ because $\omega_{n+1}(0) = \omega_n(0) = (-1,0,0)$. Let σ be an integer smaller than $4^n - 1$. We assume that on $[0, \frac{\sigma}{4^n}]$ the curves ω_n and ω_{n+1} only coincide at the times $\frac{\sigma'}{4^n}$ for $\sigma' = 0, \cdots, \sigma$. Let us now focus on what happen on $[\frac{\sigma}{4^n}, \frac{\sigma+1}{4^n}]$. The curves are both starting from $\omega_n(\frac{\sigma}{4^n}) = \omega_{n+1}(\frac{\sigma}{4^n})$ and respectively lift $\omega_n^{\mathbb{C}}$ and $\omega_{n+1}^{\mathbb{C}}$. The previous planar curves coincide at $\frac{\sigma}{4^n}$, at $\frac{\sigma+1}{4^n}$ and at the midpoint $\frac{\sigma}{4^n} + \frac{1}{2\cdot 4^n}$. Then these are the only possible meeting points for ω_n and ω_{n+1} on $[\frac{\sigma}{4^n}, \frac{\sigma+1}{4^n}]$. Now, We consider two $\mathbb{H}$-lifts, starting from $\omega_{n+1}(\frac{\sigma}{4^n})$ and we will use Lemma 1.6 for them. On the one hand we lift horizontally $\omega_{n+1}^{\mathbb{C}}$ on $\left[\frac{\sigma}{4^n}, \frac{\sigma}{4^n} + \frac{1}{2\cdot 4^n}\right]$ and on the other hand we lift $\omega_n^{\mathbb{C}}$ on the same interval. Both planar curves arrive in the same point and the associated closed planar curve sweeps the positive area $(\frac{l_n^2 \cdot \tan(\theta_{n+1})}{16})$ of a triangle. This quantity is the difference for the third coordinate of the end points of the $\mathbb{H}$-lifts. We have

$$\omega_{n+1}(\frac{\sigma}{4^n} + \frac{1}{2\cdot 4^n}) \neq \omega_n(\frac{\sigma}{4^n} + \frac{1}{2\cdot 4^n}).$$

If we make the similar operation lifting $\omega_{n+1}^{\mathbb{C}}$ and $\omega_n^{\mathbb{C}}$ on $\left[\frac{\sigma}{4^n}, \frac{\sigma+1}{4^n}\right]$, we contrarily obtain an algebraic area equal to zero and can conclude that

$$\omega_{n+1}(\frac{\sigma+1}{4^n}) = \omega_n(\frac{\sigma+1}{4^n}).$$

■

A corollary of this lemma is that for any integer $m \geq n$, $\omega(\frac{\sigma}{4^n}) = \omega_m(\frac{\sigma}{4^n})$.

Remark 3.4. In the previous lemma, we noticed that $\omega_{n+1}(\frac{\sigma}{4^n} + \frac{1}{2\cdot 4^n})$ has the same first coordinates as $\omega_n(\frac{\sigma}{4^n} + \frac{1}{2\cdot 4^n})$ but the t-coordinate difference is $\frac{l_n^2 \cdot \tan(\theta_{n+1})}{16}$. Then the Carnot-Carathéodory distance between them is greater than $\frac{K}{4^n \cdot \sqrt{n}}$ for some constant K. It is an indication that the linear segments of ω_n are not a good approximation of ω. This observation can be the key of a heuristic computation in order to convice oneself of Theorem 0.4. As we will see in Proposition 4.1, the approximations by the other $\mathbb{H}$-lines are not better that the one by the segments used in the construction. Indeed in this proposition, the lower bound has order $\sqrt{\theta}$ which is also the order of the distance between (AE) and C.

Remark 3.5. We can observe that $\omega^{\mathbb{C}}$ is not differentiable in any point $\frac{\sigma}{4^n}$ for any n and $\sigma \leq 4^n$. Around these points, the curve is making a spiral because $\sum_{m=n}^{+\infty} \theta_m = +\infty$. However, $\omega^{\mathbb{C}}$ is a Lipschitz curve and is then almost everywhere derivable. In fact it seems that for a time $s \in [0,1]$, written $\overline{0, a_1 a_2 \ldots}^4$ in basis 4, the curve $\omega^{\mathbb{C}}$ is differentiable in s if and only if the series $\sum_{m=1}^{+\infty} \frac{\varepsilon(\overline{a_m}^4)}{m}$ converge. Here, ε is defined by

$$\varepsilon(0) = \varepsilon(3) = 1 \quad \text{and} \quad \varepsilon(1) = \varepsilon(2) = -1.$$

4. Counterexample for the inverse implication in [5]

In this section we prove Theorem 0.4, i.e., that $B^{\Delta}_{\mathbb{H}}(\omega([0,1]))$ is infinite. With the notations of the beginning of this paper, the first step will consist in estimating the cardinal of Δ_k. In the second step, we will estimate from below the value of $\beta_{\mathbb{H}}(x, A\cdot 2^{-k})$ for a $x \in \Delta_k$. For this we will require the geometric lemmas of Section 2.

Because of the second property of the net, $\omega \subset \bigcup_{x\in\Delta_k} \mathcal{B}^{\mathbb{H}}(x, 2^{-k})$. The projection of a ball for the Heisenberg metric on the complex plane is a ball of $\mathbb{R}^2$ with the same radius. That is why

$$\omega^{\mathbb{C}} \subset \bigcup_{x\in\Delta_k} \mathcal{B}^{\mathbb{C}}(x^{\mathbb{C}}, 2^{-k}).$$

If we perform a second projection on the real axis, we obtain that the segment $[-1,1]$ is covered by a family of segments of length 2^{-k+1} which is indexed by Δ_k. We conclude that the cardinal of Δ_k is greater than 2^k.

In this paragraph, we examine what is the right fractal scale of the portion of $\omega([0,1])$ intercepted by a ball $\mathcal{B}^{\mathbb{H}}(x, A\cdot 2^{-k})$ with center in Δ_k. Let us compare $A\cdot 2^{-k}$ to $\frac{L_\infty}{4^n} \leq \frac{2.4}{4^n}$ and assume $A = 5$ for the rest of this proof. We observe that for every $k > 0$ and $n = \lceil k/2 \rceil$, $\frac{2.4}{4^n}$ is smaller than $A\cdot 2^{-k}$. It follows that there is a $\sigma \in \{0,1,\cdots,4^n-1\}$ such that $\omega\left([\frac{\sigma}{4^n}, \frac{\sigma+1}{4^n}]\right) \subset \mathcal{B}(x, A\cdot 2^{-k})$.

If we rescale correctly the last portion of curve using the similitudes of the Heisenberg group (Subsection 1.3), we obtain a curve that could have been ω if we had chosen the sequence of angle $(\theta_{n+m})_{m=1}^{+\infty}$. In particular this curve includes the set Λ_θ made of the five points

$$\left\{(-1;0), (-\frac{1+i\tan(\theta)}{2}; \frac{\tan(\theta)}{4}), (0; \frac{\tan(\theta)}{4}), (\frac{1+i\tan(\theta)}{4}; \frac{\tan(\theta)}{2}), (1;0)\right\}$$

for $\theta = \theta_{n+1}$. We are interested in the maximal distance of one point of Λ_θ to a given $\mathbb{H}$-line l. We denote this distance by $d_\theta(l)$ and by D_θ the minimum of $d_\theta(l)$ over all the $\mathbb{H}$-lines l. We noticed that there is a similitude mapping Λ_θ on a part of $\omega \cap \mathcal{B}(x, A\cdot 2^{-k})$. This map multiplies the distances by $\frac{l_n}{2}$ where we recall that l_n is the length of the 4^n segments composing ω_n. Then the distance of $\omega \cap \mathcal{B}(x, A\cdot 2^{-k})$ to the closest $\mathbb{H}$-line is greater than $\frac{l_n}{2} D_\theta$ and

$$\begin{aligned} \beta_{\mathbb{H}}(x, A\cdot 2^{-k}) &\geq \frac{l_n}{2} \cdot \frac{D_\theta}{A\cdot 2^{-k}} \\ &\geq \frac{2.4\cdot D_\theta}{4^n\cdot A\cdot 2^{-k}} \\ &\geq \frac{D_\theta}{A}. \end{aligned} \tag{4.1}$$

Proposition 4.1. *Let $\theta < 0.2$ be a positive angle and l a $\mathbb{H}$-line. Then the maximum distance of one of the five points of Λ_θ to l is greater than $K \cdot \sqrt{\theta}$ for some constant K independent of l and θ. In other words*

$$D_\theta \geq K\sqrt{\theta}.$$

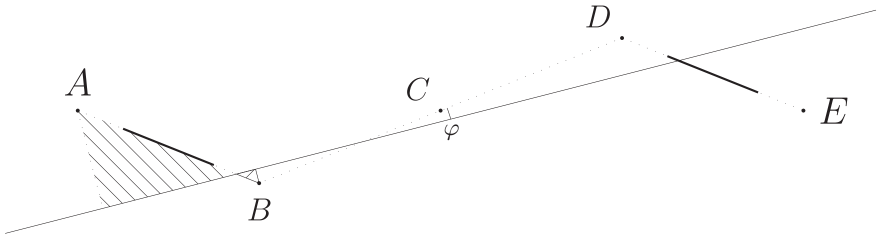

Figure 3: Some of the five points are far from a $\mathbb{H}$-line.

Proof. In this proof the points of $\mathbb{H}$ will be denoted with capital letters. We will write A, B, C, D, E where we would have wrote a, b, c, d, e before (and A is different from the real constant $A \geq 5$ introduced before). Let us first denote the five points by A, B, C, D, E where $A = (-1, 0, 0)$ and $E = (1, 0, 0)$ like on Figure 3. Thanks to the two geometric lemmas, Lemma 2.4 and Lemma 2.5, we will just have to consider the projections

$$\begin{aligned} A^{\mathbb{C}} &= -1 \\ B^{\mathbb{C}} &= -\frac{1}{2} - i\frac{\tan(\theta)}{2} \\ C^{\mathbb{C}} &= 0 \\ D^{\mathbb{C}} &= \frac{1}{2} + i\frac{\tan(\theta)}{2} \\ E^{\mathbb{C}} &= 1 \end{aligned}$$

and a planar line $l^{\mathbb{C}}$ together with the fact that some points are on a same $\mathbb{H}$-line. It is the case of the couples (A, B), (D, E) and (A, E). The three points B, C and D are also on a same $\mathbb{H}$-line.

In this proof, we will sort the possible planar lines $l^{\mathbb{C}}$ by the geometric angle $\varphi \in [0, \frac{\pi}{2}]$ they make with the line $(B^{\mathbb{C}}D^{\mathbb{C}})$. If $\varphi \geq \sqrt{\theta}$, then one of the point $B^{\mathbb{C}}$ or $D^{\mathbb{C}}$ is more distant than $l_\theta \sin\sqrt{\theta}$ to the line $l^{\mathbb{C}}$ where l_θ is the distance between $B^{\mathbb{C}}$ and $C^{\mathbb{C}}$ (it is also the distance between B and C in $\mathbb{H}$ or between $A^{\mathbb{C}}$ and $B^{\mathbb{C}}$ for example in $\mathbb{C}$). Then because of Lemma 2.4, the distance of the line l to the farthest point is greater than $\frac{1}{2} \cdot (\sqrt{\theta}\frac{2}{\pi})$.

If $\varphi \in [\frac{\theta}{4}, \sqrt{\theta}]$, we consider one of the segment $[B^{\mathbb{C}}C^{\mathbb{C}}]$ or $[C^{\mathbb{C}}D^{\mathbb{C}}]$ that the line $l^{\mathbb{C}}$ does not intersect. Let assume for example, $l^{\mathbb{C}}$ does not intersect $[B^{\mathbb{C}}C^{\mathbb{C}}]$. Then the area of the trapezoid obtained when we project $B^{\mathbb{C}}$ and $C^{\mathbb{C}}$ on $l^{\mathbb{C}}$ is greater that $\frac{l_\theta^2 \sin(\varphi)\cdot\cos(\varphi)}{2} \geq \frac{\sin(2\varphi)}{16}$. But $2\varphi \leq 2\sqrt{0.2} \leq \frac{\pi}{2}$. It follows that $\sin(2\varphi) \geq \frac{2\cdot 2\varphi}{\pi}$ and

$$\sqrt{|\mathcal{U}(B^{\mathbb{C}}, B^{l,\mathbb{C}}, C^{l,\mathbb{C}}, C^{\mathbb{C}})|} \geq \sqrt{\frac{\varphi}{4\pi}} \geq \sqrt{\frac{\theta}{16\pi}},$$

which thanks to Lemma 2.5 provides a lower bound for the distance to l with the right exponent of θ.

The last case, $\varphi \in [0, \frac{\theta}{4}]$ is the more intricate. Here, the line $l^{\mathbb{C}}$ can be very close to $(B^{\mathbb{C}}D^{\mathbb{C}})$. We will prove that it composes a great enough area when projecting orthogonally one of the segments $[A^{\mathbb{C}}B^{\mathbb{C}}]$ or $[D^{\mathbb{C}}E^{\mathbb{C}}]$ on $l^{\mathbb{C}}$. Unlike in the previous case, $l^{\mathbb{C}}$ can intersect both $[A^{\mathbb{C}}B^{\mathbb{C}}]$ and $[C^{\mathbb{C}}D^{\mathbb{C}}]$. Let assume for a while that $l^{\mathbb{C}}$ cannot intersect the central segment of $[A^{\mathbb{C}}B^{\mathbb{C}}]$ and the central segment of $[D^{\mathbb{C}}E^{\mathbb{C}}]$ where we mean by central segment the points on the segment obtained as barycenter of the ends with coefficients between $\frac{1}{4}$ and $\frac{3}{4}$. This assumption is true and we postpone it to Lemma 4.2. Assume for example that $l^{\mathbb{C}}$ does not intercept the central segment of $[A^{\mathbb{C}}B^{\mathbb{C}}]$. Then projecting $A^{\mathbb{C}}$ and $B^{\mathbb{C}}$ on $l^{\mathbb{C}}$, we compose a trapezoid (self-intersecting in the more difficult case as on Figure 3). The angle ψ between $l^{\mathbb{C}}$ and $(A^{\mathbb{C}}B^{\mathbb{C}})$ is contained in $[2\theta - \varphi, 2\theta + \varphi]$. Then $\frac{7\theta}{4} \leq \psi \leq \frac{\pi}{4}$. Hence we can estimate the algebraic area of the trapezoid in a similar way as in the previous case.

$$\begin{aligned}|\mathcal{U}(A^{\mathbb{C}}B^{\mathbb{C}}B^{l,\mathbb{C}}A^{l,\mathbb{C}})| &\geq \left(\frac{3\cdot l_\theta}{4}\right)^2 \frac{\sin(2\psi)}{4} - \left(\frac{l_\theta}{4}\right)^2 \frac{\sin(2\psi)}{4} \\ &\geq \frac{\sin(2\psi)}{32} \geq \frac{2\cdot(2\psi)}{\pi\cdot 32} \geq \frac{7\theta}{32\pi}.\end{aligned}$$

Then we have $\sqrt{|\mathcal{U}(A^{\mathbb{C}}B^{\mathbb{C}}B^{l,\mathbb{C}}A^{l,\mathbb{C}})|} \geq \sqrt{\theta\frac{7}{32\pi}}$ and Lemma 2.5 concludes the proof.

Lemma 4.2. *A planar line $l^{\mathbb{C}}$ that makes an angle $\varphi < \frac{\theta}{4}$ with $(B^{\mathbb{C}}D^{\mathbb{C}})$ can not intercept both the central segments of $[A^{\mathbb{C}}B^{\mathbb{C}}]$ and the one of $[D^{\mathbb{C}}E^{\mathbb{C}}]$.*

Proof. We argue by contradiction and assume that $l^{\mathbb{C}}$ intercepts both the central segment of $[A^{\mathbb{C}}B^{\mathbb{C}}]$ and the central segment of $[D^{\mathbb{C}}E^{\mathbb{C}}]$. We can suppose that $l^{\mathbb{C}}$ goes through $C^{\mathbb{C}}$. Actually as $[A^{\mathbb{C}}B^{\mathbb{C}}]$ is the image of $[D^{\mathbb{C}}E^{\mathbb{C}}]$ by central symmetry, the image $l'^{\mathbb{C}}$ of $l^{\mathbb{C}}$ by the same symmetry has the same property as $l^{\mathbb{C}}$. Namely it goes through the central segments. Moreover, because both central segments of $[A^{\mathbb{C}}B^{\mathbb{C}}]$ and $[D^{\mathbb{C}}E^{\mathbb{C}}]$ are convex, the parallel lines between $l^{\mathbb{C}}$ and $l'^{\mathbb{C}}$ also intercept these two sets. That is why we can assume that $l^{\mathbb{C}}$ is one of the two lines making an angle φ with $(B^{\mathbb{C}}D^{\mathbb{C}})$ and going through $C^{\mathbb{C}}$. It's not difficult to convince oneself that $l^{\mathbb{C}}$ can not cross the central segment of $[A^{\mathbb{C}}B^{\mathbb{C}}]$. Indeed, assume that we divide uniformly $[A^{\mathbb{C}}B^{\mathbb{C}}]$ in four equal parts and join the five points with $C^{\mathbb{C}}$, the greatest of the four angles is the one involving the line $(B^{\mathbb{C}}C^{\mathbb{C}})$. Then it is greater than $\theta/4$ which is the angle average and it is also greater than φ. This implies a contradiction. ■

By (4.1) and Proposition 4.1, we finally get

$$\begin{aligned} B^{\Delta}_{\mathbb{H}}(\omega([0,1])) &\geq \sum_{k\in\mathbb{N}} 2^{-k} \sum_{x\in\Delta_k} \beta^2_{\mathbb{H}}(x, A\cdot 2^{-k})(\omega([0,1])) \\ &\geq \sum_{k\in\mathbb{N}} 2^{-k}2^{k} \left(\frac{D_{\theta_{\lceil k/2\rceil+1}}}{A}\right)^2 \\ &\geq C\sum_{k\in\mathbb{N}} \frac{1}{\lceil k/2\rceil+1} \geq +\infty. \end{aligned}$$

Hence we have proved Theorem 0.4.

Remark 4.3. As we wrote in the introduction there is an analogue of the theory of the geometric salesman problem for metric space, using Menger curvature in the definition of beta numbers [11]. As we did in this paper for the theorem of [5], Schul presented in [16] a counterexample to the converse implication: the criterion of Haholamaa would not be necessary. However the counterexample of Schul should not be completely satisfactory (see [16, Subsection 3.3.1]). Notice that it seems that Ω, described in this paper cannot be turned into a counterexample for the approach of Haholamaa.

Appendix: Dido's problem

In Proposition 1.4 and Section 2 we use this kind of statement:

Proposition. (Dual of Dido's problem) *Over the absolutely continuous closed planar curves*

- *that are the concatenation of a segment with a path α,*
- *of given algebraic area $\mathcal{A}$,*

the half circles minimize the length $\mathcal{L}$ of the path α.

In a variant of this problem, the length of the segment is now given. Then $\mathcal{L}$ is minimized when α is a circle arc.

This proposition is actually the dual statement of a very old problem called Dido's problem [18]. Instead of proving the dual statement, we will state and prove the direct one. For that purpose we will suppose that we know that the solutions of the planar isoperimetric problem are circles. Dido's problem is related to the foundation of Carthage in Tunisia. It is written that Queen Dido and her followers arrived on a coast by the sea and that the local inhabitants allowed her to stay in as much land as can be encompassed in an oxhide. Then Dido made a rope by cutting the oxhide into fine strips and encircle a wide domain of land. Finding the way to limit this piece of land is a variant of the isoperimetric problem and we will see that the optimal way is to make a circle arc. However, the full circle is not optimal because it does not take advantage of the fact that the coast is a natural border. This classical result of calculus of variation can be reformulate in the following way:

Proposition. (Dido's problem) *Over the absolutely continuous closed planar curves*

- *that are the concatenation of a segment with a path* α
- *such that the length* $\mathcal{L}$ *of* α *is given,*

the half circles maximize the algebraic area $\mathcal{A}$ *of the curve.*

In a variant of this problem, the length of the segment is now given. Then $\mathcal{A}$ *is maximized when* α *is a circle arc.*

We present here a demonstration of this proposition.

Proof. We can fix the segment to be a part of the real axis $y = 0$. Consider the curves $\alpha : [0,1] \to \mathbb{C}$ of given length $\mathcal{L}$ such that $\alpha(0) = 0_{\mathbb{C}}$ and $\Im(\alpha(1)) = 0$ where $\Im$ is the imaginary part of a complex number. Then the problem is to maximize the algebraic area $\mathcal{A} = \frac{1}{2}\int_0^1 \alpha \times \dot{\alpha}$ for a given $\mathcal{L} = \int_0^1 |\dot{\alpha}|$. In the variant, $\alpha(1)$ is given. We will treat it just after the next paragraph.

The key idea is to close the curve α by connecting it with its symmetric curve with respect to the real axis. We obtain a closed curve whose swept area is twice the initial one.

$$\frac{1}{2}\int_0^1 \alpha \times \dot{\alpha} + \frac{1}{2}\int_1^0 \bar{\alpha} \times \dot{\bar{\alpha}} = 2 \cdot \frac{1}{2}\int_0^1 \alpha \times \dot{\alpha} = 2\mathcal{A}$$

(Here, $\bar{\alpha}$ is the complex conjugated curve. It is not the curve with inverse parametrization defined at the beginning of Subsection 1.5.) The length of this curve is also twice the initial one. If the new curve is a positively oriented circle, its algebraic area is the maximum among all closed curves with length $2\mathcal{L}$. This is in particular true among the curves symmetric with respect to $y = 0$. It follows that the solution of the authentic Dido's problem is a half circle. There are for a given starting point and a given area (positive or negative) exactly two solutions to the problem. These solutions are symmetric with respect to the starting point $0_{\mathbb{C}}$.

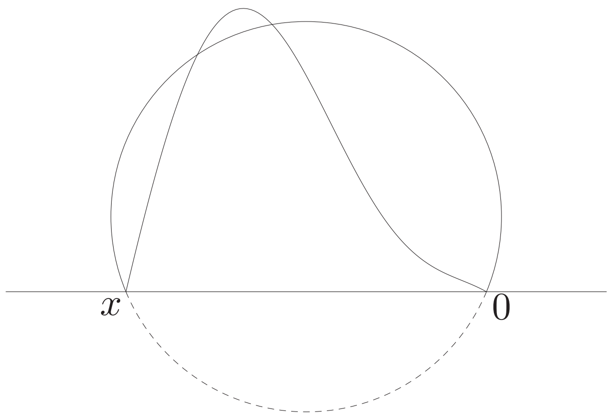

Figure 4: Two curves of same length.

A scheme of the variant of Dido's problem is presented on Figure 4. Here we fix the two ends of the curve so that the length of the segment is given. Let us assume for example $\alpha(0) = 0_{\mathbb{C}}$ and $\alpha(1) = x$ for a given $x \in [-\mathcal{L}, \mathcal{L}]\backslash\{0\}$. There is an unique circle arc of length $\mathcal{L}$ from the first to the second point that encloses a positive algebraic area. Indeed, the radius of the circle arcs is a strictly increasing and continuous function of $\mathcal{L}$. We prove that this circle arc enclose the greatest possible area $\mathcal{A}$. Compare our candidate with another curve and connect both of them with the rest of the circle. Hence we have two closed curves with the same length and one of them is a circle. The area of the circle is greater. Then the circle arc also encloses a greater area as the other curve. Thus we proved that the circle arcs of a given length provide the greatest area in Dido's problem with constraint. In the critical case $x = 0$, the problem is the classical isoperimetric problem. An infinite number of circles are solution. ■

The solutions of the dual problem presented at the beginning of this appendix can be proved in a similar way as we did for Dido's problem.

References

[1] Ambrosio, L. and Kirchheim, B.: Rectifiable sets in metric and Banach spaces. *Math. Ann.* **318** (2000), no. 3, 527–555.

[2] Ambrosio, L. and Rigot, S.: Optimal mass transportation in the Heisenberg group. *J. Funct. Anal.* **208** (2004), no. 2, 261–301.

[3] David, G. and Semmes, S.: *Analysis of and on uniformly rectifiable sets*. Mathematical Surveys and Monographs **38**. American Mathematical Society, Providence, RI, 1993.

[4] Federer, H.: *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften **153**. Springer-Verlag New York, New York, 1969.

[5] Ferrari, F., Franchi, B. and Pajot, H.: The geometric traveling salesman problem in the Heisenberg group. *Rev. Mat. Iberoam.* **23** (2007), no. 2, 437–480.

[6] Folland, G. B. and Stein, E. M.: *Hardy spaces on homogeneous groups*. Mathematical Notes **28**. Princeton University Press, Princeton, NJ, 1982.

[7] Franchi, B., Serapioni, R. and Serra Cassano, F.: Rectifiability and perimeter in the Heisenberg group. *Math. Ann.* **321** (2001), no. 3, 479–531.

[8] Franchi, B., Serapioni, R. and Serra Cassano, F.: On the structure of finite perimeter sets in step 2 Carnot groups. *J. Geom. Anal.* **13** (2003), no. 3, 421–466.

[9] Gromov, M.: Carnot-Carathéodory spaces seen from within. In *Sub-Riemannian geometry*, 79–323. Progr. Math. **144**. Birkhäuser, Basel, 1996.

[10] Hahlomaa, I.: Menger curvature and Lipschitz parametrizations in metric spaces. *Fund. Math.* **185** (2005), no. 2, 143–169.

[11] Hahlomaa, I.: *Menger curvature and Lipschitz parametrizations in metric spaces*. Report. University of Jyväskylä Department of Mathematics and Statistics **98**. University of Jyväskylä, Jyväskylä, 2005.

[12] Jones, P. W.: Rectifiable sets and the traveling salesman problem. *Invent. Math.* **102** (1990), no. 1, 1–15.

[13] Montgomery, R.: *A Tour of Subriemannian Geometries, their Geodesics and Applications*. Mathematical Surveys and Monographs **91**. American Mathematical Society, Providence, RI, 2002.

[14] Monti, R.: Some properties of Carnot-Carathéodory balls in the Heisenberg group. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* **11** (2001), no. 3, 155–167.

[15] Okikiolu, K.: Characterization of subsets of rectifiable curves in $\mathbb{R}^n$. *J. London Math. Soc. (2)* **46** (1992), no. 2, 336–348.

[16] Schul, R.: Analyst's traveling salesman theorems. A survey. In *In the tradition of Ahlfors-Bers. IV*, 209–220. Contemp. Math. **432**. Amer. Math. Soc., Providence, RI, 2007.

[17] SCHUL, R.: Subsets of rectifiable curves in Hilbert space–the analyst's TSP. *J. Anal. Math.* **103** (2007), 331–375.

[18] VERGIL: *Aeneis*. Reclam, 2003.

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