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Valiron's construction in higher dimension

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Abstract

We consider holomorphic self-maps φ of the unit ball $\mathbb{B}^N$ in $\mathbb{C}^N$ ($N = 1, 2, 3, \dots$). In the one-dimensional case, when φ has no fixed points in $\mathbb{D} := \mathbb{B}^1$ and is of hyperbolic type, there is a classical renormalization procedure due to Valiron which allows to semi-linearize the map φ, and therefore, in this case, the dynamical properties of φ are well understood. In what follows, we generalize the classical Valiron construction to higher dimensions under some weak assumptions on φ at its Denjoy-Wolff point. As a result, we construct a semi-conjugation σ, which maps the ball into the right half-plane of $\mathbb{C}$, and solves the functional equation $\sigma \circ \varphi = \lambda\sigma$, where $\lambda > 1$ is the (inverse of the) boundary dilation coefficient at the Denjoy-Wolff point of φ.

1. Introduction

1.1. The one-dimensional case

Let φ be a holomorphic map on $\mathbb{D}$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$. If φ has no fixed points in $\mathbb{D}$, then by the classical Wolff lemma (see, *e.g.*, [1]) there exists a unique point $\tau \in \partial\mathbb{D}$, called *the Denjoy-Wolff point of* φ, such that the sequence of iterates $\{\varphi^{\circ n}\}$ of φ converges uniformly on compacta to the constant map $\zeta \mapsto \tau$, $\forall \zeta \in \mathbb{D}$. Also, by the classical Julia-Wolff-Caratheodory theorem, τ is a fixed point (as non-tangential limit) for φ and the first derivative φ' has non-tangential limit $c \in (0, 1]$ at τ; moreover,

$$c = \liminf_{\zeta \to \tau} \frac{1 - |\varphi(\zeta)|}{1 - |\zeta|}.$$

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The number c is called the *multiplier* of φ or the *boundary dilatation coefficient* at τ. The map φ is called *hyperbolic* if $c < 1$ and *parabolic* if $c = 1$.

Geometrically, one defines the horodisks $H(t) := \{z \in \mathbb{D} : |\tau - z|^2/(1 - |z|^2) < 1/t\}$, which are disks in $\mathbb{D}$ internally tangent to $\partial\mathbb{D}$ at τ, and which get smaller as t gets larger. Then the following mapping property holds: $\phi(H(t)) \subset H(t/c)$. In formulas:

$$\frac{|\tau - \varphi(z)|^2}{1 - |\varphi(z)|^2} \le c\frac{|\tau - z|^2}{1 - |z|^2},$$

for every $z \in \mathbb{D}$.

In 1931 G. Valiron [14] (see also [15] and [4]) proved that if φ is hyperbolic then there exists a nonconstant holomorphic map $\theta : \mathbb{D} \to \mathbb{H} := \{w \in \mathbb{C} : \operatorname{Re} w > 0\}$ which solves the so-called *Schröder equation*:

$$\theta \circ \varphi = \frac{1}{c}\,\theta. \tag{1.1}$$

Valiron constructs the map θ as follows. First, in order to simplify notations, one can move to the right half-plane $\mathbb{H}$ via the Cayley map $C(\zeta) = (\tau + \zeta)/(\tau - \zeta)$, which takes τ to ∞ and conjugates φ to a self-map $\phi := C \circ \varphi \circ C^{-1}$ of $\mathbb{H}$, with Denjoy-Wolff point ∞ and multiplier $1/c$. Then, one considers the orbit $x_n + iy_n := \phi^{\circ n}(1)$ of the point $w = 1$, and studies the sequence of renormalized iterates:

$$\sigma_n(w) := \frac{\phi^{\circ n}(w)}{x_n}. \tag{1.2}$$

Valiron showed that $\{\sigma_n\}$ converges to a holomorphic map $\sigma : \mathbb{H} \to \mathbb{H}$ such that $\sigma \circ \phi = \frac{1}{c}\sigma$. Thus $\theta := \sigma \circ C$ solves (1.1).

After Valiron's construction, Ch. Pommerenke [11], [12], C. Cowen [6] and P. Bourdon and J. Shapiro [5] exploited other constructions to solve (1.1) (and the corresponding Abel's equation for the parabolic case). In particular, Pommerenke's approach in [11] is based on a slightly different, but equivalent, renormalization which replaces (1.2). The approach in [12], which works for random iteration sequences, needs some regularity hypothesis. On the other hand, Cowen's construction [6] is based on an abstract model relying strongly on the Riemann uniformization theorem. Finally, Bourdon and Shapiro's construction is based upon a different renormalization process which works only with some further regularity of φ at τ, but also guarantees some stronger regularity properties for the semi-conjugation θ.

In [4, Prop. 6] the first and last named authors proved that actually all those different methods (when applicable) provide essentially the same solution. Namely, if $\tilde{\sigma} : \mathbb{D} \to \mathbb{H}$ is another (nonconstant) solution of the functional equation (1.1) then there exists $\lambda > 0$ such that $\tilde{\sigma} = \lambda\sigma$.

Moreover, Valiron showed that σ comes with some guaranteed, but weak, regularity properties at τ. In function theory language, σ is semi-conformal (or isogonal) at τ, namely, σ fixes $\infty \in \partial\mathbb{H}$ non-tangentially and $\mathsf{Arg}\,\sigma$ has non-tangential limit 0 at ∞. As showed in [4], the semi-conformality of σ is essentially responsible for the uniqueness properties of σ and for the following dynamical properties of ϕ: for every orbit $z_n := \phi^{\circ n}(z_0)$, $\mathsf{Arg}\,z_n$ tends to a limit $\alpha(z_0) \in (-\pi/2, \pi/2)$ which depends harmonically on z_0, and conversely, given an angle $\alpha \in (-\pi/2, \pi/2)$ one can always find an orbit whose limiting argument is α.

1.2. Valiron's method in higher dimensions

In $\mathbb{C}^N$, $N = 2, 3, \ldots$, we let $\pi_j : \mathbb{C}^N \to \mathbb{C}$, $j = 1, \ldots, N$, be the coordinate mappings; the usual inner product is $\langle z_1, z_2 \rangle := \sum_{j=1}^N z_{1,j}\overline{z_{2,j}}$, where $z_{n,j} = \pi_j(z_n)$; the norm is $\|z\|^2 := \langle z, z \rangle$. The unit ball $\mathbb{B}^N$ is $\{z \in \mathbb{C}^N : \|z\|^2 < 1\}$.

Let φ be a holomorphic self-map of $\mathbb{B}^N$. If φ has no fixed points in $\mathbb{B}^N$ then B. MacCluer [10] proved that the Denjoy-Wolff theorem still holds. Namely, the sequence of iterates of φ, $\{\varphi^{\circ n}\}$, converges uniformly on compacta to the constant map $z \mapsto \tau$, $\forall z \in \mathbb{B}^N$, for a (unique) point $\tau \in \partial\mathbb{B}^N$ (called again the *Denjoy-Wolff point* of φ). Like in the one-dimensional case, the number

$$c := \liminf_{z \to \tau} \frac{1 - \|\varphi(z)\|}{1 - \|z\|},$$

belongs to $(0, 1]$ and is called the *multiplier* of φ or the *boundary dilatation coefficient* of φ at τ. Also, τ is a fixed point in the sense of non-tangential limits (and actually in the sense of K-limits as we define below). However, in this case the differential of φ might not have non-tangential limit at τ. The map φ is called *hyperbolic* if $c < 1$ and *parabolic* if $c = 1$.

Here too φ preserves certain ellipsoids internally tangent to $\partial\mathbb{B}^N$ at τ: defining

$$E(t) := \left\{ z \in \mathbb{B}^N : \frac{|1 - \langle z, \tau \rangle|^2}{1 - \|z\|^2} < 1/t \right\}, \tag{1.3}$$

then $\varphi(E(t)) \subset E(t/c)$. In formulas,

$$\frac{|1 - \langle \varphi(z), \tau \rangle|^2}{1 - \|\varphi(z)\|^2} \le c \frac{|1 - \langle z, \tau \rangle|^2}{1 - \|z\|^2}, \tag{1.4}$$

for every $z \in \mathbb{B}^N$.

Assuming some regularity for φ at τ, in the spirit of Bourdon-Shapiro, in [3] the first and the second named authors proved that, if φ is hyperbolic, one can solve the following functional equation:

$$\sigma \circ \varphi = A\sigma,$$

where $\sigma : \mathbb{B}^N \to \mathbb{C}^N$ is a nonconstant holomorphic map with good regularity properties at τ, and where A is the matrix $d\varphi_\tau$. Recently such a result has been improved in $\mathbb{B}^2$ by F. Bayart assuming less regularity for φ at τ (see [2] where also the parabolic case is considered).

On the other hand, the first and third named author in [4] have shown, that for all hyperbolic self-maps φ, i.e., with no regularity assumptions at τ, and for each orbit $z_n = \varphi^{\circ n}(z_0)$, there is a Koranyi region $K(\tau, R)$ such that z_n will tend to τ while staying in $K(\tau, R)$. Recall that, for $R > 1/2$, the *R-Koranyi approach region* at τ is a region of the form

$$K(\tau, R) := \{z \in \mathbb{B}^N : |1 - \langle z, \tau\rangle| < R(1 - \|z\|^2)\}. \tag{1.5}$$

The original aim, when looking for semi-conjugations in the one-dimensional case, was to show that general hyperbolic self-maps do indeed have a similar dynamical behavior as the hyperbolic automorphisms that share the same attracting fixed point.

In higher dimensions however, it is easy to construct maps whose image lies in a sub-variety with non-zero codimension, and thus automorphisms alone don't seem to be enough to model the dynamics of such maps (although one may try to consider automorphisms of lower dimensional balls). Also the fact that the differential of φ does not in general have a non-tangential limit at τ, shows that before trying to semi-conjugate φ to an automorphism on an higher-dimensional ball, it is preferable to study the following "one-dimensional" equation first.

Problem 1.1. *Find a nonconstant holomorphic map* $\Theta : \mathbb{B}^N \to \mathbb{H} \subset \mathbb{C}$ *such that*

$$\Theta \circ \varphi = \frac{1}{c}\Theta. \tag{1.6}$$

The aim of this paper is to try to solve Problem 1.1 by generalizing the method of Valiron to higher dimensions.

As in the one-dimensional case, it is more convenient to move to the Siegel domain

$$\mathbb{H}^N := \{(z, w) \in \mathbb{C} \times \mathbb{C}^{N-1} : \mathsf{Re}\, z > \|w\|^2\} \tag{1.7}$$

which is biholomorphic to $\mathbb{B}^N$ via the Cayley transform $\mathcal{C} : \mathbb{B}^N \to \mathbb{H}^N$ defined as

$$\mathcal{C}(\zeta_1, \zeta') := \left(\frac{1+\zeta_1}{1-\zeta_1}, \frac{\zeta'}{1-\zeta_1}\right). \tag{1.8}$$

Thus, if $\phi : \mathbb{H}^N \to \mathbb{H}^N$ is a hyperbolic holomorphic map with Denjoy-Wolff point ∞ and multiplier $1/c$, we define the following sequence

$$\sigma_n(z,w) := \frac{\pi_1 \circ \phi^{\circ n}(z,w)}{x_n}, \tag{1.9}$$

where, $\pi_1(z,w) := z$ is the projection on the first component and we set $x_n = \mathsf{Re}\,\pi_1(\phi^{\circ n}(1,0))$. For short we will say that the *Valiron method works* whenever the sequence $\{\sigma_n\}$ converges uniformly on compacta.

Our main result is the following:

Main Theorem. *Let $\varphi : \mathbb{B}^N \to \mathbb{B}^N$ be a hyperbolic holomorphic self-map with Denjoy-Wolff point $\tau \in \partial\mathbb{B}^N$ and multiplier $c < 1$. If*

(1) *there exists $z_0 \in \mathbb{B}^N$ such that the sequence $\{\varphi^{\circ n}(z_0)\}$ is special and*

(2) *the K-$\displaystyle\lim_{z\to\tau} \frac{1-\langle\varphi(z),\tau\rangle}{1-\langle z,\tau\rangle}$ exists,*

then the Valiron method works and there exists a nonconstant holomorphic function $\Theta : \mathbb{B}^N \to \mathbb{H}$ such that $\Theta \circ \varphi = \frac{1}{c}\Theta$.

In order to explain our hypotheses (1) and (2), we recall that a sequence $\{z_n\} \subset \mathbb{B}^N$ converging to a point $\tau \in \partial\mathbb{B}^N$ is said to be *special* if

$$\lim_{n\to\infty} \frac{\|z_n - \langle z_n,\tau\rangle\tau\|^2}{1-|\langle z_n,\tau\rangle|^2} = 0,$$

or, equivalently, the Kobayashi distance $k_{\mathbb{B}^N}(z_n, \langle z_n,\tau\rangle\tau)$, between $\{z_n\}$ and the projection of z_n along τ, tends to zero as $n \to \infty$. For the definition and properties of the Kobayashi distance we refer to [9] or [1]; we will only use the fact that the Kobayashi distance is invariant under biholomorphisms and that $k_{\mathbb{B}^N}(0,z) = \tanh^{-1}(\|z\|)$.

Moreover, a function $h : \mathbb{B}^N \to \mathbb{C}$ has *K-limit* L at $\tau \in \partial\mathbb{B}^N$, K-$\lim_{z\to\tau} h(z) = L$, if for any $R > 1/2$ and any sequence $\{z_n\} \subset K(\tau,R)$ converging to τ it follows that $\lim_{n\to\infty} h(z_n) = L$ (see [1] or [13]).

Notice that if $\varphi : \mathbb{B}^N \to \mathbb{B}^N$ is a hyperbolic holomorphic self-map with Denjoy-Wolff point $\tau \in \partial\mathbb{B}^N$ and multiplier $c < 1$, then Rudin's version of the classical Julia-Wolff-Caratheodory theorem (see [13, Thm. 8.5.6] or [1, Thm. 2.2.29]) implies that

$$\lim_{n\to\infty} \frac{1-\langle\varphi(z_n),\tau\rangle}{1-\langle z_n,\tau\rangle} = c \tag{1.10}$$

for all sequences $\{z_n\} \subset \mathbb{B}^N$ converging to τ such that $\{z_n\}$ is special and $\{\langle z_n,\tau\rangle\}$ converges to 1 non-tangentially in $\mathbb{D}$. Such a limit is called *restricted K-limit*. Unfortunately, it is easy to show that K-limits imply restricted K-limits, but not the converse. Thus, hypothesis (2) is a non-trivial requirement.

Condition (1) is not always easy to verify, unless, say, the map φ happens to fix (as a set) a slice ending at τ. For instance, under the regularity assumptions of [3] it follows that (2) holds, but it is not clear, *ex ante*, that (1) must also hold. On the other hand, once the semi-conjugation is established in [3], with good regularity properties, then it is easy to verify that (1) had to hold, *ex post.* In fact, we don't know of any explicit examples where (1) fails. So it could be the case that (1) is actually a superfluous hypothesis for the Main Theorem.

1.3. An example

The following is an example of a map as in the Main Theorem satisfying condition (1) but not (2) and for which the Valiron method still works.

Consider the map

$$\phi : \mathbb{H}^2 \ni (z, w) \mapsto (Az + Aw^2\psi(z), 0)$$

where $\psi : \mathbb{H} \to \mathbb{D}$ is any holomorphic function and $A > 1$. Then clearly, $\phi(\mathbb{H}^2) \subset \mathbb{H}^2$, ∞ is the Denjoy-Wolff point of ϕ, the multiplier is $A > 1$ and the sequence $\{\phi^{\circ n}(1, 0)\} = \{(A^n, 0)\}$ is special. Moreover,

$$\pi_1 \circ \phi^{\circ n}(z, w) = A^n z + A^n w^2 \psi(z).$$

Hence

$$\sigma_n(z, w) := \frac{\pi_1 \circ \phi^{\circ n}(z, w)}{x_n} = z + w^2\psi(z).$$

Therefore $\{\sigma_n\}$ does not depend on n and it can be checked easily that the map $\sigma(z, w) := z + w^2\psi(z)$ solves $\sigma \circ \phi = A\sigma$. Thus the Valiron method works. However, the K-limit of $\frac{\phi_1(z,w)}{z}$ at ∞ does not exist if ψ doesn't have a non-tangential limit at ∞. In particular, for such ψ, hypothesis (2) in the Main Theorem is not satisfied.

It is interesting to note that for such an example, the crucial equation (3.15) below becomes

$$\frac{Ax_n z + Ax_n w^2\psi(x_n z)}{x_n z} = A + A\frac{w^2}{z}\psi(x_n z),$$

and the limit for $n \to \infty$ does not exists if $w \neq 0$.

In particular, the regularity hypothesis (2) in the Main Theorem, while necessary in our proof, is not necessary for Valiron's method to work.

Our Main Theorem is proved in Section 3. In order to prove it, in Section 2 we introduce a new characterization of K-limits for functions, which we then develop in the Appendix into the notion of *E-limits*. We believe that the new understanding of K-limits which comes from the study of our E-limits might be a useful tool for other results. In the last section we include some further comments and open questions.

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2. Preliminaries on K-Limits

As mentioned before, we work in the Siegel domain (1.7). A direct computation using (1.5) and (1.8) shows that the Koranyi region $K(\tau, R)$ with vertex at τ and amplitude R in $\mathbb{B}^N$ corresponds to one with vertex at ∞ and amplitude $M := 2R > 1$ in $\mathbb{H}^N$ given by

$$K(\infty, M) := \left\{(z,w) \in \mathbb{H}^N : \|w\|^2 < \mathsf{Re}\, z - \frac{|z+1|}{M}\right\}. \tag{2.1}$$

To get a geometric feeling for these objects, notice that the ellipsoids $E(t)$ defined in (1.3) correspond in $\mathbb{H}^N$ to the sets

$$\mathcal{E}(T) := \left\{(z,w) \in \mathbb{H}^N : \mathsf{Re}\, z - \|w\|^2 > T\right\}$$

for some $T > 0$ depending on t. So, in particular, a sequence in $K(\infty, M)$ tending to infinity will eventually be contained in every $\mathcal{E}(T)$ for T large, because z tends to infinity when $(z,w) \in \mathbb{H}^N$ tends to infinity.

Notice also that the property (1.4) for a hyperbolic map $\phi = (\phi_1, \phi') : \mathbb{H}^N \to \mathbb{H}^N$ with multiplier $A > 1$ reads as follows:

$$\mathsf{Re}\, \phi_1(z,w) - \|\phi'(z,w)\|^2 > A(\mathsf{Re}\, z - \|w\|^2)$$

for every $(z,w) \in \mathbb{H}^N$.

We will find it convenient to use an equivalent characterization of K-limits. First we need a few definitions.

For $Z = (z,w) \in \mathbb{H}^N$, let $p(Z) := (z,0)$ be the projection of Z onto the complex line $L := \{(z,0) : z \in \mathbb{H}\} \subset \mathbb{H}^N$. If $k_{\mathbb{H}^N}$ denotes the Kobayashi distance on $\mathbb{H}^N$, a calculation shows that

$$k_{\mathbb{H}^N}(Z, p(Z)) = k_{\mathbb{H}^N}(Z, L).$$

Definition 2.1. Let $\{Z_n\} = \{(z_n, w_n)\} \subset \mathbb{H}^N$ converge to ∞.

(i) We say the convergence is *C-special* if there exists $0 \leq C < \infty$ such that

$$\limsup_{n\to\infty} k_{\mathbb{H}^N}(Z_n, p(Z_n)) \leq C,$$

where $k_{\mathbb{H}^N}$ is the Kobayashi distance on $\mathbb{H}^N$.

(ii) We say the convergence is *restricted* if $\{z_n\}$ converges non-tangentially to ∞ in $\mathbb{H}$.

Remark 2.2. Let $\{Z_n\} \subset \mathbb{H}^N$ be a sequence which converges to ∞. Then there exists $0 \le C' < +\infty$ such that $k_{\mathbb{H}^N}(Z_n, p(Z_n)) \le C'$ for all $n \in \mathbb{N}$ if and only if $\{Z_n\}$ is C-special for some $C \le C'$ (such C is in general strictly smaller than C').

Remark 2.3. The concepts just introduced of C-special and restricted sequences are formulated using the complex geodesic $z \in \mathbb{H} \mapsto (z, 0) \in \mathbb{H}^N$ and the projection associated to it. It turns out that being C-special and restricted do not depend on the chosen complex geodesic with ∞ in its boundary. This is used in the proof of the Main Theorem and could be useful in domains other than $\mathbb{H}^N$ and $\mathbb{B}^N$. For this reason, in the Appendix, Section 5, we provide a rigorous proof of this fact.

Remark 2.4. A 0-special sequence is simply referred to as *special*, see also [1] and [13].

Lemma 2.5. *Let $Z_n = (z_n, w_n) \in \mathbb{H}^N$ converge to ∞. Then, the following are equivalent:*

(1) *Z_n stays inside a Koranyi region $K(\infty, M)$ for some $1 < M < \infty$;*
(2) *Z_n is C-special, for some $C < \infty$, and is restricted;*
(3) *There is $0 < a < 1$ and $0 < T < \infty$, such that*

$$\|w_n\|^2 \le a \mathsf{Re}\, z_n \qquad \text{and} \qquad |\mathsf{Im}\, z_n| \le T \mathsf{Re}\, z_n.$$

The proof of Lemma 2.5 rests on the following computation. For $Z = (z, w) \in \mathbb{H}^N$, we compute the Kobayashi distance in $\mathbb{H}^N$ between Z and $p(Z)$. Set $z = x + iy$ and notice that the map $T(u, v) = (\frac{u-iy}{x}, \frac{v}{\sqrt{x}})$ is an automorphism of $\mathbb{H}^N$. Thus by invariance, we have

$$\begin{aligned} k_{\mathbb{H}^N}\big((z,0),(z,w)\big) &= k_{\mathbb{H}^N}\big((1,0),T(z,w)\big) \\ &= k_{\mathbb{B}^N}\big(0,\mathcal{C}^{-1}(T(z,w))\big) = \tanh^{-1}\|\mathcal{C}^{-1}(T(z,w))\| \\ &= \tanh^{-1}\|(0,\frac{w}{\sqrt{x}})\| = \tanh^{-1}\frac{\|w\|}{\sqrt{x}}. \end{aligned} \tag{2.2}$$

In other words, $k_{\mathbb{H}^N}(Z, p(Z)) = \tanh^{-1}(\|w\|/\sqrt{\mathsf{Re}\, z})$ and it is useful to recall that $\tanh^{-1}(s)$ is a positive increasing function on $(0, 1)$ with a vertical asymptote at 1.

Proof of Lemma 2.5. By (2.2), a sequence $\{Z_n\} = \{(z_n, w_n)\} \subset \mathbb{H}^N$ is C-special for some $0 < C < \infty$ if and only if $\limsup_{n\to\infty} \|w_n\|^2/(\mathsf{Re}\, z_n) \le a$ for some $0 < a < 1$. In fact, $a = \tanh C$. Thus, since $|\mathsf{Im}\, z_n| \le T\mathsf{Re}\, z_n$ is an usual formulation of non-tangentiality in $\mathbb{H}$, we have that (2) and (3) are equivalent.

Assuming (3) and writing $z_n = x_n + iy_n$, we have $|z_n+1|^2 \leq (1+T^2)x_n^2 + 2x_n + 1$. Thus

$$x_n - \frac{|z_n+1|}{M} \geq \left(1 - \frac{\sqrt{1+T^2}}{M}\right)x_n + o\left(\frac{1}{x_n}\right),$$

as x_n tends to infinity. Choose M large enough, so that $1 - \sqrt{1+T^2}/M < a < 1$. This ensures that $Z_n \in K(\infty, M)$ for all n large. So (3) implies (1).

Conversely, assume that $Z_n \in K(\infty, M)$ for some $1 < M < \infty$. Then, since

$$x_n - |z_n+1|/M \leq (1-1/M)x_n,$$

by (2.1), we have $\|w_n\|^2 \leq a\mathsf{Re}\, z_n$ with $a = 1-1/M$. Also, $|z_n+1| \leq M\mathsf{Re}\, z_n$, so $|\mathsf{Im}\, z_n| \leq M\mathsf{Re}\, z_n$. Hence, (1) implies (3). ■

3. The proof of the Main Theorem

We start by reformulating it in the context of $\mathbb{H}^N$.

Main Theorem (Siegel domain version)**.** *Let $\phi = (\phi_1, \phi') : \mathbb{H}^N \to \mathbb{H}^N$ be holomorphic, with Denjoy-Wolff point ∞ and multiplier $\lambda > 1$. Assume that*

(1) *There exists $Z_0 \in \mathbb{H}^N$ such that the sequence $\{\phi^{\circ n}(Z_0)\}$ is special.*

(2) $K\text{-}\lim_{\mathbb{H}^N \ni (z,w) \to \infty} \frac{\phi_1(z,w)}{z}$ *exists.*

Then Valiron's method works and there exists a non-constant holomorphic map $\sigma : \mathbb{H}^N \to \mathbb{H}$ such that

$$\sigma \circ \phi = \lambda \sigma.$$

Remark 3.1. By considering $T \circ \phi \circ T^{-1}$, where T is an automorphism of $\mathbb{H}^N$ fixing ∞ and such that $T(Z_0) = (1, 0)$, we can always assume that it is the sequence $\phi^{\circ n}(1, 0)$ that is special, see the proof of Lemma 5.2. So we will make this assumption in the sequel.

Remark 3.2. The Valiron method is invariant under conjugation, namely, let $\phi : \mathbb{H}^N \to \mathbb{H}^N$ be hyperbolic holomorphic with Denjoy-Wolff point ∞, let T be an automorphism of $\mathbb{H}^N$ fixing ∞ and let $\tilde{\phi} := T \circ \phi \circ T^{-1}$. Then the sequence $\{\sigma_n\} := \{(\pi_1 \circ \phi^{\circ n})/x_n\}$ given by (1.9) converges if and only if the sequence $\{\tilde{\sigma}_n\} := \{(\pi_1 \circ \tilde{\phi}^{\circ n})/\tilde{x}_n\}$ converges (here $\tilde{x}_n = \mathsf{Re}\, \pi_1(\tilde{\phi}^{\circ n}(1, 0)) = \mathsf{Re}\, \pi_1 \circ T \circ \phi^{\circ n} \circ T^{-1}(1, 0))$. In fact, by a direct computation, it turns out that if $\sigma_n \to \sigma$ as $n \to \infty$ then $\tilde{\sigma}_n \to (x_0 - \|w_0\|^2)\sigma \circ T^{-1}$, where $(x_0 + iy_0, w_0) := T^{-1}(1, 0)$. We leave the details of such a computation to the reader.

We need a preliminary result.

Lemma 3.3. *Let $\phi = (\phi_1, \phi') : \mathbb{H}^N \to \mathbb{H}^N$ be holomorphic, with Denjoy-Wolff point ∞ and multiplier $\lambda \geq 1$. Assume the sequence $\{\phi^{\circ n}(1,0)\}$ is special. Write $\phi^{\circ n}(1,0) = (z_n, w_n)$ and $z_n = x_n + iy_n$. Then*

(1) $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda.$

(2) *There exists $L \in \mathbb{R}$ such that* $\lim_{n\to\infty} \frac{y_n}{x_n} = L.$

Proof. As proved in [4, section 3.5], for any fixed $Z \in \mathbb{H}^N$, the orbit $\{\phi^{\circ n}(Z)\}$ stays in a Koranyi region with vertex at ∞ and so, in particular, it is restricted. Therefore, there exists $C > 0$ such that for all $n \in \mathbb{N}$

$$|y_n| \leq Cx_n. \tag{3.1}$$

By Rudin's version of the classical Julia-Wolff-Caratheodory theorem (1.10), reformulated in $\mathbb{H}^N$ (see Theorem 5.5 in the Appendix), since (z_n, w_n) is special and restricted, it follows that

$$\lim_{n\to\infty} \frac{z_{n+1}}{z_n} = \frac{\phi_1(z_n, w_n)}{z_n} = \lambda.$$

In particular we can write

$$z_{n+1} = \lambda z_n + o(1) z_n. \tag{3.2}$$

Dividing (3.2) by x_n and taking the real part, we obtain $\frac{x_{n+1}}{x_n} = \lambda + \mathsf{Re}\, o(1) - \frac{y_n}{x_n} \mathsf{Im}\, o(1)$. Taking the limit for $n \to \infty$, by (3.1), we get

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda, \tag{3.3}$$

which proves (1).

In order to prove (2), let $\{\frac{y_{n_k}}{x_{n_k}}\}$ be any convergent subsequence and let L be its limit. By (3.1), L is finite. Moreover,

$$\frac{z_{n+1}}{z_n} = \frac{x_{n+1}}{x_n} \frac{1 + i\frac{y_{n+1}}{x_{n+1}}}{1 + i\frac{y_n}{x_n}} \tag{3.4}$$

and by (3.2) and (3.3) we see that $\{\frac{y_{n_k+1}}{x_{n_k+1}}\}$ is also a convergent sequence with the same limit L. Assume by contradiction that there exists a converging subsequence $\{\frac{y_{m_k}}{x_{m_k}}\}$ with limit $L' \neq L$. Let

$$q_n := \frac{x_{n+1}}{x_n} + i\frac{y_{n+1} - y_n}{x_n}.$$

By (3.3), we have

$$\mathsf{Im}\, q_{n_k} = \frac{y_{n_k+1} - y_{n_k}}{x_{n_k}} = \frac{y_{n_k+1}}{x_{n_k+1}} \frac{x_{n_k+1}}{x_{n_k}} - \frac{y_{n_k}}{x_{n_k}} \longrightarrow L(\lambda - 1),$$

and similarly $\mathsf{Im}\, q_{m_k} \to L'(\lambda-1)$. Therefore $\{q_{n_k}\}$ converges to $\lambda + iL(\lambda-1)$ while $\{q_{m_k}\}$ converges to $\lambda + iL'(\lambda - 1)$.

We claim that $\{q_n\}$ can have at most two accumulation points, say a, a' (which must be necessarily $a = \lambda + iL(\lambda - 1)$ and $a' = \lambda + iL'(\lambda - 1)$). Assuming the claim is true, let U, U' be two open neighborhoods of a and a' respectively such that $U \cap U' = \emptyset$. Since $\{q_n\}$ has only a, a' as accumulation points by our claim, there exists n_0 such that for all $n > n_0$ then either $q_n \in U$ or $q_n \in U'$. Moreover, since $\{q_{n_k}\} \subset U$ for $n_k > n_0$ and $\{q_{m_k}\} \subset U'$ for $m_k > n_0$, one can select a subsequence $\{q_{l_k}\} \subset U$ such that $\{q_{l_k+1}\} \subset U'$. But this implies that $\{\frac{y_{l_k}}{x_{l_k}}\}$ converges to $L(\lambda - 1)$ while $\{\frac{y_{l_k+1}}{x_{l_k+1}}\}$ converges to $L'(\lambda - 1)$, contradicting our previous argument in (3.4).

We are left to show that $\{q_n\}$ can have at most two accumulation points. We already know that $\mathsf{Re}\, q_n \to \lambda > 1$. We are going to show that the (real) sequence $\{k_{\mathbb{H}}(1, q_n)\}$ of hyperbolic distances between 1 and q_n has limit, say d. Thus the accumulation points of $\{q_n\}$ must belong to the intersection between the real line $\{\zeta \in \mathbb{H} : \mathsf{Re}\, \zeta = \lambda\}$ and the boundary of the hyperbolic disc of center 1 and radius d, and this intersection consists of at most two points.

To see that $\{k_{\mathbb{H}}(1, q_n)\}$ converges, let us introduce the family of automorphisms of $\mathbb{H}^N$ given by

$$T_n(z, w) := \left(\frac{z - iy_n}{x_n}, \frac{w}{\sqrt{x_n}} \right). \tag{3.5}$$

Notice that $T_n(z_n, 0) = (1, 0)$ and $T_n \circ T_{n+1}^{-1}(1, 0) = (q_n, 0)$, from which we obtain that

$$\begin{aligned} k_{\mathbb{H}}\big(1, q_n\big) &= k_{\mathbb{H}^N}\big((1,0), (q_n, 0)\big) = k_{\mathbb{H}^N}\big((1,0), T_n \circ T_{n+1}^{-1}(1,0)\big) \\ &= k_{\mathbb{H}^N}\big(T_n^{-1}(1,0), T_{n+1}^{-1}(1,0)\big) = k_{\mathbb{H}^N}\big((z_n, 0), (z_{n+1}, 0)\big). \end{aligned} \tag{3.6}$$

Now, by the contracting property of Kobayashi's distance,

$$\begin{aligned} k_{\mathbb{H}^N}\big((z_n, 0), (z_{n+1}, 0)\big) &= k_{\mathbb{H}}\big(z_n, z_{n+1}\big) \\ &= k_{\mathbb{H}}\big(\pi_1(z_n, w_n), \pi_1(z_{n+1}, w_{n+1})\big) \\ &\le k_{\mathbb{H}^N}\big((z_n, w_n), (z_{n+1}, w_{n+1})\big). \end{aligned} \tag{3.7}$$

On the other hand, by the triangle inequality,

$$\begin{aligned} k_{\mathbb{H}^N}\big((z_n, 0), (z_{n+1}, 0)\big) &\ge k_{\mathbb{H}^N}\big((z_n, w_n), (z_{n+1}, w_{n+1})\big) \\ &\quad - k_{\mathbb{H}^N}\big((z_n, 0), (z_n, w_n)\big) - k_{\mathbb{H}^N}\big((z_{n+1}, 0), (z_{n+1}, w_{n+1})\big). \end{aligned} \tag{3.8}$$

Since $\{(z_n, w_n)\}$ is special, both the function $k_{\mathbb{H}^N}((z_n, 0), (z_n, w_n))$ and the function $k_{\mathbb{H}^N}((z_{n+1}, 0), (z_{n+1}, w_{n+1}))$ tend to 0 as $n \to \infty$. Therefore, from (3.6), (3.7) and (3.8) it follows

$$\begin{aligned}\lim_{n\to\infty} k_{\mathbb{H}}\big(1, q_n\big) &= \lim_{n\to\infty} k_{\mathbb{H}^N}\big((z_n, 0), (z_{n+1}, 0)\big) \\ &= \lim_{n\to\infty} k_{\mathbb{H}^N}\big((z_n, w_n), (z_{n+1}, w_{n+1})\big),\end{aligned}$$

and the latter limit exists because the sequence $\{k_{\mathbb{H}^N}((z_n, w_n), (z_{n+1}, w_{n+1}))\}$ is non-increasing in n since the Kobayashi distance is contracted by holomorphic maps. ■

Proof of the Main Theorem. As mentioned in Remark 3.1 and Remark 3.2, after conjugating ϕ with some automorphism of $\mathbb{H}^N$ we can suppose that $Z_0 = (1, 0)$, and as we saw in the proof of Lemma 3.3, the orbit of $(1, 0)$ is thus both special and restricted. Moreover, see Proposition 5.8 in the Appendix, the conjugation made does not effect our regularity hypothesis, namely

$$K\text{-}\lim_{\mathbb{H}^N \ni (z,w)\to\infty} \frac{\phi_1(z, w)}{z} = \lambda. \tag{3.9}$$

Letting $(z_n, w_n) := \phi^{\circ n}(1, 0)$, $z_n = x_n + iy_n$, and using Lemma 2.5, we see that

$$\lim_{n\to\infty} \frac{\|w_n\|}{\sqrt{x_n}} = 0. \tag{3.10}$$

Now we consider the Valiron-like sequence $\{\sigma_n\}$ of holomorphic maps from $\mathbb{H}^N$ to $\mathbb{H}$ defined by

$$\sigma_n(z, w) := \frac{\pi_1 \circ \phi^{\circ n}(z, w)}{x_n},$$

where, as usual, $\pi_1(z, w) := z$ is the projection on the first component. Notice that

$$\sigma_n \circ \phi = \frac{\pi_1 \circ \phi^{\circ(n+1)}}{x_n} = \frac{x_{n+1}}{x_n}\sigma_{n+1}. \tag{3.11}$$

If we can prove that the sequence $\{\sigma_n\}$ converges uniformly on compacta to a non-constant map $\sigma : \mathbb{H}^N \to \mathbb{H}$ (which is necessarily holomorphic), then by taking the limit for $n \to \infty$ in (3.11), and by Lemma 3.3 (1), we obtain that $\sigma \circ \phi = \lambda\sigma$.

We will now show that $\{\sigma_n\}$ is uniformly convergent on compacta to a non-constant function.

First of all, we notice that by Lemma 3.3 (2),

$$\sigma_n(1, 0) = 1 + i\frac{y_n}{x_n} \longrightarrow 1 + iL, \quad \text{as } n \to \infty.$$

And, on the other hand, again by Lemma 3.3

$$\sigma_n(\phi(1,0)) = \frac{\pi_1 \circ \phi^{\circ(n+1)}(1,0)}{x_n} = \frac{x_{n+1}+iy_{n+1}}{x_n}$$
$$= \frac{x_{n+1}}{x_n} + i\frac{x_{n+1}}{x_n}\frac{y_{n+1}}{x_{n+1}} \to \lambda + i\lambda L,$$

as $n \to \infty$. Since $\lambda > 1$, the above proves that any limit of the sequence $\{\sigma_n\}$ cannot be constant.

Now we are going to prove that for any $(z,w) \in \mathbb{H}^N$

$$\lim_{n\to\infty} k_{\mathbb{H}}(\sigma_n(z,w), \sigma_{n+1}(z,w)) = 0. \tag{3.12}$$

To this aim, we first notice that the set $\{\sigma_n(z,w)\}$ is relatively compact in $\mathbb{H}$. Indeed, let $\pi_w : \mathbb{C}^N \to \mathbb{C}^{N-1}$ be the projection $\mathbb{C} \times \mathbb{C}^{N-1} \ni (z,w) \mapsto w \in \mathbb{C}^{N-1}$ and define

$$S_n(z,w) := \left(\sigma_n(z,w), \frac{\pi_w(\phi^{\circ n}(z,w))}{\sqrt{x_n}}\right). \tag{3.13}$$

Notice that $S_n = L_n \circ \phi^{\circ n}$, where L_n is the automorphism of $\mathbb{H}^N$ defined by $L_n(z,w) = (z/x_n, w/\sqrt{x_n})$. Therefore $S_n : \mathbb{H}^N \to \mathbb{H}^N$. Moreover, by Lemma 3.3 (1) and (3.10)

$$S_n(1,0) = \left(\sigma_n(1,0), \frac{w_n}{\sqrt{x_n}}\right) = (1 + i\frac{y_n}{x_n}, \frac{w_n}{\sqrt{x_n}}) \to (1+iL, 0), \tag{3.14}$$

as $n \to \infty$.

In particular there exists $C > 0$ such that $k_{\mathbb{H}^N}(S_n(1,0), (1+iL,0)) < C$ for all $n \in \mathbb{N}$. Therefore, by the triangle inequality and the contraction property,

$$k_{\mathbb{H}^N}\big(S_n(z,w), (1+iL,0)\big) \le k_{\mathbb{H}^N}\big(S_n(z,w), S_n(1,0)\big)$$
$$+ k_{\mathbb{H}^N}\big(S_n(1,0), (1+iL,0)\big)$$
$$\le k_{\mathbb{H}^N}\big((z,w),(1,0)\big) + C,$$

which proves that $\{S_n(z,w)\}$ is relatively compact in $\mathbb{H}^N$.

Now, notice that

$$\sigma_{n+1} = \pi_1 \circ L_{n+1} \circ \phi \circ L_n^{-1} \circ S_n.$$

Since we already proved that the sequence $\{S_n(z,w)\}$ is relatively compact in $\mathbb{H}^N$, (3.12) will follow if we prove that $\pi_1 \circ L_{n+1} \circ \phi \circ L_n^{-1} \to \pi_1$ as $n \to \infty$. A direct computation shows that

$$\pi_1 \circ L_{n+1} \circ \phi \circ L_n^{-1}(z,w) = \frac{\pi_1(\phi(x_n z, \sqrt{x_n}w))}{x_n z}\frac{x_n z}{x_{n+1}}. \tag{3.15}$$

Now for all $n \in \mathbb{N}$, by (2.2)

$$k_{\mathbb{H}^N}\big((x_n z, \sqrt{x_n} w), (x_n z, 0)\big) = \tanh^{-1} \frac{\|w\|}{\sqrt{\mathsf{Re}\, z}} < \infty,$$

and clearly $\{x_n z\}$ converges to ∞ non-tangentially in $\mathbb{H}$. Thus the sequence $\{(x_n z, \sqrt{x_n} w)\}$ is C-special and restricted. Hence, by applying (3.9) and Lemma 3.3 (1) to the limit as $n \to \infty$ in (3.15), we get $\pi_1 \circ L_{n+1} \circ \phi \circ L_n^{-1}(z, w) \to z$ as $n \to \infty$, as needed.

At this point, let $\{\sigma_{n_k}\}$ be a convergent subsequence of $\{\sigma_n\}$ and let σ be its limit, which we know is non-constant. By (3.12), $\{\sigma_{n_k+1}\}$ also converges to σ. By (3.11) and Lemma 3.3 (1) we see that

$$\sigma \circ \phi = \lambda \sigma. \tag{3.16}$$

It remains to show that the Valiron method works, namely, that the sequence $\{\sigma_n\}$ converges. By the very definition, $\{\sigma_n\}$ converges if and only if $\{\pi_1 \circ S_n\}$ does with S_n defined in (3.13). We already proved that $\{S_n\}$ is bounded on compacta of $\mathbb{H}^N$, thus it is a normal family. Let S be a limit of $\{S_n\}$. Let $Z \in \mathbb{H}^N$. Since the Kobayashi distance is contracted by holomorphic maps, the sequence $\{k_{\mathbb{H}}(S_n(1,0), S_n(Z))\}$ is decreasing in n and must have a limit. Therefore, by (3.14), for all $Z \in \mathbb{H}^N$,

$$\lim_{n\to\infty} k_{\mathbb{H}}(S_n(1,0), S_n(Z)) = k_{\mathbb{H}}((1+iL, 0), S(Z)).$$

This implies that if $\tilde{S}$ is another limit of $\{S_n\}$ then $k_{\mathbb{H}}((1+iL,0), S(Z)) = k_{\mathbb{H}}((1+iL,0), \tilde{S}(Z))$ for all $Z \in \mathbb{H}^N$. Thus, conjugating both $S, \tilde{S}$ with a Cayley map $\mathcal{C}'$ which maps $(1+iL, 0)$ into $O \in \mathbb{B}^N$, we find two holomorphic maps $S', \tilde{S}' : \mathbb{B}^N \to \mathbb{B}^N$ with the property that $\|S'(Z)\| = \|\tilde{S}'(Z)\|$ for all $Z \in \mathbb{B}^N$. Hence (see, *e.g.*, [7, Prop. 3, p. 102]) there exists a unitary matrix U such that $S' = U\tilde{S}'$. Translating into $\mathbb{H}^N$ this means that $\tilde{S} = T \circ S$ for some automorphism $T : \mathbb{H}^N \to \mathbb{H}^N$ fixing $(1+iL, 0)$. We claim that $\pi_1 \circ T(z,w) = z$, hence $\pi_1 \circ S = \pi_1 \circ \tilde{S}$ which implies that $\{\pi_1 \circ S_n\}$ –and hence $\{\sigma_n\}$– is converging. In order to prove that $\pi_1 \circ T(z,w) = z$, it is enough to prove that $T(z,0) = (z,0)$ for some point $z \in \mathbb{H} \setminus \{1+iL\}$, because then by the classical theory of automorphisms (see [1] or [13]) T must fix pointwise the complex geodesic $\mathbb{H} \times \{0\}$. To this aim, let $Z_1 := \phi(1,0)$. Let $\{S_{n_k}\}$ be a sub-sequence of $\{S_n\}$ converging to S. By (3.16),

$$(\pi_1 \circ S)(Z_1) = (\pi_1 \circ S)(\phi(1,0)) = \lambda\sigma(1,0) = \lambda(1+iL).$$

On the other hand, setting as before $(z_n, w_n) := \phi^{\circ n}(1,0)$, we get

$$(\pi_w \circ S)(Z_1) = \lim_{k\to\infty} \frac{\pi_w(\phi^{\circ n_k}(Z_1))}{\sqrt{x_{n_k}}} = \lim_{k\to\infty} \frac{\pi_w(\phi^{\circ(n_k+1)}(1,0))}{\sqrt{x_{n_k}}}$$
$$= \lim_{k\to\infty} \frac{w_{n_k+1}}{\sqrt{x_{n_k}}} = \lim_{k\to\infty} \frac{w_{n_k+1}}{\sqrt{x_{n_k+1}}}\sqrt{\frac{x_{n_k+1}}{x_{n_k}}} = 0,$$

where the last equality follows from (3.10) and Lemma 3.3 (1). Thus $S(Z_1) = (\lambda(1+iL), 0)$. Similarly, we have $\tilde{S}(Z_1) = (\lambda(1+iL), 0)$. Therefore

$$T(\lambda(1+iL), 0) = (T \circ S)(Z_1) = \tilde{S}(Z_1) = (\lambda(1+iL), 0),$$

which proves that $\pi_1 \circ T(z, 0) = z$ as needed. ■

4. Further remarks and open questions

1. In order to make the Valiron construction to work, in the Main Theorem we need the technical hypothesis (1), namely that ϕ possesses a 0-special orbit. We do not know whether any hyperbolic holomorphic self-map of the ball always has such an orbit or not. Clearly, if the self-map has an invariant complex geodesic (whose closure must necessarily contain the Denjoy-Wolff point) then such a condition is satisfied for all points on such a complex geodesic. For instance, if $T : \mathbb{B}^N \to \mathbb{B}^N$ is a hyperbolic automorphism with Denjoy-Wolff point e_1 and other fixed point $-e_1$, then the orbit of any point $(z, 0')$ is (obviously) special, and conversely, the orbit of any point of the form (z, z') with $z' \neq 0'$ is not special.

2. As shown by the Example in section 1.3, hypothesis (2) in the Main Theorem is not necessary for the Valiron construction to work in higher dimension.

3. Along the lines of the one-dimensional Valiron construction (see, *e.g.*, [4, p. 47]) one can prove that if σ is the intertwining map given by the Main Theorem, then $\mathbb{H} \ni \zeta \mapsto \sigma(\zeta, 0)$ is *semi-conformal* at ∞. However, no further regularity on σ at ∞ seems to follow from the construction.

4. Uniqueness (up to composition with linear fractional maps) of intertwining mappings in higher dimension –without assigning further conditions– does not hold. The main theoretical reason is that in dimension one the centralizer of a given hyperbolic automorphism consists of hyperbolic automorphisms while in higher dimension this is no longer so (see [8]). For example, if $H : \mathbb{B}^N \to \mathbb{B}^N$ is a hyperbolic automorphism, then any holomorphic self-map $F : \mathbb{B}^N \to \mathbb{B}^N$ such that $F \circ H = H \circ F$ solves the (trivial) Schröder equation $\sigma \circ H = H \circ \sigma$. By [8], if $N > 1$, then there exist mappings F which are not linear fractional maps.

5. Appendix: E-limits

In this appendix, we introduce the notion of E-limit in $\mathbb{H}^N$ and show that it is equivalent to that of K-limit. However, this new definition might be useful in more general domains. We also prove a couple of routine facts that were needed in the proof of the Main Theorem.

A *complex geodesic* $f : \mathbb{H} \to \mathbb{H}^N$ is a holomorphic map which is an isometry between the Poincaré distance on $\mathbb{H}$ and the Kobayashi distance on $\mathbb{H}^N$. It is well known (see, *e.g.*, [1]) that for $\mathbb{H}^N$ the image of a complex geodesic is the intersection of $\mathbb{H}^N$ with an affine complex line. A *linear projection* $\rho : \mathbb{H}^N \to \mathbb{H}^N$ is a holomorphic map such that $\rho^2 = \rho$, the image $\rho(\mathbb{H}^N)$ is the intersection of $\mathbb{H}^N$ with an affine complex line (namely it is a complex geodesic) and $\rho^{-1}(\rho(Z))$ is an affine hyperplane in $\mathbb{H}^N$ for all $Z \in \mathbb{H}^N$. To any complex geodesic it is associated a unique linear projection and conversely, to any linear projection it is associated a unique (up to parametrization) complex geodesic.

Given any complex geodesic $f : \mathbb{H} \to \mathbb{H}^N$ there exists an automorphism G of $\mathbb{H}^N$ such that $f(\zeta) = G^{-1}(\zeta, 0)$. The linear projection associated to f is then given by $\rho(z,w) = G^{-1}(\pi_1(G(z,w)), 0)$, where $\pi_1(z,w) := z$. The map $\widetilde{\rho} := f^{-1} \circ \rho : \mathbb{H}^N \to \mathbb{H}$ is called the *left inverse* of f.

If $\rho : \mathbb{H}^N \to \mathbb{H}^N$ is a linear projection such that $\overline{\rho(\mathbb{H}^N)}$ contains ∞, for short we say that ρ is a linear projection at ∞.

We will denote by $p_1 : \mathbb{H}^N \to \mathbb{H}^N$ the linear projection at ∞ given by $p_1(z,w) = (z,0)$, associated to the complex geodesic $f(\zeta) = (\zeta, 0)$ and left inverse $\pi_1(z,w) = z$.

Definition 5.1. Let $\rho : \mathbb{H}^N \to \mathbb{H}^N$ be a linear projection at ∞. A sequence $\{Z_k\} \subset \mathbb{H}^N$ converging to ∞ is said *C-special with respect to ρ* if there exists $C \geq 0$ such that

$$\limsup_{k\to\infty} k_{\mathbb{H}^N}(Z_k, \rho(Z_k)) \leq C.$$

The sequence $\{Z_k\}$ converging to ∞ is said to be *ρ-restricted* if $\{\rho(Z_k)\}$ converges non-tangentially to ∞ in $\rho(\mathbb{H}^N)$.

Lemma 5.2. *Let $\{Z_k\} \subset \mathbb{H}^N$ be a sequence converging to ∞. Let $\rho_0 : \mathbb{H}^N \to \mathbb{H}^N$ be a linear projection at ∞. Then $\{Z_k\}$ is C-special ($C \geq 0$) with respect to ρ_0 if and only if it is C-special (same C) with respect to any linear projection at ∞ ρ. The sequence $\{Z_k\}$ is ρ_0-restricted if and only if it is ρ-restricted with respect to any linear projection at ∞ ρ.*

Proof. Let T_0 be an automorphism of $\mathbb{H}^N$ fixing ∞ and with the property that $\rho(z,w) = T_0^{-1}(p_1(T_0(z,w))$. Since T_0 is an isometry for $k_{\mathbb{H}^N}$, then

$\{Z_k\}$ is C-special with respect to ρ_0 (*respectively* ρ_0-restricted) if and only if $\{T_0(Z_k)\}$ is C-special with respect to p_1 (*respect.* p_1-restricted). Therefore it is enough to prove that if $\{Z_k\}$ is C-special with respect to p_1 (*respectively* p_1-restricted) then it is C-special with respect to any linear projection at ∞ ρ (*respect.* ρ-restricted).

Given a linear projection at ∞ ρ, there exists $a \in \mathbb{C}^{N-1}$ and an automorphism $T \in \mathsf{Aut}(\mathbb{H}^N)$ of the type

$$T(z,w) = (z + \|a\|^2 + 2\langle w, a\rangle, w + a)$$

such that $\rho = T^{-1} \circ p_1 \circ T$. A direct computation shows that

$$\rho(z,w) = (z + 2\|a\|^2 + 2\langle w, a\rangle, -a). \tag{5.1}$$

Therefore, writing $Z_k = (z_k, w_k) = (x_k + iy_k, w_k)$ and, arguing similarly to (2.2), we obtain

$$\begin{aligned} k_{\mathbb{H}^N}\big(p_1(Z_k), \rho(Z_K)\big) &= k_{\mathbb{H}^N}\big((z_k, 0), (z_k + 2\|a\|^2 + 2\langle w_k, a\rangle, -a)\big) \\ &= k_{\mathbb{H}^N}\left((1,0), (\frac{z_k + 2\|a\|^2 + 2\langle w_k, a\rangle - iy_k}{x_k}, \frac{-a}{\sqrt{x_k}})\right) \\ &= \tanh^{-1}\sqrt{\frac{|2\|a\|^2 + 2\langle w_k, a\rangle|^2 + 4x_k\|a\|^2}{x_k^2\left|2 + \frac{2\|a\|^2}{x_k} + 2\frac{\langle w_k, a\rangle}{x_k}\right|^2}}. \end{aligned}$$

The last term tends to 0 as $x_k \to \infty$, which is the case if $k \to \infty$ because $Z_k \to \infty$ and $x_k = \mathsf{Re}\, z_k > \|w_k\|^2$. Thus

$$\lim_{k\to\infty} k_{\mathbb{H}^N}(p_1(Z_k), \rho(Z_K)) = 0. \tag{5.2}$$

Now, using the triangle inequality and (5.2) we see that if $\{Z_k\}$ is C-special with respect to p_1, then

$$\begin{aligned} &\limsup_{k\to\infty} k_{\mathbb{H}^N}(Z_k, \rho(Z_K)) \\ &\qquad \le \limsup_{k\to\infty} k_{\mathbb{H}^N}(Z_k, p_1(Z_K)) + \limsup_{k\to\infty} k_{\mathbb{H}^N}(p_1(Z_k), \rho(Z_K)) \le C, \end{aligned}$$

as stated.

On the other hand, if $\{Z_k\}$ is p_1-restricted (namely, $\mathsf{Re}\, z_k \ge c\mathsf{Im}\, z_k$ for some $c > 0$), from (5.1) and since $\mathsf{Re}\, z_k > \|w_k\|^2$ it follows that $\{Z_k\}$ is also ρ-restricted ■

Remark 5.3. It is worth to note explicitly that by Lemma 5.2 the condition of being C-special and that of being restricted do not depend on the chosen linear projection.

Definition 5.4. Let $h : \mathbb{H}^N \to \mathbb{C}$ be holomorphic. We say that h has E-limit $A \in \mathbb{C}$ at ∞, and we write

$$E\text{-}\lim_{\mathbb{H}^N \ni (z,w) \to \infty} h(z,w) = A,$$

if for any sequence $\{Z_k\} \subset \mathbb{H}^N$ converging to ∞ which is C-special for some $C \geq 0$ (C depending on $\{Z_k\}$) and restricted, it follows that $\lim_{k\to\infty} h(Z_k) = A$.

If the limit holds only for 0-special, restricted sequences we write

$$E^0\text{-}\lim_{\mathbb{H}^N \ni (z,w) \to \infty} h(z,w) = A.$$

Next we state a version of the Julia-Wolff-Carathéodory theorem due to Rudin for the unit ball $\mathbb{B}^n$ ([13, Thm. 8.5.6]), using our previous notations:

Theorem 5.5. *Let $\phi = (\phi_1, \phi') : \mathbb{H}^N \to \mathbb{H}^N$ be holomorphic, with Denjoy-Wolff point ∞ and multiplier $\lambda \geq 1$. Let $\rho : \mathbb{H}^N \to \mathbb{H}^N$ be a linear projection at ∞ and let $\widetilde{\rho} : \mathbb{H}^N \to \mathbb{H}$ be an associated left inverse. Then*

$$E^0\text{-}\lim_{\mathbb{H}^N \ni (z,w) \to \infty} \frac{\widetilde{\rho} \circ \phi(z,w)}{\widetilde{\rho}(z,w)} = \lambda.$$

As a corollary we have the following:

Lemma 5.6. *Let $\phi = (\phi_1, \phi') : \mathbb{H}^N \to \mathbb{H}^N$ be holomorphic, with Denjoy-Wolff point ∞ and multiplier $\lambda \geq 1$. Assume $E\text{-}\lim_{\mathbb{H}^N \ni (z,w) \to \infty} \frac{\phi_1(z,w)}{z}$ exists. Then*

(1) $E\text{-}\lim_{\mathbb{H}^N \ni (z,w) \to \infty} \frac{\phi_1(z,w)}{z} = \lambda$,

(2) $E\text{-}\lim_{\mathbb{H}^N \ni (z,w) \to \infty} \frac{\|\phi'(z,w)\|}{|z|} = 0$.

Proof. (1) It follows directly from Theorem 5.5.

(2) Since $\phi(\mathbb{H}^N) \subseteq \mathbb{H}^N$ then $\mathsf{Re}\, \phi_1(z,w) \geq \|\phi'(z,w)\|^2$ for all $(z,w) \in \mathbb{H}^N$. Thus dividing by $|z|^2$ and taking limits, (2) follows from (1). ■

Remark 5.7. Rudin's Julia-Wolff-Carathéodory theorem for the unit ball ([13, Thm. 8.5.6]) has also another implication similar to (2) of Lemma 5.6. Namely, it implies that if a holomorphic map $\phi : \mathbb{H}^N \to \mathbb{H}^N$ has Denjoy-Wolff point at ∞, then $E^0\text{-}\lim_{\mathbb{H}^N \ni (z,w) \to \infty} \frac{\|\phi'(z,w)\|}{|z|^{1/2}} = 0$. Note that, since $\|(z,w)\| \to \infty$ in $\mathbb{H}^N$ implies in particular that $|z| \to \infty$, it follows that $E^0\text{-}\lim_{\mathbb{H}^N \ni (z,w) \to \infty} \frac{\|\phi'(z,w)\|}{|z|} = 0$. However, as shown in the proof of Lemma 5.6, (2) follows directly from (1) and, on the other hand, it does not seem to be clear how to get (2) from Rudin's theorem and the hypotheses of Lemma 5.6 without using (1).

The following technical proposition is needed in the proof of the Main Theorem.

Proposition 5.8. *Let* $\phi = (\phi_1, \phi') : \mathbb{H}^N \to \mathbb{H}^N$ *be holomorphic, with Denjoy-Wolff point* ∞ *and multiplier* $\lambda \geq 1$. *Let* ρ_0 *be a linear projection at* ∞ *with a left inverse* $\widetilde{\rho}_0$. *Suppose* $E\text{-}\lim_{\mathbb{H}^N \ni (z,w) \to \infty} \frac{\widetilde{\rho}_0(\phi(z,w))}{\widetilde{\rho}_0(z,w)}$ *exists. Then for any complex geodesic* $f : \mathbb{H} \to \mathbb{H}^N$ *with* $f(\infty) = \infty$ *with left inverse* $\widetilde{\rho}_f : \mathbb{H}^N \to \mathbb{H}$ *it follows*

$$E\text{-}\lim_{\mathbb{H}^N \ni (z,w) \to \infty} \frac{\widetilde{\rho}_f \circ \phi(z,w)}{\widetilde{\rho}_f(z,w)} = \lambda.$$

Proof. Up to conjugation we can assume that $\rho_0 = p_1$ and then by hypothesis we know that $E\text{-}\lim_{\mathbb{H}^N \ni (z,w) \to \infty} \frac{\phi_1(z,w)}{z}$ exists and equals λ by Lemma 5.6. Given the complex geodesic f, there exists $a \in \mathbb{C}^{N-1}$ and an automorphism $T \in \mathsf{Aut}(\mathbb{H}^N)$ of the type

$$T(z,w) = (z + \|a\|^2 + 2\langle w, a\rangle, w + a)$$

such that $T \circ f(\zeta) = (\zeta, 0)$ and $\widetilde{\rho}(z,w) = \pi_1 \circ T(z,w) = z + \|a\|^2 + 2\langle w, a\rangle$, where, as usual $\pi_1(z,w) = z$. Thus

$$\begin{aligned} \frac{\widetilde{\rho}_f \circ \phi(z,w)}{\widetilde{\rho}_f(z,w)} &= \frac{\phi_1(z,w) + \|a\|^2 + 2\langle \phi'(z,w), a\rangle}{z + \|a\|^2 + 2\langle w, a\rangle} \\ &= \frac{\phi_1(z,w) + \|a\|^2 + 2\langle \phi'(z,w), a\rangle}{z(1 + \|a\|^2/z + 2\langle w, a\rangle/z)}. \end{aligned}$$

Taking into account that $1 + \|a\|^2/z + 2\langle w, a\rangle/z = 1 + o(|z|^{-1})$ since $|z| \geq \mathsf{Re}\, z \geq \|w\|^2$, the result follows from Lemma 5.6. ∎

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