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The volume near the zeroes of a smooth function

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Abstract

We show that if a smooth function that never vanishes to infinite order, then the set of points within the distance δ from the zeroes of this function has volume $O(\delta)$.

1. Statement of Result

Let $B(x,r)$ denote the open ball of radius r about x in $\mathbb{R}^n$. In this note we prove the following result.

Theorem 1. *Let F be a real-valued C^m function on $B(0,1)$, with*

1. $c_0 < \max_{|\alpha|=m-1} |\partial^\alpha F(0)| < C_0$, *and with*
2. $|\partial^\alpha F| \le C_1$ *on* $B(0,1)$ *for* $|\alpha| = m$.

Let

3. $V(F) = \{x \in B(0,1) : F(x) = 0\}$, *and let*
4. $V(F,\delta) = \{x \in B(0,c_1) : \text{ distance}(x, V(F)) < \delta\}$,

where c_1 is a small enough constant determined by c_0, C_0, C_1, m, n.

Then we have

$$\mathrm{Vol}\{V(F,\delta)\} \le C_2\delta \text{ for } 0 < \delta < c_1,$$

where C_2 is a large constant determined by c_0, C_0, C_1, m, n.

Thus, if F is a smooth function that never vanishes to infinite order, then the set of points within the distance δ from the zeroes of F has volume $O(\delta)$. If we allow F to vanish to infinite order then the corresponding assertion is plainly wrong. For the level sets of polynomials this statement is proven in [1].

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2. A Convenient Reduction

In this section, we reduce Theorem 1 to the following result which is seemingly a bit less general.

Theorem 2. *Let F be a real-valued C^m function on $B(0,1)$, with*

1. $c_0 < |\partial^\alpha F| < C_0$ *everywhere on* $B(0,1)$, *for every multi-index* α *of order* $m-1$,
2. $|\partial^\alpha F| \le C_1$ *everywhere on* $B(0,1)$ *for every multi-index* α *of order* m.

Let

3. $V(F) = \{x \in B(0,1) : F(x) = 0\}$, *and let*
4. $V(F,\delta) = \{x \in B(0,c_1) : \text{ distance}(x, V(F)) < \delta\}$,

where c_1 is a small enough constant determined by c_0, C_0, C_1, m, n.

Then we have

$$\mathrm{Vol}\{V(F,\delta)\} \le C_2\delta \quad \text{for } 0 < \delta < c_1,$$

where C_2 is a large constant determined by c_0, C_0, C_1, m, n.

To reduce Theorem 1 to Theorem 2, we use the following elementary result.

Proposition 3. *Let F satisfy the hypotheses of Theorem 1. Then there exists a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$, and constants c and C, with the following properties:*

1. *c and C are determined by c_0, C_0, C_1, m, n,*
2. *the maps A and A^{-1} have norms at most C,*
3. *$F \circ A$ is well-defined on $B(0,c)$,*
4. *$c < |\partial^\alpha(F \circ A)| < C$ on $B(0,c)$ for all α with $|\alpha| = m-1$,*
5. *$|\partial^\alpha(F \circ A)| < C$ on $B(0,c)$ for all α with $|\alpha| = m$.*

Once the proposition is proven, then the Theorem 1 follows by applying Theorem 2 to the function $\tilde{F}(x) = (F \circ A)(cx)$, $x \in B(0,1)$.

Proof of the Proposition. In this proof, we say that a constant is *controlled*, if it is determined by c_0, C_0, C_1, m and n; and we write c, C, C', etc. to denote the controlled constants.

Pick a vector $v \in \mathbb{R}^n$ of length 1 to maximize $|(v \cdot \nabla)^{m-1} F(0)|$. Without loss of generality, we may assume that $v = e_n$, the n'th unit vector in $\mathbb{R}^n$. Then we have

$$c < \left|\left(\frac{\partial}{\partial x_n}\right)^{m-1} F(0)\right| < C, \qquad \text{and} \qquad |\partial^\alpha F(0)| < C \text{ for } |\alpha| = m-1\,.$$

Consequently, for any $\lambda \in (0,1)$, and for any multi-index $\alpha = (\alpha_1, \dots \alpha_n)$ with $|\alpha| = \alpha_1 + \dots + \alpha_n = m-1$, we have

$$\left(\frac{\partial}{\partial x_n} + \lambda \frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n} + \lambda \frac{\partial}{\partial x_{n-1}}\right)^{\alpha_{n-1}} \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} F(0) = \sum_{k=0}^{m-1} A_k^{(\alpha)} \lambda^k,$$

with $c < |A_0^{(\alpha)}| < C$ and $|A_k^{(\alpha)}| < C$ for all k.

Therefore, if we take $\lambda = \bar{c}$ for small enough controlled constant $\bar{c}$, then we obtain

$$c < \left| \left(\frac{\partial}{\partial x_n} + \lambda \frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n} + \lambda \frac{\partial}{\partial x_{n-1}}\right)^{\alpha_{n-1}} \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} F(0) \right| < C$$

for all $\alpha = (\alpha_1, \dots \alpha_n)$ with $|\alpha| = m-1$.

We define

$$A : (x_1, \dots x_n) \mapsto (x_n + \lambda x_1, \dots, x_n + \lambda x_{n-1}, x_n).$$

Thus

$$||A||, ||A^{-1}|| \le C \tag{2.1}$$

and

$$c < |\partial^\alpha (F \circ A)(0)| < C \text{ for all } \alpha \text{ with } |\alpha| = m-1. \tag{2.2}$$

From (2.1), and from hypothesis (2) of Theorem 1, we see then

$$F \circ A \text{ is well-defined on } B(0,c), \quad \text{and} \tag{2.3}$$

$$|\partial^\alpha (F \circ A)| < C \text{ on } B(0,c), \quad \text{for all } \alpha \text{ with } |\alpha| = m. \tag{2.4}$$

From (2.2), (2.3), (2.4), we obtain

$$c' < |\partial^\alpha (F \circ A)| < C' \text{ on } B(0,c''), \quad \text{for all } \alpha \text{ with } |\alpha| = m-1. \tag{2.5}$$

Since c, C, c', C', c'' are controlled constants, the conclusion of our proposition follows at once from (2.1), (2.3), (2.4), (2.5). The proof of the proposition is complete. ■

Thus we have reduced Theorem 1 to Theorem 2.

3. An Elementary Remark

For $i = 1, \dots n$, let e_i denote the i'th unit vector in $\mathbb{R}^n$. In this section we recall the following elementary result.

Proposition 4. *Let M_1, M_2, a_1, δ, Γ be positive real numbers and let G be a real-valued C^2 function on $B(x^0, 2\delta)$. Assume that*

1. $|\frac{\partial}{\partial x_i} G| \le M_1 \Gamma \delta^{-1}$ *and* $|\frac{\partial^2}{\partial x_i \partial x_j} G| \le M_2 \Gamma \delta^{-2}$ *on* $B(x^0, 2\delta)$;
2. $|\frac{\partial}{\partial x_{i_0}} G(x^0)| \ge a_1 \Gamma \delta^{-1}$ *and*
3. $|G(x^0)| \le a_* \Gamma$ *for all small enough* a_*, *determined by* M_1, M_2, a_1, n.

Then, for any $x \in B(x^0, a_*\delta)$, there exists $\tau \in (-\delta, \delta)$ such that $G(x + \tau e_{i_0}) = 0$.

Proof. By rescaling, we may suppose $\Gamma = \delta = 1$. Integrating $|\nabla G|$ and $|\nabla \frac{\partial}{\partial x_{i_0}} G|$ on the line segment joining x^0 to x, we find that

$$|G(x) - G(x^0)| \le \sqrt{n} M_1 |x - x^0| \le \sqrt{n} M_1 a_*, \quad \text{and}$$

$$\left| \frac{\partial}{\partial x_{i_0}} G(x) - \frac{\partial}{\partial x_{i_0}} G(x^0) \right| \le \sqrt{n} M_2 |x - x^0| \le \sqrt{n} M_2 a_*$$

Hence,

$$|G(x)| \le (1 + \sqrt{n} M_1) a_*, \quad \text{and} \tag{3.1}$$

$$\left| \frac{\partial}{\partial x_{i_0}} G(x) \right| \ge a_1 - \sqrt{n} M_2 a_* \ge 1/2 a_1, \quad \text{(if we take } a_* \text{ small enough)} \tag{3.2}$$

Since also $|(\frac{\partial}{\partial x_{i_0}})^2 G| \le M_2$ on $B(x^0, 2)$, (3.2) implies that

$$\left| \frac{\partial}{\partial x_{i_0}} G(x + \tau e_{i_0}) \right| \ge 1/2 a_1 - M_2 |\tau| \ge 1/4 a_1, \tag{3.3}$$

for $\tau \in [-\frac{a_1}{4M_2}, \frac{a_1}{4M_2}] \cap (-1, 1) = I$.

Let $g(\tau) = G(x + \tau e_{i_0})$ for $\tau \in I$. Then g is a C^2-function on I; and (3.1), (3.3) yield

$$|g(0)| \le (1 + \sqrt{n} M_1) a_*, \quad \text{and} \quad |g'| \ge 1/4 a_1 \text{ on } I \tag{3.4}$$

If a_* is taken small enough, then (3.4) easily implies $g(\tau) = 0$ for some $\tau \in I$. In particular, $G(x + \tau e_{i_0}) = 0$ for some $\tau \in (-1, 1)$, proving the proposition. ∎

4. Two Main Lemmas

From now on, we assume that the function F and the constants c_0, C_0, C_1 satisfy the hypothesis of the Theorem 2. We say that a constant is *controlled*, if it is determined by c_0, C_0, C_1, m and n; and we write c, C, C', etc. to denote the controlled constants.

As in the Section 2, we write $e_1, \dots, e_n$ for the unit vectors in $\mathbb{R}^n$.

Lemma 5. *For a small enough controlled constant $\bar{c}$, the following holds. Suppose $x^0 \in V(F) \cap B(0, 1/2)$, and suppose $0 < \delta < \bar{c}$. Then, for any $x \in B(x^0, \bar{c}\delta)$, there exist β, i_0, τ with*

1. $|\beta| \leq m - 2,\ 1 \leq i_0 \leq n;$
2. $\tau \in [-\delta, \delta]$ *and*
3. $\partial^\beta F(x + \tau e_{i_0}) = 0.$

Proof. Let $A_m, A_{m-1}, \dots A_0$ be constants to be picked later. We write $C(A_m, \dots A_k)$ to denote a constant determined by $A_m, \dots A_k$ and c_0, C_0, C_1, m, n. We define

$$\Omega = \max_{|\gamma| \leq m-1} A_{|\gamma|} \delta^{|\gamma|} |\partial^\gamma F(x^0)|, \tag{4.1}$$

and we suppose that the max in (4.1) is attained at $\gamma = \bar{\gamma}$. From the hypothesis (1) of the Theorem 2, we have

$$\Omega \geq A_{m-1} c_0 \delta^{m-1} \tag{4.2}$$

In particular, $\Omega \neq 0$. Since $x^0 \in V(F)$, we have $F(x^0) = 0$, so the maximum in (4.1) is not attained at $\gamma = 0$. Hence, $\bar{\gamma} \neq 0$, and consequently, we may write $\bar{\gamma} = 1_{i_0} + \beta$, where $|\beta| \leq m - 2$, and 1_{i_0} is the i_0-th unit multi-index. In particular, i_0 and β satisfy (1). By the definition of Ω, $\bar{\gamma}$, i_0, β, we have

$$|\partial^\gamma F(x^0)| \leq A_{|\gamma|}^{-1} \Omega \delta^{-|\gamma|} \text{ for } |\gamma| \leq m - 1, \text{ and} \tag{4.3}$$

$$\left|\frac{\partial}{\partial x_{i_0}} (\partial^\beta F)(x^0)\right| = A_{|\beta|+1}^{-1} \Omega \delta^{-|\beta|-1} \tag{4.4}$$

Also, for $|\gamma| = m$, $x \in B(0, 1)$, estimate (4.2) and the hypothesis (2) of the Theorem 2 yield

$$|\partial^\gamma F(x)| \leq C_1 \leq C_1 c_0^{-1} A_{m-1}^{-1} \Omega \delta^{-(m-1)} \tag{4.5}$$

If

$$0 < \delta < A_m^{-1} c_0 A_{m-1} C_1^{-1}, \tag{4.6}$$

then (4.5) implies

$$|\partial^\gamma F| \leq A_m^{-1} \Omega \delta^{-|\gamma|} \text{ on } B(0, 1), \text{ for } |\gamma| = m. \tag{4.7}$$

From (4.3), (4.7) and Taylor's theorem, we obtain

$$|\partial^\gamma F| \leq C(A_m, \dots A_{|\gamma|})\Omega\delta^{-|\gamma|} \quad \text{on } B(x^0, 2\delta), \text{ for } |\gamma| \leq m, \tag{4.8}$$

provided

$$\delta < 1/4 \tag{4.9}$$

(Condition (4.9) guarantees that $B(x^0, 2\delta) \subset B(0,1)$, since $x^0 \in B(0, 1/2)$)
In particular, (4.8) gives

$$\left|\frac{\partial}{\partial x_i}\left[\partial^\beta F\right]\right| \leq C(A_m, \dots, A_{|\beta|+1})\Omega\delta^{-|\beta|-1} \quad \text{on } B(x^0, 2\delta), \tag{4.10}$$

and

$$\left|\frac{\partial^2}{\partial x_i \partial x_j}\left[\partial^\beta F\right]\right| \leq C(A_m, \dots, A_{|\beta|+2})\Omega\delta^{-|\beta|-2} \text{ on } B(x^0, 2\delta) \text{ for all } i, j. \tag{4.11}$$

Also, (4.3) and (4.4) give

$$\left|\frac{\partial}{\partial x_{i_0}}\left[\partial^\beta F\right](x^0)\right| = A^{-1}_{|\beta|+1}\Omega\delta^{-|\beta|-1} \tag{4.12}$$

and

$$|\left[\partial^\beta F\right](x^0)| \leq A^{-1}_{|\beta|}\Omega\delta^{-|\beta|} \tag{4.13}$$

Note that $A_{|\beta|}$ appears in (4.13), but not in (4.10), (4.11), (4.12). Suppose that

$$A_{|\beta|} \text{ exceeds a large enough constant } C(A_m, \dots, A_{|\beta|+1}), \tag{4.14}$$

Then (4.10)-(4.14) are the hypotheses of the proposition 3 with $G = \partial^\beta F$, $\Gamma = \Omega\delta^{-|\beta|}$, $M_1 = C(A_m, \dots, A_{|\beta|+1})$, $M_2 = C(A_m, \dots, A_{|\beta|}+2)$, $a_1 = A^{-1}_{|\beta|+1}$, $a_* = A^{-1}_{|\beta|}$. Applying the proposition, we learn the following:

$$\begin{aligned} &\text{Given } x \in B(x^0, A^{-1}_{|\beta|}\delta), \text{ there exists } \tau \in (-\delta, \delta), \\ &\text{such that } \partial^\beta F(x + \tau e_{i_0}) = 0. \end{aligned} \tag{4.15}$$

We now take $A_m = A_{m-1} = 1$, and successively pick the controlled constants $A_{m-2}, A_{m-3}, \dots A_0$, so that (4.14) holds for all $|\beta| \leq m-2$. In particular, if $\bar{c}$ is a small enough controlled constant, and if $0 < \delta < \bar{c}$, then (4.6) and (4.9) are satisfied, and (4.15) applies to all $x \in B(x^0, \bar{c}\delta)$. Since we have already noted, that (1) holds, the conclusions of the lemma 5 are obvious from (4.15). ■

From now on, we fix $\bar{c}$ as in the Lemma 5.

We prepare to state our second Lemma. Let $0 < \delta < \bar{c}$ be given. Fix a cube Q^0 centered at the origin, such that

$$1/4 \leq \text{diameter} Q^0 < 1/2, \tag{4.16}$$

and such that diameterQ^0 is an integer multiple of δ. Then we can partition Q^0 into cubes $\{Q_\nu\}$ of diameter $\bar{c}\delta$.

Let x_ν be the center of Q_ν. Note that $Q^0 \subset B(0, 1/2)$, thanks to (4.16).

We define a *label* to be an ordered pair (i_0, β) satisfying condition (1) of Lemma 5. We say, that the cube Q_ν *carries the label* (i_0, β), provided we have $\partial^\beta F(x_\nu + \tau e_{i_0}) = 0$ for some $\tau \in [-\delta, \delta]$. From Lemma 5 (applied to $x = x_\nu$), we learn the following basic fact:

(4.17) Every Q_ν containing a zero of F must carry some label.

On the other hand, we have the following result.

Lemma 6. *Fix a label (i_0, β). Then there are at most $C\delta^{-(n-1)}$ cubes Q_ν that carry the given label.*

Proof. Without loss of generality, we may suppose that $i_0 = n$. We arrange the cubes Q_ν into columns", by saying that Q_ν and $Q_{\nu'}$ belong to the same "column" if their centers x_ν and $x_{\nu'}$ differ at most in the n-th coordinate. There are at most $C\delta^{-(n-1)}$ distinct columns. Hence, to prove lemma 6, it is enough to show that any given column contains at most C distinct Q_ν that carry the label (i_0, β).

Fix a column $\mathcal{C}$. For a suitable $\bar{x} \in \mathbb{R}^{n-1}$, the cubes Q_ν in $\mathcal{C}$ have centers $(\bar{x}, t_1), \dots (\bar{x}, t_N)$, where $t_1, \dots t_N$ form an arithmetic progression with the step $c\delta$. For each i $(1 \leq i \leq N)$, we have $(\bar{x}, t_i) \in Q_\nu \subset Q^0 \subset B(0, 1/2)$.

Therefore, for $\tau \in [-\delta, \delta]$ and $i = 1, \dots, N$, we have

$$t_i + \tau \in I, \tag{4.18}$$

where I is the interval $\{t \in \mathbb{R} : (\bar{x}, t) \in B(0, 1)\}$.

Let Q_ν be one of the cubes in $\mathcal{C}$, with center $(\bar{x}, t_i)$. By definition, Q_ν carries the label (i_0, β) (with $i_0 = n$) if and only if $\partial^\beta F(\bar{x}, t_i + \tau) = 0$ for some $\tau \in [-\delta, \delta]$. In view of (4.18), it follows that the number of $Q_\nu \in \mathcal{C}$ that carry the label (i_0, β) is equal to the number of t_i $(i = 1, \dots, N)$ that lie within the distance δ from a zero of the function $g(t) = \partial^\beta F(\bar{x}, t)$, defined for $t \in I$. Hence, to prove Lemma 6, it is enough to show:

(4.19) There are at most C distinct i $(i = 1, \dots N)$, such that t_i lies within the distance δ from a zero of $g(t)$ $(t \in I)$.

Moreover, since $t_1, \dots, t_N$ form an arithmetic progression with the step $c\delta$, assertion (4.19) will follow, if we can prove that

(4.20) the function g has at most C distinct zeroes in I.

Thus, Lemma 6 is reduced to the task of proving (4.20).

For $t \in I$, we have $(\bar{x}, t) \in B(0,1)$ by definition of I, and therefore

$$\left(\frac{d}{dt}\right)^{m-1-|\beta|} g(t) = \left(\frac{\partial}{\partial x_n}\right)^{m-1-|\beta|} \partial^\beta F(\bar{x}, t) \neq 0,$$

thanks to the hypothesis (1) of Theorem 2. That is,

$$\left(\frac{d}{dt}\right)^{m-1-|\beta|} g(t) \text{ vanishes nowhere on } I. \tag{4.21}$$

A standard argument, repeatedly applying Rolle's theorem from elementary calculus, shows that any function satisfying (4.21) can have at most $m - 1 - |\beta|$ distinct zeroes in I. Hence, (4.20) holds, completing the proof of Lemma 6. ■

5. Conclusion

We retain the notation and the setting of Section 4.

Let c_1 be a small enough controlled constant, and suppose we are given $x \in V(F, \delta)$ with $0 < \delta < c_1$. By definition of $V(F, \delta)$, we have $x \in B(0, c_1)$, and $|x - x^0| < \delta$ for some $x^0 \in V(F)$. In particular, $x^0 \in B(0, c_1 + \delta) \subset B(0, 2c_1) \subset Q^0$, so $x^0 \in Q_\nu$ for some ν. Thus, Q_ν contains a point of $V(F)$, and $|x - x_\nu| \leq |x - x^0| + |x^0 - x_\nu| < \delta + \text{diameter } Q_\nu = (1 + \bar{c})\delta$, i.e., $x \in B(x_\nu, (1 + \bar{c})\delta)$. We have therefore proven the following:

(5.1) For $0 < \delta < c_1$, the set $V(F, \delta)$ is contained in the union of the balls $B(x_\nu, (1 + \bar{c})\delta)$ over all ν such that Q_ν contains a point of $V(F)$.

From Section 4 (conclusion (4.17) and lemma 6), we see that there are at most $C\delta^{-(n-1)}$ distinct ν such that Q_ν contains a point of $V(F)$. Since each $B(x_\nu, (1 + \bar{c})\delta)$ has volume $C\delta^n$, it follows from (5.1) that

$$\text{Vol}\{V(F, \delta)\} \leq C_2\delta \quad \text{for } 0 < \delta < c_1, \tag{5.2}$$

where C_2 is a controlled constant. Estimate (5.2) is precisely the conclusion of Theorem 2. We recall from Section 2 that Theorem 1 follows from Theorem 2. Hence, the proofs of Theorems 1 and 2 are complete.

References

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