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Nonvariational layer potentials with respect to Hölder continuous vector fields

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Abstract

Nontangential a.e. vanishing of the oblique derivative of a harmonic function with respect to a Hölder continuous vector field on a Lipschitz boundary is shown to imply that the harmonic function is constant.

In this article we prove a uniqueness result for an oblique derivative problem with respect to a transverse Hölder continuous vector field defined on the boundary of a Lipschitz domain $\Omega \subset \mathbb{R}^n$. The result is for harmonic functions when vanishing data is prescribed nontangentially almost everywhere, with respect to surface measure, in $L^p(\partial\Omega)$ rather than everywhere as in the classical formulation. It is motivated in part by the following result due to A. P. Calderón.

Theorem 0.1 ([1]). *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded Lipschitz domain with connected complement. Let $\vec{\alpha}$ be a continuous transverse unit vector field on $\partial\Omega$. Then there exist a finite number of linearly independent functions $f_1, \ldots, f_l \in L^2(\partial\Omega)$ so that if $g \in L^2(\partial\Omega)$ satisfies*

$$\int_{\partial\Omega} g f_j \, ds = 0 \quad (j = 1, \ldots, l) \tag{0.1}$$

then there exists a harmonic function u in Ω with ∇u nontangentially in $L^2(\partial\Omega)$ such that

$$\text{n.t.} \lim_{X \to Q} \vec{\alpha}(Q) \cdot \nabla u(X) = g(Q) \ \ a.e.(ds(Q)) \tag{0.2}$$

Furthermore u will be unique up to the addition of l solutions to the homogeneous oblique derivative problem, i.e. the dimension of the space of solutions to the problem (0.2) *for vanishing g equals the number of linear conditions* (0.1) *imposed on (a nonvanishing) g for solvability.*

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Moreover, Calderón shows that this theorem holds when L^2 is replaced by L^p for p in a (small) interval about 2 depending on the Lipschitz geometry of Ω. I.e., given a Lipschitz domain Ω, there exists a positive number $\epsilon(\Omega)$ such that $|p-2| < \epsilon(\Omega)$ implies Calderón's theorem holds for oblique data in $L^p(\partial\Omega)$.

By virtue of Theorem 2.6 below and Nadirashvili's extension to Lipschitz domains [12] of the Hopf boundary point lemma it follows, when $\vec{\alpha}$ is Hölder continuous, that $l = 1$ in Calderón's theorem. Thus the oblique derivative problem for Hölder continuous vector fields in a bounded Lipschitz domain for p near 2 is like the Neumann problem (a variational problem) where data must meet one condition (mean value zero) and there is one nontrivial solution to the homogeneous problem (the constant solution).

In [17] it is shown by a variety of examples that the uniqueness result here cannot hold in general when $p < 2$, even for smooth vector fields. Given $p < 2$ there exist Lipschitz domains and vector fields with nonconstant solutions to the homogeneous problem. It is also shown, in the plane but not in higher dimensions, that the uniqueness result here can be improved to include any continuous transverse vector field.

Calderón proves his theorem by an integral equation method utilizing the *nonvariational* layer potentials

$$(0.3)\quad (-\frac{1}{2}\vec{\alpha}\cdot N + K_\alpha^*)f(P) = \\ -\frac{1}{2}\vec{\alpha}(P)\cdot N_P f(P) + p.v.\frac{1}{\omega_n}\int_{\partial\Omega}\frac{\vec{\alpha}(P)\cdot(P-Q)}{|P-Q|^n}f(Q)ds(Q)\,.$$

Here N_P denotes the outer unit normal vector defined at almost every $P \in \partial\Omega$. By the theorem of [2] K_α^* is bounded on $L^p(\partial\Omega)$ for $1 < p < \infty$.

As argued in [17] the operators (0.3) are not Fredholm in general for $p < 2$ nor for $p > 2$ (see also [11]). The results here and in [17], however, show that Calderón's operators can be as well behaved as the classical layer potentials based on the normal vector field [14, 6] when p is near 2. Combining Calderón's proof of Theorem 0.1 with Corollary 2.7 below,

Theorem 0.2. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded Lipschitz domain with connected boundary and let $|p-2| < \epsilon(\Omega)$. Let the transverse unit vector field $\vec{\alpha}$ be (*continuous *when $n = 2$) Hölder continuous on $\partial\Omega$. Then*

$$-\frac{1}{2}\vec{\alpha}\cdot N + K_\alpha^* : L^p(\partial\Omega) \to L^p(\partial\Omega)$$

is invertible from a codimension 1 subspace to a codimension 1 subspace.

(Corollary 2.7 does not require connected boundary. More generally it is the *range* of $-\frac{1}{2}\vec{\alpha}\cdot N + K_\alpha^*$ that must satisfy the same number of linear conditions as there are components of the boundary.)

How well behaved boundary layer potentials for higher order equations are, when smooth transverse fields are substituted for the normal field, is a question which arises in [16]. That paper is an attempt to generalize the Neumann problem and the corresponding variational potentials for the biharmonic equation [15] to the general 4th order case. However, a difficulty arises when attempting to prove the closed range of the variational potentials in $L^2(\partial\Omega)$. It can be avoided by substituting a differentiable field for the normal. Calderón's potentials serve as a model for this kind of substitution.

In Section 1 a Lipschitz domain setup is described, results of Dahlberg and Stein reviewed, and a localization lemma proved. In Section 2, after recalling results of Hunt-Wheeden and Dahlberg, two lemmas are proved and then the main theorem. The theorem states that almost everywhere non-tangential vanishing of the oblique derivative implies everywhere vanishing of the classical oblique derivative. The hypotheses of Nadirashvili's theorem are thus satisfied and uniqueness follows as a corollary.

1. Preliminaries and a localization

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded Lipschitz domain. Let $\vec{\alpha}$ denote a *continuous unit* vector field defined on the boundary $\partial\Omega$ with the property that $\vec{\alpha} \cdot N$ taken over the boundary is uniformly bounded from below by a positive constant

$$\vec{\alpha} \cdot N \geq c(\vec{\alpha}) > 0 \tag{1.1}$$

Here N denotes the *outer unit normal* vector to Ω defined *a.e.* with respect to surface measure ds on $\partial\Omega$. Consider a point $P \in \partial\Omega$. By the definition of Lipschitz domain and (1.1) $\vec{\alpha}(P)$ can be taken to be the normal vector to a hyperplane with respect to which $\partial\Omega$, *locally* about P, is the graph of a Lipschitz function $\varphi = \varphi_P$ with Lipschitz constant M independent of P depending only on Ω and $\vec{\alpha}$. After a rotation of space the hyperplane can be taken to be $\mathbb{R}^{n-1}$, $\vec{\alpha}(P)$ taken to be the basis vector $\vec{e_n}$, and, writing points $X = (X_1, \ldots, X_n) \in \mathbb{R}^n$ in rectangular coordinates as $X = (x, y)$ where $x \in \mathbb{R}^{n-1}$ and $X_n = y \in \mathbb{R}$,

$$|\varphi(x) - \varphi(x')| \leq M|x - x'| \tag{1.2}$$

for all $x, x' \in \mathbb{R}^{n-1}$.

Write $P = (x_0, y_0)$. After a scaling of space ($\vec{\alpha}$ and N remain unit vectors, M is scale invariant and the scaled φ still satisfy (1.2)) one obtains the setup, uniform in $P \in \partial\Omega$,

$$\begin{aligned} 100Z_P &:= \{(x, y) : |x - x_0| < 100\sqrt{n},\ |y - y_0| < 10^4 nM\} \cap \overline{\Omega} \\ &= \{(x, y) : |x - x_0| < 100\sqrt{n},\ -10^4 nM + y_0 < y \leq \varphi(x)\} \cap \overline{\Omega} \end{aligned} \tag{1.3}$$

M will always be taken to be at least 1. More generally

$$\theta Z_P := \{(x,y) : |x - x_0| < \theta\sqrt{n}, |y - y_0| < 100\theta nM\} \cap \overline{\Omega}$$

for $0 < \theta \leq 100$. Each θZ_P is a starlike Lipschitz domain with *starcenter* $(x_0, y_0 - \theta 50nM)$ and with Lipschitz geomery uniform in θ and P. Moreover, by the uniform continuity of $\vec{\alpha}$, this scaling can be done so that for all $Q \in \partial\Omega \cap Z_P$

$$\vec{\alpha}(P) \cdot \vec{\alpha}(Q) > \frac{100nM}{\sqrt{1 + (100nM)^2}}$$

For any θ let $\mathbf{n}(\theta, \Omega, \vec{\alpha})$ be the smallest number of θZ_P needed to cover $\partial\Omega$. Scaling of space will not be used again.

Given $a > 1$ a *nontangential cone* $\Gamma_a(P)$ is defined using the above local coordinate systems for each $P = (x_0, y_0) \in \partial\Omega$ by

$$\Gamma_a(P) = \{(x,y) : y_0 - y > aM|x - x_0|, y_0 - y < 100M\}\,.$$

The cone at P is right-circular with axis along the direction $\vec{\alpha}(P)$. For each a the collection of all such cones forms a regular family of nontangential cones as in [5, p. 298].

Given an $a > 1$ and any function v defined in Ω the nontangential maximal function of v is defined by

$$N_a(v)(Q) = \sup_{X \in \Gamma_a(Q)} |v(X)|$$

We will say that v is *nontangentially in* $L^p(\partial\Omega)$ if $N_a(v) \in L^p(\partial\Omega)$.

When $p > 1$ and $a < b$, a geometric argument [8] shows that

$$\|N_a(v)\|_{L^p(\partial\Omega)} \leq C\|N_b(v)\|_{L^p(\partial\Omega)}$$

where C depends on a, b and p.

In the case v is harmonic in Ω Dahlberg [4] proved that there is an $\epsilon(\Omega) > 0$ so that, given any $p > 2 - \epsilon(\Omega)$, the condition $N_a(v) \in L^p(\partial\Omega)$ implies that v has nontangential boundary values $v(Q)$ for *a.e.*(ds) $Q \in \partial\Omega$ and $v \in L^p(\partial\Omega)$. In addition there is a constant depending only on p, a and the Lipschitz geometry of Ω so that

$$\|N_a(v)\|_{L^p(\partial\Omega)} \leq C\|v\|_{L^p(\partial\Omega)} \tag{1.4}$$

In any Lipschitz domain $D \subset \mathbb{R}^n$ nontangential cones may be similarly defined. A collection of domains that have comparable Lipschitz geometries, e.g. the boundaries can be covered, up to a uniform scaling factor, by cylinders in one-to-one correspondence described by the same Lipschitz constant M of (1.2), will yield bounds (1.4) uniform over the collection.

The *Lusin area integral* or *square function* for a $C^1(D)$ function v may be defined as

$$S_a(v)(Q) = \left(\int_{\Gamma_a(Q)} \frac{|\nabla v(X)|^2}{|X-Q|^{n-2}} dX\right)^{\frac{1}{2}}$$

When v is *harmonic* in D and $0 < p < \infty$ Dahlberg proved [5] that there is a constant C depending only on p, a and the Lipschitz geometry of D so that

$$C^{-1}\int_{\partial D} N_a(v)^p ds \leq \int_{\partial D} S_a(v)^p ds \leq C\int_{\partial D} N_a(v)^p ds \tag{1.5}$$

where the left-hand inequality also requires the normalization that all v vanish at some fixed point in D.

Both the Lusin area integral and the nontangential maximal function are lower semicontinuous functions.

Define nontangential cones that have been *truncated* by an amount $h < 100M$ by

$$\Gamma_a^h(Q) = \{X \in \Gamma_a(Q) : |X-Q| < h\}$$

The following slightly more precise statement than either [13] or [14, p. 600] can be proved by following the arguments of the former pp. 213-216.

Lemma 1.1 (E.M. Stein). *Let v be harmonic in Ω. Let $1 < b < a < 100$ and $h < k < 100M$. Fix any point $P \in \partial\Omega$.*

Then there exists a constant C, independent of P, depending only on a, b, $\frac{h}{k}$ and M such that for every $Q \in 100Z_P \cap \partial\Omega$

$$\int_{\Gamma_a^h(Q)} \frac{|\nabla v(X)|^2}{|X-Q|^{n-2}} dX \leq C^2\left(\int_{\Gamma_b^k(Q)} \frac{|\vec{\alpha}(P)\cdot\nabla v(X)|^2}{|X-Q|^{n-2}} dX + h^2 \max_{|X-Q|=h} |\nabla v(X)|^2\right)$$

This is used to prove the following localization lemma.

Lemma 1.2. *Let $p > 2-\epsilon(\Omega)$ and suppose u is harmonic in Ω with $N_2(\nabla u) \in L^p(\partial\Omega)$. Let $\lambda > 0$. Then there are constants $A > 1$ and $C > 0$ independent of λ depending only on p, $\vec{\alpha}$, and the Lipschitz geometry of Ω, there is a positive integer $\overline{\mathbf{n}}$ depending only on $\vec{\alpha}$ and the Lipschitz geometry of Ω, and there is a sequence of Lipschitz cylinders $\theta_j Z_{P_j}$ with interiors Ω_j so that if*

$$\lambda > C\|N_2(\nabla u)\|_p \tag{1.6}$$

then

$$\bigcup_j \partial\Omega_j \cap \partial\Omega \subset \{Q \in \partial\Omega : N_2(\nabla u) + S_2(\nabla u) > \lambda\}, \tag{1.7}$$

no point of $\partial\Omega$ *is contained in more than* $\overline{\mathbf{n}}$ *of the* $\partial\Omega_j$*, and*

$$(1.8) \qquad |\{Q \in \partial\Omega : S_{40}(\nabla u) > A\lambda\}| \le \frac{C}{\lambda^p}\sum_j \int_{\partial\Omega_j} |\vec{\alpha}(P_j)\cdot\nabla u(Q)|^p ds(Q)$$

Proof. By (1.5) and the Chebychev inequality there is a constant large enough depending only on p and the Lipschitz geometry of Ω such that (1.6) implies $\{N_2 + S_2 > \lambda\} := \{N_2(\nabla u) + S_2(\nabla u) > \lambda\} \subset \partial\Omega$ has measure less than 1. Consider any Lipschitz cylinder $100Z_{(x_0,y_0)}$ together with its associated Lipschitz function φ and coordinate system. Consider the open set $G = \{x \in \mathbb{R}^{n-1} : |x - x_0| < 100\sqrt{n}$ and $(x, \varphi(x)) \in \{N_2 + S_2 > \lambda\}\}$. Let $G = \cup_j K_j$ be a dyadic Whitney decomposition [13, p. 167] with the property that if a cube K_j has side length l_j then

$$(1.9) \qquad K_j \cap \{x : 2l_j\sqrt{n-1} \le dist(x, G^c) \le 4l_j\sqrt{n-1}\} \ne \emptyset$$

For each K_j let P_j be the projection of its center into $\partial\Omega$ and consider *the collection of Lipschitz cylinders* $l_jZ_{P_j}$ *such that* $l_jZ_j \cap 1Z_{(x_0,y_0)} \ne \emptyset$. Because λ is large $l_j < 1$ and all cylinders retained are in $3Z_{(x_0,y_0)}$.

For $A > 1$, $\{S_{40}(\nabla u) > A\lambda\} \subset \{N_2 + S_2 > \lambda\}$.

Denote the interiors of the retained $l_jZ_{P_j}$ by Ω_j. Define $E_j = \{Q = (x, \varphi(x)) : S_{40}(\nabla u) > A\lambda$ *and* $x \in K_j\}$. By (1.9) there is a boundary point $Q_j^* \notin \{N_2 + S_2 > \lambda\}$ with distance to $\partial\Omega_j \cap \partial\Omega$ less than $4Ml_j\sqrt{n-1}$. Consequently for $h = 15l_jM\sqrt{n-1}$ and every $Q \in \partial\Omega_j \cap \partial\Omega$

$$\int_{\Gamma_{40}(Q)\setminus\Gamma_{40}^h(Q)} \frac{|\nabla\nabla u(X)|^2}{|X - Q|^{n-2}} dX \le$$
$$C\int_{\Gamma_{40}(Q)\setminus\Gamma_{40}^h(Q)} \frac{|\nabla\nabla u(X)|^2}{|X - Q_j^*|^{n-2}} dX \le CS_2^2(\nabla u)(Q_j^*) \le C\lambda^2$$

where C depends only on the Lipschitz geometry of Ω. Thus for A large enough depending on C, and for $Q \in E_j$

$$(1.10) \qquad S_{40,h}(\nabla u)(Q) := \left(\int_{\Gamma_{40}^h(Q)} \frac{|\nabla\nabla u(X)|^2}{|X - Q|^{n-2}} dX\right)^{\frac{1}{2}} > \frac{A}{2}\lambda$$

The cones Γ_{40}^h are also defined on the projection of K_j into $\partial\Omega_j$. From there they can be extended to be a regular family for the Lipschitz domains Ω_j uniformly in j and the square function (1.10) likewise extended. The inequalities (1.5) for the corresponding nontangential maximal function will be uniform in j and independent of the point (x_0, y_0).

For each Ω_j recall the vector $\vec{\alpha}(P_j)$. The functions $H_j(X) = \vec{\alpha}(P_j) \cdot \nabla u(X)$ are harmonic in the original domain Ω for all j and

$$\frac{\partial}{\partial X_i} H_j = \vec{\alpha}(P_j) \cdot \nabla \frac{\partial}{\partial X_i} u$$

for $i = 1, \ldots, n$. For every $Q \in E_j$, Lemma 1.1 applies to $v = \frac{\partial}{\partial X_i} u$ over the cones $\Gamma_{40}^h(Q)$. In addition, by interior estimates and the fact that $|\nabla u| \leq \lambda$ in $\Gamma_{40}(Q) \backslash \Gamma_{40}^h(Q)$ since $N_2(\nabla u)(Q_j^*) \leq \lambda$, it follows that

$$h \max_{|X-Q|=h} |\nabla \frac{\partial u}{\partial X_i}| \leq C' \lambda\,.$$

Thus by (1.10), applying Lemma 1.1 for some $k > h$ and utilizing the constant there

$$|E_j| \leq \frac{2^p}{A^p \lambda^p} \int_{E_j} S_{40,h}^p(\nabla u) ds \leq \frac{2^p C^p}{A^p \lambda^p} \int_{E_j} S_{30,k}^p(H_j) ds + \frac{2^p C^p C'^p}{A^p} |E_j|\,.$$

Consequently for A again large enough and by the right side of (1.5) followed by (1.4)

$$|E_j| \leq \frac{C''}{\lambda^p} \int_{E_j} S_{30,h}^p(H_j) ds \leq \frac{C'''}{\lambda^p} \int_{\partial\Omega_j} |H_j|^p ds\,.$$

Because

$$\{S_{40} > A\lambda\} \cap 1Z_{(x_0,y_0)} \subset \cup E_j$$

and a finite number $\mathbf{n}(1, \Omega, \vec{\alpha})$ of $1Z_{(x_0,y_0)}$ cover $\partial\Omega$,and because of (1.9), the conclusions of the lemma follow by summing over j and $\mathbf{n}$. ■

2. Uniqueness for the almost everywhere oblique derivative problem

In this section it will first be assumed that the vector field $\vec{\alpha}$ is itself Lipschitz continuous, i.e. there is a constant M_α so that for all $P, Q \in \partial\Omega$

$$|\vec{\alpha}(P) - \vec{\alpha}(Q)| \leq M_\alpha |P - Q| \tag{2.1}$$

Following the exposition of [10] let ω denote harmonic measure on $\partial\Omega$ with respect to some fixed point of Ω. Harmonic measure of a set $E \subset \partial\Omega$ with respect to an arbitrary point $X \in \Omega$ is then $\int_E K(X,Q) d\omega(Q)$ where $K(X,Q) \geq 0$ is the *kernel function*.

Given a point $Q \in \partial\Omega$ the surface ball of radius r is defined by $\Delta(Q,r) = \{Q' : |Q' - Q| < r\} \cap \partial\Omega$. The kernel function satisfies the following lemma (formulated as in Lemma 5.13 of [10]) of Hunt and Wheeden [9, p. 315].

Lemma 2.1. (Hunt and Wheeden) *Let* $dist(X, \partial\Omega) = r < 1$ *and suppose* $X \in \Gamma_2(Q)$. *Define* $\Delta_j = \Delta(Q, 2^j r)$ *and dyadic surface rings* $R_j = \Delta_j \backslash \Delta_{j-1}$, $j = 1, 2, \dots$

Then there is a constant C *and an exponent* $\beta > 0$ *depending only on the Lipschitz geometry of* Ω *so that*

$$\sup_{Q \in R_j} K(X, Q) \leq \frac{C 2^{-\beta j}}{\omega(\Delta_j)},\ j = 1, 2, \dots$$

The Hardy-Littlewood maximal function with respect to harmonic measure ω on $\partial\Omega$ of a function f is defined with respect to surface balls

$$\mathcal{M}_\omega f(Q) = \sup_{\Delta \ni Q} \frac{1}{\omega(\Delta)} \int_\Delta |f| d\omega,\ Q \in \partial\Omega$$

Dahlberg's reverse Hölder inequality for the density of harmonic measure [3] shows that

$$\mathcal{M}_\omega^{\tilde{p}} f(Q) \leq C_{\tilde{p}} \mathcal{M}_s |f|^{\tilde{p}}(Q) \tag{2.2}$$

for any $\tilde{p} > 2 - \epsilon(\Omega)$, where on the right side is the Hardy-Littlewood maximal function of $|f|^{\tilde{p}}$ with respect to surface measure ds, and the constant depends also on the Lipschitz geometry of Ω.

Dahlberg also showed [4] that if H is a harmonic function in Ω with nontangential maximal function in L^p for some $p > 2 - \epsilon(\Omega)$ then it has the Poisson representation

$$H(X) = \int_{\partial\Omega} H(Q) K(X, Q) d\omega(Q) \tag{2.3}$$

Nontangentially vanishing oblique derivative means

$$\lim_{\Gamma_a(P) \ni X \to P} \vec{\alpha}(P) \cdot \nabla u(X) = 0$$

for almost every $P \in \partial\Omega$. The vanishing is independent of $a > 1$ when the gradient is nontangentially in L^p.

Lemma 2.2. *Let* $p > 2 - \epsilon(\Omega)$ *and* u *be as in Lemma 1.2. Suppose* u *has oblique derivative with respect to* $\vec{\alpha}$ *nontangentially vanishing* a.e.(ds) *on* $\partial\Omega$ *and suppose* $\vec{\alpha}$ *is Lipschitz continuous satisfying* (2.1). *Let* $\delta > 0$ *be given. Fix any point* $P \in \partial\Omega$ *and define* $H(X) = \vec{\alpha}(P) \cdot \nabla u(X)$.

Then there are constants $c > 0$ *and* $C < \infty$ *that depend only on the Lipschitz geometry of* Ω *so that if* $|X - P| < c\delta^{1+\frac{1}{\beta}}$ *where* $\beta > 0$ *is from Lemma 2.1 and if* $X \in \Gamma_2(Q_0)$ *with* $\mathrm{dist}(X, \partial\Omega) = r$, *then*

$$|H(X)| \leq C\delta M_\alpha \max_j \frac{1}{\omega(\Delta_j)} \int_{\Delta_j} |\nabla u| d\omega \leq C\delta M_\alpha \mathcal{M}_\omega(\nabla u)(Q_0) \tag{2.4}$$

where $\Delta_j = \Delta(Q_0, 2^j r)$ *as in Lemma* 2.1 *(Hunt and Wheeden).*

Proof. Because β can be assumed close to 0, given any $1 > \delta > 0$ there is a $J = J_\delta$ so that

$$\delta \leq \sum_{j=J}^{\infty} 2^{-\beta j} \leq 2\delta \tag{2.5}$$

or equivalently

$$\delta^{-\frac{1}{\beta}} 2^{-\frac{1}{\beta}} (1-2^{-\beta})^{-\frac{1}{\beta}} \leq 2^J \leq \delta^{-\frac{1}{\beta}} (1-2^{-\beta})^{-\frac{1}{\beta}} \tag{2.6}$$

Take $c = (1-2^{-\beta})^{\frac{1}{\beta}}$.

By (2.3), Lemma 2.1 and the right-hand inequality (2.5)

$$|H(X)| \leq C\delta \max_j \frac{1}{\omega(\Delta_j)} \int_{\Delta_j} |\nabla u| d\omega + C \sum_{j=0}^{J-1} 2^{-\beta j} \frac{1}{\omega(\Delta_j)} \int_{\Delta_j} |H(Q)| d\omega(Q)$$

Now $|H(Q)| = |(\vec{\alpha}(P) - \vec{\alpha}(Q)) \cdot \nabla u(Q)|$ $a.e.(ds)$. By the choice of c, the right-hand inequality (2.6) and the fact that $r \leq |X-P|$, every Q contained in the surface balls under the finite sum satisfies $|Q - Q_0| < \delta$. In addition $|Q_0 - P| \leq Cr$ because $X \in \Gamma_2(Q_0)$. Consequently $|Q-P| \leq C\delta$ and thus by (2.1) $|H(Q)| \leq C\delta M_\alpha |\nabla u(Q)|$ for every Q contained in the surface balls under the finite sum. ∎

Lemma 2.3. *Let p, u and $\vec{\alpha}$ be as in Lemma 2.2. Then $N_2(\nabla u) \in L^q(\partial\Omega)$ for all $q < \infty$.*

Proof. Let λ satisfying (1.6) and Ω_j be as in Lemma 1.2. Because $N_2(\nabla u) \in L^1$ and because it suffices to take the L^1-norm less than or equal to 1, it follows from (1.7) that $diam\Omega_j \leq C\lambda^{-\frac{1}{n-1}}$ for all j. By choosing $\delta = (\frac{C}{c})^{\frac{\beta}{\beta+1}} \lambda^{-\frac{1}{n-1}\frac{\beta}{\beta+1}}$, the diameter of every Ω_j is bounded by the amount $c\delta^{1+\frac{1}{\beta}}$ from Lemma 2.2. Points on the sides $\partial\Omega_j \cap \Omega$ are contained in cones $\Gamma_2(Q_0)$ for $Q_0 \in \partial\Omega_j \cap \partial\Omega$. Thus by Lemma 1.2 followed by Lemma 2.2 and a geometric argument

$$\begin{aligned} |\{S_{40}(\nabla u) > A\lambda\}| &\leq \frac{C}{\lambda^p} \sum_j \int_{\partial\Omega_j} |\vec{\alpha}(P_j) \cdot \nabla u(Q)|^p ds(Q) \\ &\leq \frac{C'}{\lambda^{p(1+\frac{1}{n-1}\frac{\beta}{\beta+1})}} \sum_j \int_{\partial\Omega_j \cap \partial\Omega} \mathcal{M}_\omega^p(\nabla u)(Q) ds(Q)\,. \end{aligned}$$

By choosing $2 - \epsilon(\Omega) < \tilde{p} < p$, utilizing (2.2) and the theorem of Hardy and Littlewood, the summation is controlled by $\|\nabla u\|_p^p$ on the boundary of Ω. Consequently $S_{40}(\nabla u)$ is in $L^q(\partial\Omega)$ for every $q < p(1 + \frac{1}{n-1}\frac{\beta}{\beta+1})$ and so are $N_{40}(\nabla u)$ and then $N_2(\nabla u)$. The same statement follows for any given $q < \infty$ by a finite number of repetitions of the argument. ∎

Remark 2.4. It is only at the end of the above proof that the left side of (1.5) is used, and then the required normalization plays no rôle.

Remark 2.5. Replacing (2.1) with a Hölder condition

$$|\vec{\alpha}(P) - \vec{\alpha}(Q)| \leq M_\alpha |P - Q|^\gamma \tag{2.7}$$

for some $0 < \gamma \leq 1$, yields (2.4) of Lemma 2.2 with δ replaced by δ^γ. Lemma 2.3 remains true under the Hölder condition, the proof altered only by $\frac{\gamma}{n-1}\frac{\beta}{\beta+1}$ replacing $\frac{1}{n-1}\frac{\beta}{\beta+1}$.

Theorem 2.6. *Let $p > 2 - \epsilon(\Omega)$ and let $\vec{\alpha}$ be a Hölder continuous unit vector field defined on $\partial\Omega$ and satisfying* (2.7)*. If u is harmonic in Ω with gradient nontangentially in $L^p(\partial\Omega)$ and with oblique derivative with respect to $\vec{\alpha}$ vanishing nontangentially a.e.(ds), then*

(i) $u \in C(\overline{\Omega})$ and

(ii) for each *$Q \in \partial\Omega$ the classical directional derivative*

$$\lim_{t \uparrow 0} \frac{u(Q) - u(Q + t\vec{\alpha}(Q))}{-t}$$

exists and is equal to zero.

Proof. The first conclusion follows by Sobolev embedding from Lemma 2.3. (See Remark 2.5.)

Fix a point on the boundary. By translation it may be taken to be the *origin*. Define $H(X) = \vec{\alpha}(0) \cdot \nabla u(X)$ harmonic in Ω. By the mean value theorem for differentiable functions on an interval, the second conclusion will follow if $\lim_{\overline{X} \to 0} H(\overline{X}) = 0$ where $\overline{X}$ denotes the points $t\vec{\alpha}(0)$. In Lemma 2.2 take both P and Q_0 to be the *origin* and $|\overline{X}| = \frac{c}{2}\delta^{1+\frac{1}{\beta}}$. Because $\overline{X} \in \Gamma_2(0)$ this last quantity is also equivalent to $r = dist(\overline{X}, \partial\Omega)$. By Dahlberg's reverse Hölder inequality (e.g. see (2.2)) followed by Jensen's inequality

$$\frac{1}{\omega(\Delta_j)} \int_{\Delta_j} |\nabla u| d\omega \leq C\left(\frac{1}{|\Delta_j|} \int_{\Delta_j} |\nabla u|^2 ds\right)^{\frac{1}{2}} \leq C\left(\frac{1}{|\Delta_j|} \int_{\Delta_j} |\nabla u|^q ds\right)^{\frac{1}{q}}$$

so that C depends only on the Lipschitz geometry of Ω.

Choose $q > (1+\frac{1}{\beta})\frac{(n-1)}{\gamma}$. The right-hand side of the above inequality is finite by Lemma 2.3. Since $|\Delta_j| \geq c_n r^{n-1}$ for every surface ball of Lemma 2.2, while r^{n-1} is equivalent to $\delta^{(1+\frac{1}{\beta})(n-1)}$, it follows from Lemma 2.2 and Remark 2.5 that

$$|H(\overline{X})| \leq C\delta^{\gamma - (1+\frac{1}{\beta})\frac{(n-1)}{q}} \|\nabla u\|_{L^q(\partial\Omega)}$$

which proves the theorem. ■

Under conditions on Ω and $\vec{\alpha}$ more general than those here, N. S. Nadirashvili showed that a harmonic function, which is continuous up to the boundary and which has vanishing classical oblique derivative everywhere on the boundary, is constant.

Corollary 2.7. (Theorem 2.6 and [12, Theorem 1]) *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $p > 2 - \epsilon(\Omega)$. A harmonic function having gradient nontangentially in $L^p(\partial\Omega)$ and having oblique derivative, with respect to a unit transverse Hölder continuous vector field, vanishing nontangentially a.e.(ds) on on the boundary is constant.*

Remark 2.8. Because the L^p-Dirichlet problem (1.4) is solvable in a C^1 domain for the range $1 < p$ [7, 4], both Lemmas 1.2 and 2.2 hold for C^1 domains and $p > 1$. Lemma 2.3 also follows in this setting because harmonic measure satisfies a reverse Hölder inequality for all $q < \infty$ [4] thereby yielding (2.2) for all $\tilde{p} > 1$. Both Theorem 2.6 and Corollary 2.7 hold for $p > 1$ when Ω is C^1.

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