Rev. Mat. Iberoamericana **22** (2006), no. 3, 955–962

Asymptotic behaviour of monomial ideals on regular sequences

Monireh Sedghi

Abstract

Let R be a commutative Noetherian ring, and let $\mathbf{x} = x_1, \ldots, x_d$ be a regular R-sequence contained in the Jacobson radical of R. An ideal I of R is said to be a monomial ideal with respect to $\mathbf{x}$ if it is generated by a set of monomials $x_1^{e_1} \ldots x_d^{e_d}$. The monomial closure of I, denoted by $\widetilde{I}$, is defined to be the ideal generated by the set of all monomials m such that $m^n \in I^n$ for some $n \in \mathbb{N}$. It is shown that the sequences $\mathrm{Ass}_R R/\widetilde{I^n}$ and $\mathrm{Ass}_R \widetilde{I^n}/I^n$, $n = 1, 2, \ldots$, of associated prime ideals are increasing and ultimately constant for large n. In addition, some results about the monomial ideals and their integral closures are included.

1. Introduction

Let R be a commutative Noetherian ring and I an ideal of R. In [9], L. J. Ratliff Jr., conjectured about the asymptotic behaviour of $\mathrm{Ass}_R R/I^n$ when R is a domain. Subsequently, M. Brodmann [1] showed that if R is Noetherian and M is a finitely generated R-module, then $\mathrm{Ass}_R M/I^nM$ is ultimately constant for large $n \in \mathbb{N}$. Furthermore, in [9], L. J. Ratliff Jr. showed that the sequence $\{\mathrm{Ass}_R R/(I^n)_a\}_{n\in\mathbb{N}}$ is increasing and ultimately constant, where $(I^n)_a$ denotes the classical integral closure of the ideal I^n. (We use $\mathbb{N}$ to denote the set of positive integers.)

The results of Brodmann and Ratliff have led to a large body of research. For example, McAdam and Ratliff (see [7, 3.9 and 11.16]) showed that, if the ideal I of R contains a non-zero divisor on R, then the sequences

$$\{\mathrm{Ass}_R R/(I^n)_a\}_{n\in\mathbb{N}} \quad \text{and} \quad \{\mathrm{Ass}_R (I^n)_a/I^n\}_{n\in\mathbb{N}}$$

are increasing and eventually constant.

2000 Mathematics Subject Classification: 13B20, 13B21.
Keywords: Monomial ideals, integral closures, monomial closures.

There is considerable current interest in the concept of monomial ideal in a commutative Noetherian ring. Let R be a commutative Noetherian ring and let $\mathbf{x} := x_1, \dots, x_d$ be a regular R-sequence which is contained in the Jacobson radical of R. An ideal I of R is called a *monomial ideal* with respect to $\mathbf{x}$ if it can be generated by monomials $x_1^{e_1} \dots x_d^{e_d}$. For any monomial ideal I with respect to $\mathbf{x}$, the *monomial closure* of I, denoted by $\widetilde{I}$, is defined the ideal generated by all monomials m with $m^l \in I^l$ for some $l \in \mathbb{N}$.

The above-mentioned results of Brodmann, McAdam and Ratliff raise questions about asymptotic behaviour on monomial ideals on R-sequences. As the main result of the second section we shall prove the following:

Theorem 1.1. *Let I denote a monomial ideal of R with respect to a regular R-sequence contained in the Jacobson radical of R. Then the sequences*

$$\{\operatorname{Ass}_R R/\widetilde{I^n}\}_{n\in\mathbb{N}} \quad \text{and} \quad \{\operatorname{Ass}_R \widetilde{I^n}/I^n\}_{n\in\mathbb{N}}$$

of associated primes are increasing for large n and eventually constant.

For any regular R-sequence $\mathbf{x} := x_1, \dots, x_d$ contained in the Jacobson radical of R, let $(\mathbf{x})$ be a prime ideal of R. In third section we will obtain some results about the integral closure and monomial closure of a certain monomial ideal in R. In this section, among other things, we prove the following theorem.

Theorem 1.2. *Let $\mathbf{x} := x_1, \dots, x_d$ be a regular R-sequence contained in the Jacobson radical of R, let $(\mathbf{x})$ be a prime ideal of R and let τ be a permutation of $\{1, \dots, d\}$. Let $I := (x_{\tau(1)}^{\alpha_1}, \dots, x_{\tau(s)}^{\alpha_s})$ be a monomial ideal of R where $1 \le s \le d$ and $\alpha_i \in \mathbb{N}$. Then for any $n \in \mathbb{N}$, I^n and $\widetilde{I^n}$ are $(x_{\tau(1)}, \dots, x_{\tau(s)})$-primary ideals and $(I^n)_a = \widetilde{I^n}$.*

The proof of Theorem 1.2 is given in 3.4. For any unexplained notation and terminology we refer the reader to [6] and [2].

2. Finiteness of associated primes

In this section we show that if $\mathbf{x} := x_1, \dots, x_d$ is a regular R-sequence contained in the Jacobson radical of R, and I is a monomial ideal of R with respect to $\mathbf{x}$ then the sets $\operatorname{Ass}_R R/\widetilde{I^n}$ and $\operatorname{Ass}_R \widetilde{I^n}/I^n$ are constant for large n. The main results are Theorems 2.5 and 2.7. We begin with

Definition 2.1. Let $\mathbf{x} := x_1, \dots, x_d$ be a regular R-sequence of elements of R. Then:

(i) An element m of R is called a *monomial element w.r.t.* $\mathbf{x}$ if there exist non-negative integers $e_1, \dots, e_d$ such that $m = x_1^{e_1} \dots x_d^{e_d}$. In view of [6, Thm. 16.2] it is easy to see that $e_1, \dots, e_d$ are determined uniquely by m.

(ii) Suppose that $m = x_1^{e_1} \dots x_d^{e_d}$ is a monomial w.r.t. $\mathbf{x}$. The *support of* m, denoted by $\mathrm{supp}(m)$, is defined to be the set $\{j \mid j \in \{1, \dots, d\}$ and $e_j \neq 0\}$.

(iii) Let $\mathcal{M}$ denote the set of all monomials of R w.r.t. $\mathbf{x}$. An ideal I of R is called *monomial w.r.t.* $\mathbf{x}$ if it is generated by elements in $\mathcal{M}$. It follows from this that the zero ideal and R itself are monomial ideals.

(iv) Let I be a monomial ideal of R w.r.t. $\mathbf{x}$. The *monomial closure of* I, denoted by $\widetilde{I}$, is defined to be the ideal generated by all monomials m in $\mathcal{M}$ such that $m^n \in I^n$ for some $n \in \mathbb{N}$. Let $(I)_a$ denote the classical integral closure of I introduced by Northcott and Rees [8]. Then obviously $I \subseteq \widetilde{I} \subseteq (I)_a$. Also, if J is a second monomial ideal of R w.r.t. $\mathbf{x}$, then $\widetilde{\widetilde{I}\widetilde{J}} = \widetilde{IJ}$, and if $I \subseteq J$ then $\widetilde{I} \subseteq \widetilde{J}$. Recall that, by [5, Proposition 1], the set of all monomials of R is closed under finite products.

(v) Suppose that s is an integer such that $1 \leq s \leq d$, and let τ be a permutation of $\{1, \dots, d\}$ and let $e_1, \dots, e_s$ be positive integers. The monomial ideal generated by $x_{\tau(1)}^{e_1}, \dots, x_{\tau(s)}^{e_s}$ is called a *generalized-parameter ideal.*

Throughout this section, $\mathbf{x} := x_1, \dots, x_d$ is a regular R-sequence contained in the Jacobson radical of R w.r.t. $\mathbf{x}$, unless otherwise specified.

The following lemma plays a key role in the proof of the main results.

Lemma 2.2. *Let I be a non-zero monomial ideal of R. Then for all large n,*

$$(\widetilde{I^n} :_R \widetilde{I^k}) = (\widetilde{I^n} :_R I^k) = \widetilde{I^{n-k}},$$

for every integer k.

Proof. As $\widetilde{I^{n-k}} I^k \subseteq \widetilde{I^{n-k}} \widetilde{I^k} \subseteq \widetilde{I^n}$, it is enough to show that $(\widetilde{I^n} :_R I^k) \subseteq \widetilde{I^{n-k}}$. To this end, it will suffice to show that $(\widetilde{I^{n+1}} :_R I) \subseteq \widetilde{I^n}$ for all large n. Let $x \in R$ be such that $Ix \subseteq \widetilde{I^{n+1}}$ and let I be generated by s elements. Then there exists an integer $l \geq 1$ such that $x^l I^{sl} \subseteq I^{(n+s)l}$. Hence $x^l \in (I^{nl+sl} :_R I^{sl})$, and so in view of [7, Lemma 8.1], $x^l \in I^{nl}$. Therefore by definition $x \in \widetilde{I^n}$, as desired. ■

Before bringing the next result we fix some notations. Let U be an arbitrary subset of $\{1, \dots, d\}$. We use $\mathfrak{q}_U$ to denote the ideal of R generated by the set $\{x_i : i \in U\}$ and $\mathcal{P}_U$ to denote the set $\mathrm{Ass}_R R/\mathfrak{q}_U$. In particular, if $U = \{1, \dots, d\}$ then we write,

$$\mathfrak{q} := \mathfrak{q}_U \text{ and } \mathcal{P} := \mathrm{Ass}_R R/\mathfrak{q}.$$

The following lemma is originally shown by K. Kiyek and J. Stückrad in [5, Lemma 1], we give a direct proof by using the important notion of the generalized-parameter monomial ideals introduced by W. Heinzer *et al.* [3].

Lemma 2.3. *Let I be a monomial ideal of R generated by monomials $m_1, \dots, m_r$. Then*

$$\mathrm{Ass}_R R/I \subseteq \bigcup \{\mathcal{P}_U : U \subseteq \mathrm{supp}(m_1) \cup \dots \cup \mathrm{supp}(m_r)\}.$$

Proof. Since $\mathrm{Ass}_R R = \mathcal{P}_\emptyset$, it is enough to consider $I \neq 0$. Then by [5, Lemma 3] and [3, Theorem 4.9] I is an irredundant finite intersection of generalized-parameter monomial ideals which is unique up to the order of the factors. Hence without loss of generality we may assume that I is a generalized-parameter monomial ideal w.r.t. $\mathbf{x}$. Because $\mathbf{x}$ is contained in the Jacobson radical of R, we can assume $I = (x_1^{e_1}, \dots, x_s^{e_s})$ where $1 \leq s \leq d$ and $e_i \in \mathbb{N}$ for all $i = 1, \dots, s$. If $I = (x_1, \dots, x_s)$ then the assertion follows. Now, let $e_i \geq 2$ for some $i = 1, \dots, s$. As $\mathbf{x}$ is contained in the Jacobson radical of R, we may assume that $i = 1$. Consider $J = (x_1^{e_1-1}, x_2^{e_2}, \dots, x_s^{e_s})$ and let $\mu : R/J \longrightarrow R/I$ denote the multiplication homomorphism by x_1. One can easily see that μ is a monomorphism, and then from the exact sequence,

$$0 \longrightarrow R/J \longrightarrow R/I \longrightarrow R/(x_1, x_2^{e_2}, \dots, x_s^{e_s}) \longrightarrow 0$$

we have

$$\mathrm{Ass}_R R/I \subseteq \mathrm{Ass}_R R/J \cup \mathrm{Ass}_R R/(x_1, x_2^{e_2}, \dots, x_s^{e_s}).$$

Hence the assertion follows by induction. ■

Lemma 2.4. *Let I be a non-zero monomial ideal of R. Then for all large n,*

$$\mathrm{Ass}_R R/\widetilde{I^n} = \mathrm{Ass}_R \widetilde{I^{n-1}}/\widetilde{I^n}.$$

Proof. For large n, let $\mathfrak{p} \in \mathrm{Ass}_R R/\widetilde{I^n}$. Then there exists $x \in R \setminus \widetilde{I^n}$ such that $\mathfrak{p} = (\widetilde{I^n} :_R x)$. Since $I \subseteq \mathfrak{p}$, it follows that $x \in (\widetilde{I^n} :_R I)$. Hence by Lemma 2.2, $x \in \widetilde{I^{n-1}}$ and so $\mathfrak{p} \in \mathrm{Ass}_R \widetilde{I^{n-1}}/\widetilde{I^n}$. This completes the proof. ■

We can now state and prove our the main results, which concern the asymptotic stability of $\mathrm{Ass}_R R/\widetilde{I^n}$ and $\mathrm{Ass}_R \widetilde{I^n}/I^n$.

Theorem 2.5. *Let I be a monomial ideal of R. Then the sequence*

$$\{\mathrm{Ass}_R R/\widetilde{I^n}\}_{n \geq 1}$$

of associated prime ideals, is increasing for large n and eventually constant.

Proof. For large n, let $\mathfrak{p} \in \mathrm{Ass}_R R/\widetilde{I^n}$. Then there exists $x \in R \setminus \widetilde{I^n}$ such that $\mathfrak{p} = (\widetilde{I^n} :_R x)$. Hence in view of Lemma 2.2, $\mathfrak{p} = (\widetilde{I^{n+1}} :_R Ix)$. Therefore $\mathfrak{p} \in \mathrm{Ass}_R R/\widetilde{I^{n+1}}$. It follows that the sequence $\{\mathrm{Ass}_R R/\widetilde{I^n}\}_{n\geq 1}$ is increasing for all large n. On the other hand by Lemma 2.3, the set $\bigcup_{n\geq 1} \mathrm{Ass}_R R/\widetilde{I^n}$ is finite. Consequently the assertion now follows. ■

An immediate consequence of Theorem 2.5 is the following.

Corollary 2.6. *Let I be a monomial ideal of R. Then the sequences*

$$\{\mathrm{Ass}_R (I^n)_a/\widetilde{I^n}\}_{n\geq 1} \text{ and } \{\mathrm{Ass}_R \widetilde{I^{n-1}}/\widetilde{I^n}\}_{n\geq 1}$$

of associated primes are increasing for large n and eventually constant.

Theorem 2.7. *Let I be a monomial ideal of R. Then the sequence*

$$\{\mathrm{Ass}_R \widetilde{I^n}/I^n\}_{n\geq 1}$$

is increasing for large n and ultimately constant.

Proof. In view of Brodmann's result [1], it is enough to show that the sequence $\{\mathrm{Ass}_R \widetilde{I^n}/I^n\}_{n\geq 1}$ is increasing for large n. To do this, by [7, Lemma 8.1] for all large n we have $(I^{n+1} :_R I) = I^n$. Now for large n, let $\mathfrak{p} \in \mathrm{Ass}_R \widetilde{I^n}/I^n$ and we write $\mathfrak{p} = (I^n :_R x)$ for some $x \in \widetilde{I^n} \setminus I^n$. Then we have

$$\mathfrak{p} = ((I^{n+1} :_R I) :_R x) = (I^{n+1} :_R Ix).$$

Since $Ix \subseteq \widetilde{I^{n+1}}$, it follows that there exists $y \in \widetilde{I^{n+1}}$ such that $\mathfrak{p} = (I^{n+1} :_R y)$. This shows that $\mathfrak{p} \in \mathrm{Ass}_R \widetilde{I^{n+1}}/I^{n+1}$ for all large n, and so the assertion follows. ■

3. Some results about monomial ideals

Recall that $\mathbf{x} := x_1, \ldots, x_d$ is a regular R-sequence contained in the Jacobson radical of R.

Remark 3.1. Let $\mathfrak{q} := (\mathbf{x})$ be a prime ideal of R. Then it follows from [2, Theorem 1.1.8] that the associated graded ring $\oplus_{n\geq 0}\mathfrak{q}^n/\mathfrak{q}^{n+1}$ is a domain. As $\mathfrak{q}$ is contained in the Jacobson radical of R, we have $\cap_{n\geq 1}\mathfrak{q}^n = 0$, and hence R is also a domain. Indeed if x and y are non-zero elements of R with $xy = 0$, we may choose integers m, n such that $x \in \mathfrak{q}^m \setminus \mathfrak{q}^{m+1}$ and $y \in \mathfrak{q}^n \setminus \mathfrak{q}^{n+1}$. Then $(x + \mathfrak{q}^{m+1})(y + \mathfrak{q}^{n+1}) = 0$ and this is a contradiction. Moreover, since $\mathfrak{q}$ is a prime ideal of R generated by *height* $\mathfrak{q}$ elements it follows that $R_\mathfrak{q}$ is a regular local ring. So that, if $U \subseteq \{1, \ldots, d\}$ and $\mathfrak{q}_U$ is an ideal of R generated by $\{x_i : i \in U\}$, then $\mathfrak{q}_U R_\mathfrak{q}$ is a prime ideal of $R_\mathfrak{q}$.

Thus $\mathfrak{q}_U$ will be a prime ideal of R. Now, let τ be a permutation on $\{1,\dots,d\}$ and $I := (x_{\tau(1)}^{e_1},\dots,x_{\tau(s)}^{e_s})$ a generalized-parameter monomial ideal. Then $\mathrm{Ass}_R R/I = \{(x_{\tau(1)},\dots,x_{\tau(s)})\}$ by Lemma 2.3. As $(x_{\tau(1)},\dots,x_{\tau(s)})$ is a prime ideal it follows that I is a $(x_{\tau(1)},\dots,x_{\tau(s)})$-primary monomial ideal of R. Furthermore, since I is generated by a regular R-sequence it follows from [4, Thm. 125 and Ex. 13] that $\mathrm{Ass}_R R/I^n = \mathrm{Ass}_R R/I$ for any $n \in \mathbb{N}$, and so I^n is also a $(x_{\tau(1)},\dots,x_{\tau(s)})$-primary monomial ideal of R.

Proposition 3.2. *Let $\mathfrak{q} = (\mathbf{x})$ be as in Remark* 3.1 *and let I be a non-zero monomial ideal of R. Then I has a primary decomposition each primary component of which is monomial ideal.*

Proof. In view of [3, Theorem 4.9] I has a unique generalized-parametric monomial decomposition. Now since $\mathfrak{q} = (\mathbf{x})$ is a prime ideal it follows that from Remark 3.1 that the mentioned decomposition is the desired primary decomposition of I. ■

Lemma 3.3. *Let $\mathfrak{q} = (\mathbf{x})$ and let $I := (x_{\tau(1)}^{e_1},\dots,x_{\tau(s)}^{e_s})$ be as in Remark* 3.1. *Then $\widetilde{I}$ is a primary ideal of R.*

Proof. Let $\mathfrak{p} := (x_{\tau(1)},\dots,x_{\tau(s)})$. Then by Remark 3.1 we have $\mathrm{Ass}_R R/I = \{\mathfrak{p}\}$. Let $\widetilde{I} := (m_1,\dots,m_r)$ where $m_i \in \mathcal{M}$ for all $i = 1,\dots,r$. In view of Lemma 2.3 and Remark 3.1, it is enough to show that $\mathrm{supp}(m_i) \subseteq \{\tau(1),\dots,\tau(s)\}$ for each $1 \le i \le r$. Assume the contrary. Then there exists $e \in \mathbb{N}, 1 \le j \le r$ and $1 \le t \le d$ such that $m_j = x_t^e m'$ where $m' \in \mathcal{M}$, $\mathrm{supp}(m') \subseteq \{\tau(1),\dots,\tau(s)\}$, $m' \notin \widetilde{I}$ and $t \notin \{\tau(1),\dots,\tau(s)\}$. Then there is an integer k such that $m_j^k \in I^k$. Hence $x_t^{ek} \in (I^k :_R m'^k)$. As $m'^k \notin I^k$, it follows that $x_t^{ek} \in \mathfrak{p}$. So that $x_t \in \mathfrak{p}$, which is contradiction. ■

The following result, which was shown by K. Kiyek and J. Stuckrad in [5, Proposition 7], extends the original argument that the integral closure of a monomial ideal in a polynomial ring over a field in a finite number of indeterminates is a monomial ideal (see [11, section 6.6, Ex. 6.6.1]).

Theorem 3.4. *Let $\mathfrak{q} := (\mathbf{x})$ be a prime ideal of R and let τ be a permutation on $\{1,\dots,d\}$. Let $I := (x_{\tau(1)}^{e_1},\dots,x_{\tau(s)}^{e_s})$ be a generalized-parameter monomial ideal of R. Then for any $n \in \mathbb{N}$, I^n and $\widetilde{I^n}$ are $(x_{\tau(1)},\dots,x_{\tau(s)})$-primary monomial ideals of R and $(I^n)_a = \widetilde{I^n}$.*

Proof. By virtue of Remark 3.1 and Lemma 3.3, it will suffice to show that $(I^n)_a = \widetilde{I^n}$ for each $n \in \mathbb{N}$. To this end, let $\mathfrak{p} := (x_{\tau(1)},\dots,x_{\tau(s)})$. Then since $\mathfrak{p}$ is generated by a regular R-sequence it follows that $R_\mathfrak{p}$ is a regular local ring and $I^n R_\mathfrak{p}$ is a $\mathfrak{p}R_\mathfrak{p}$-primary ideal. Hence by [3, (2.2.5)] it yields that $(I^n R_\mathfrak{p})_a = \widetilde{I^n R_\mathfrak{p}}$. Now the assertion follows easily from [10, Lemma 2.3]. ■

For any monomial ideal I of R with respect to the regular R-sequence $\mathbf{x} := x_1, \dots, x_d$, we shall use $\widetilde{A}(I)$ to denote the ultimate constant values of the sequence $\{\operatorname{Ass}_R R/\widetilde{I^n}\}_{n\geq 1}$.

Proposition 3.5. *Let T be a faithfully flat Noetherian extension of R such that $\mathbf{x} := x_1, \dots, x_d$ is contained in the Jacobson radical of T. Then for any monomial ideal I of R the following conditions hold:*

(i) *IT is a monomial ideal of T and $\widetilde{IT} \cap R = \widetilde{I}$.*

(ii) *For any $\mathfrak{p} \in \widetilde{A}(I)$ there exists $\mathfrak{q} \in \widetilde{A}(IT)$ such that $\mathfrak{q} \cap R = \mathfrak{p}$.*

Proof. In order to prove (i), let m be a monomial element of T with respect to $\mathbf{x} := x_1, \dots, x_d$ (recall that $\mathbf{x}$ is a T-sequence by faithfully flatness) such that $m \in \widetilde{IT} \cap R$. Then $m^l \in I^l T$ for some $l \in \mathbb{N}$. Since T is faithful over R, it follows that $m^l \in I^l$. Hence $\widetilde{IT} \cap R \subseteq \widetilde{I}$. As the opposite inclusion is obvious, the result follows. Finally, in order to show (ii), let $\mathfrak{p} \in \widetilde{A}(I)$. Then $\mathfrak{p} \in \operatorname{Ass}_R R/\widetilde{I^n}$ for large n. Since T is a Noetherian ring and $\widetilde{I^n T} \cap R = \widetilde{I^n}$ by (i), it follows that there exists $\mathfrak{q} \in \widetilde{A}(IT)$ such that $\mathfrak{q} \cap R = \mathfrak{p}$. This completes the proof. ■

Acknowledgment. The author is deeply grateful to the referee for his or her very careful reading of the original manuscript and valuable suggestions. The material of this paper was suggested by Professor Peter Schenzel. I am deeply grateful to him for his guidance and encouragement. Also, I would like to thank Dr. R. Naghipour for his careful reading of the first draft.

References

[1] Brodmann, M.: Asymptotic stability of $\operatorname{Ass}(M/I^n M)$. *Proc. Amer. Math. Soc.* **74** (1979), 16–18.

[2] Bruns, W. and Herzog, J.: *Cohen-Macaulay rings*. Cambridge Studies in Advanced Mathematics **39**. Cambridge University Press, Cambridge, 1993.

[3] Heinzer, W., Mirbagheri, A., Ratliff, L. J., Jr. and Shah, K.: Parametric decomposition of monomial ideals II. *J. Algebra* **187** (1997), 120–149.

[4] Kaplansky, I.: *Commutative rings*. Revised edition. The University of Chicago Press, Chicago, Ill.-London, 1974.

[5] Kiyek, K. and Stückrad, J.: Integral closure of monomial ideals on regular sequences. (Proceedings of the International Conference on Algebraic Geometry and Singularities, Sevilla, 2001). *Rev. Mat. Iberoamericana* **19** (2003), 483–508.

[6] Matsumura, H.: *Commutative ring theory*. Cambridge Studies in Advanced Mathematics **8**. Cambridge University Press, Cambridge, 1986.

[7] McAdam, S.: *Asymptotic prime divisors*. Lecture Notes in Mathematics **1023**. Springer-Verlag, Berlin, 1983.

[8] Northcott, D. G. and Rees, D.: Reductions of ideals in local rings. *Proc. Cambridge Philos. Soc.* **50** (1954), 145–158.

[9] Ratliff, L. J., Jr.: On asymptotic prime divisors. *Pacific J. Math.* **111** (1984), no. 2, 395–413.

[10] Sharp, R. Y.: Linear growth of primary decompositions of integral closures. *J. Algebra* **207** (1998), 276–284.

[11] Vasconcelos, W. V.: *Computational methods in commutative algebra and algebraic geometry*. Algorithms and Computation in Mathematics **2**. Springer-Verlag, Berlin, 1998.

Recibido: 14 de septiembre de 2004
Revisado: 21 de diciembre de 2004

Monireh Sedghi
Department of Mathematics
University of Tabriz
Tabriz, Iran

and

Department of Mathematics
Azarbaidjan University of Tarbiat Moallem
Azarshar, Tabriz, Iran
m.sedghi@tabrizu.ac.ir

The author thanks the Martin-Luther-Universität Halle-Wittenberg for the hospitality and facilities offered during the preparation of this paper.